

# On spectra of 1-dimensional diffusion operators

Ichiro SHIGEKAWA  
 Kyoto University

## 1. Introduction

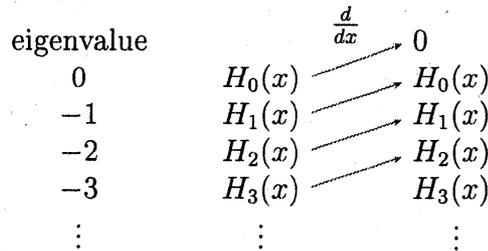
It is well-known that the Hermite polynomials are eigenfunctions of Ornstein-Uhlenbeck operator. Here the Hermite polynomials are defined by

$$(1.1) \quad H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \dots$$

They satisfy the following identity:

$$H'_n(x) = H_{n-1}(x).$$

This can be summarized as follows:



This relation suggests us that the differentiation gives rise to a correspondence between two families of eigenfunctions. In this paper, we will give a general framework of this fact for one dimensional diffusions.

The organization of the paper is as follows. In Section 2, we develop a general theory of one dimensional diffusion operators. In Section 3, we use the supersymmetry to investigate the spectrum and we give several examples in Section 4.

## 2. One dimensional diffusion operators

### 2.1 General framework of one dimensional diffusion operator

We give a general framework of one dimensional diffusion operators. We take  $I = [0, \infty)$  as a state space. Suppose we are given two continuous functions  $a, p$  on  $I$ . We assume that  $a > 0, p > 0$  on  $(0, \infty)$ . We define a measure  $\mu$  by  $\mu = p dx$ . To denote  $L^2(\mu)$ , we use  $L^2(p)$  for simplicity. We consider an operator on  $H = L^2(p)$  defined by

$$(2.1) \quad \mathcal{A}u = \frac{1}{p} (apu)'$$

The associated Dirichlet form is

$$(2.2) \quad \mathcal{E}(u, v) = \int_0^\infty u'v'ap \, dx.$$

This corresponds to the Neumann boundary condition. If we impose the Dirichlet boundary condition, we restrict the domain to functions with  $u(0) = 0$ .

Further we introduce the following functions:

$$(2.3) \quad m(x) = \int_0^x p(y) \, dy,$$

$$(2.4) \quad s(x) = \int_0^x \frac{1}{a(y)p(y)} \, dy.$$

The measure  $dm = d\mu$  is called a speed measure and  $s$  is called a scale function. We assume the integrability of  $p$  and  $\frac{1}{ap}$  near 0 and so  $m(0) = s(0) = 0$ . At infinity, we assume  $m(\infty) + s(\infty) = \infty$ . By Feller's classification of the boundary, 0 is exit and entrance and  $\infty$  is non-exit or non-entrance (here we use the terminology in Itô-McKean [4] § 4.1). We assume these assumptions as a typical case and the other cases can be treated similarly and we may choose any interval  $[a, b]$  instead of  $[0, \infty)$ .

We define another measure  $\nu = adm$  and consider  $L^2(\nu)$ . Again we use the notation  $L^2(ap)$  instead of  $L^2(\nu)$ . The domain  $\text{Dom}(\mathcal{E})$  is given by

$$(2.5) \quad \text{Dom}(\mathcal{E}) = \{u \in L^2(p); u \text{ is absolutely continuous on } (0, \infty) \text{ and } u' \in L^2(ap)\}.$$

The topology in  $\text{Dom}(\mathcal{E})$  is given by the following inner product  $\mathcal{E}_1$ :

$$(2.6) \quad \mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_\mu.$$

Here  $(u, v)_\mu$  denotes the  $L^2$  inner product in  $L^2(\mu) = L^2(p)$ .

**Proposition 2.1.** Take any  $u \in \text{Dom}(\mathcal{E})$ . Then  $u$  is absolutely continuous on  $[0, \infty)$ , i.e.,  $u(0+)$  exists. Setting  $u(0) = u(0+)$ , we have

$$(2.7) \quad |u(0)| \leq \frac{1}{m(x)^{1/2}} \|u\|_{dm} + \mathcal{E}(u, u)^{1/2} s(x)^{1/2}.$$

Further, if  $s(\infty) < \infty$  then  $u(\infty)$  exists and if  $s(\infty) < \infty$ ,  $m(\infty) = \infty$  then  $u(\infty) = 0$ .

*Proof.* Take any  $x, y$  so that  $0 < x < y$ . We have

$$\begin{aligned} |u(y) - u(x)| &\leq \left| \int_x^y u'(t) \, dt \right| \\ &= \left| \int_x^y u'(t) \sqrt{ap} \frac{1}{\sqrt{ap}} \, dt \right| \\ &\leq \sqrt{\int_x^y |u'(t)|^2 ap \, dt} \sqrt{\int_x^y \frac{1}{ap} \, dt} \\ &\leq \mathcal{E}(u, u)^{1/2} (s(y) - s(x))^{1/2}. \end{aligned}$$

From this, we can see the existence of  $u(0+)$ . Moreover, the computation above show that  $u' \in L^1([0, l])$  for any  $l > 0$ , and so  $u$  is absolutely continuous on  $[0, \infty)$ . On the other hand, since  $|u(x) - u(0)| \leq \mathcal{E}(u, u)^{1/2} s(x)^{1/2}$ , we have

$$\begin{aligned}
|u(0)|^2 m(x) &= \int_0^x |u(0)|^2 p(t) dt \\
&\leq \int \{|u(t)| + \mathcal{E}(u, u)^{1/2} s(t)^{1/2}\}^2 p(t) dt \\
&\leq \int \{|u(t)|^2 + 2|u(t)|\mathcal{E}(u, u)^{1/2} s(t)^{1/2} + \mathcal{E}(u, u) s(t)\}^2 p(t) dt \\
&\leq \|u\|_{dm} + 2 \left\{ \int_0^x |u(t)|^2 p(t) dt \right\}^{1/2} \left\{ \int_0^x \mathcal{E}(u, u) s(t) p(t) dt \right\}^{1/2} \\
&\quad + \int_0^x \mathcal{E}(u, u) s(t) p(t) dt \\
&\leq \|u\|_{dm}^2 + 2\|u\|_{dm} \mathcal{E}(u, u)^{1/2} s(x)^{1/2} m(x)^{1/2} + \mathcal{E}(u, u) s(x) m(x) \\
&\leq (\|u\|_{dm} + \mathcal{E}(u, u)^{1/2} s(x)^{1/2} m(x)^{1/2})^2.
\end{aligned}$$

Hence

$$|u(0)| \leq \frac{1}{m(x)^{1/2}} \|u\|_{dm}^2 + \mathcal{E}(u, u)^{1/2} s(x)^{1/2}$$

which is the desired result (2.7).

If  $s(\infty) < \infty$ , the computation above shows the existence of  $u(\infty)$ . If, in addition,  $m(\infty) = \infty$ , then we have  $u(\infty) = 0$ . In fact, assume that  $u(\infty) \neq 0$ . Then there exist constants  $c > 0$  and  $N > 0$  such that  $|u(x)| \geq c$  for  $x \geq N$ . Combining this with  $u \in L^2(p)$ , we have

$$\infty > \int_N^\infty |u(x)|^2 p dx \geq \int_N^\infty c^2 p dx = c^2(m(\infty) - m(N))$$

which contradicts to  $m(\infty) = \infty$ . Thus we can conclude that  $u(\infty) = 0$ .  $\square$

By setting  $u(0) = u(0+)$  for  $u \in \text{Dom}(\mathcal{E})$ ,  $u(0)$  is well-defined. Moreover, we see that the mapping  $u \mapsto u(0)$  is a continuous linear functional from  $\text{Dom}(\mathcal{E})$  into  $\mathbb{R}$ .

Now we define an operator  $V: L^2(p) \rightarrow L^2(ap)$  as follow.

$$(2.8) \quad Vu = u'.$$

The inner product in  $L^2(ap)$  is, as usual, given by

$$(2.9) \quad (u, v)_\nu = \int_0^\infty u'(x)v'(x)a(x)p(x) dx.$$

Here, for  $\text{Dom}(V)$ , we consider two cases;  $\text{Dom}(V) = \text{Dom}(\mathcal{E})$  and  $\text{Dom}(V) = \text{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$ . The former is called the Neumann boundary condition and the latter is called the Dirichlet boundary condition.

**Proposition 2.2.**  $V: L^2(p) \rightarrow L^2(ap)$  is a closed operator. Defining a bilinear form  $b$  by

$$(2.10) \quad b(u, v) = (Vu, Vv) = \mathcal{E}(u, v), \quad u, v \in \text{Dom}(V),$$

$b$  satisfies the Markovian property.

*Proof.* If  $\text{Dom}(V) = \text{Dom}(\mathcal{E})$ , then we have  $u'$  is in  $L^2(ap)$  and the closability of  $V$  follows easily. In the case that  $\text{Dom}(V) = \text{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$ , the property  $u(0) = 0$  is preserved by taking limit because of (2.7). So  $V$  is closed as well in this case.

The Markovian property is easy.  $\square$

There exists a non-positive self-adjoint operator associated with  $b$ , which is given by  $-V^*V$  (von Neumann's theorem, e.g., see [5] Theorem V.3.24). So let us compute  $V^*$ . To do this, we need

**Proposition 2.3.** Take any  $\theta \in L^2(ap)$ . If we assume that  $ap\theta$  is absolutely continuous on  $(0, \infty)$  and  $\frac{(ap\theta)'}{p} \in L^2(p)$ , then we have that  $ap\theta$  is absolutely continuous on  $[0, \infty)$ . We also have

$$(2.11) \quad |ap\theta(0+)| \leq \frac{\|\theta\|_{L^2(ap)}}{s(x)^{1/2}} + \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)} m(x)^{1/2}.$$

Further, if  $m(\infty) < \infty$ , then  $ap\theta(\infty)$  exists and, if, in addition,  $s(\infty) = \infty$ , then we have  $ap\theta(\infty) = 0$ .

*Proof.* Take any  $x, y$  so that  $0 < x < y$ . Then

$$\begin{aligned} |ap\theta(y) - ap\theta(x)| &\leq \left| \int_x^y (ap\theta)' dt \right| \\ &= \left| \int_x^y \frac{(ap\theta)'}{\sqrt{p}} \sqrt{p} dt \right| \\ &\leq \sqrt{\int_x^y \frac{(ap\theta)'^2}{p} dt} \sqrt{\int_x^y p dt} \\ &\leq \sqrt{\int_x^y \frac{(ap\theta)'^2}{p^2} p dt} (m(y) - m(x))^{1/2}. \end{aligned}$$

From this, we can see the existence of  $ap\theta(0+)$ . Moreover, the computation above shows that  $(ap\theta)' \in L^1([0, l])$  for any  $l > 0$ , and hence we have that  $ap\theta$  is absolutely continuous on  $[0, \infty)$ .

On the other hand,

$$|ap\theta(x) - ap\theta(0+)| \leq \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)} m(x)^{1/2}$$

and hence

$$|ap\theta(0+)|^2 s(x)$$

$$\begin{aligned}
&= \int |ap\theta(0+)|^2 \frac{1}{a(t)p(t)} dt \\
&\leq \int \left\{ |ap\theta(t)| + \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)} m(t)^{1/2} \right\}^2 \frac{1}{a(t)p(t)} dt \\
&\leq \int \left\{ |ap\theta(t)|^2 + 2|ap\theta(t)| \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)} m(t)^{1/2} + \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)}^2 m(t) \right\} \frac{1}{a(t)p(t)} dt \\
&\leq \|\theta\|_{L^2(ap)}^2 + 2 \left\{ \int_0^x |\theta(t)|^2 a(t)p(t) dt \right\}^{1/2} \left\{ \int_0^x \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)}^2 m(t) \frac{1}{a(t)p(t)} dt \right\}^{1/2} \\
&\quad + \int_0^x \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)}^2 m(t) \frac{1}{a(t)p(t)} dt \\
&\leq \|\theta\|_{L^2(ap)}^2 + 2\|\theta\|_{L^2(ap)} \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)} m(x)^{1/2} s(x)^{1/2} + \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)}^2 m(x)s(x) \\
&\leq (\|\theta\|_{L^2(ap)} + \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)} m(x)^{1/2} s(x)^{1/2})^2.
\end{aligned}$$

This implies

$$|ap\theta(0+)| \leq \frac{\|\theta\|_{L^2(ap)}}{s(x)^{1/2}} + \left\| \frac{(ap\theta)'}{p} \right\|_{L^2(p)} m(x)^{1/2},$$

which is the desired result (2.11).

If  $m(\infty) < \infty$ , we easily see the existence of  $ap\theta(\infty)$  similarly. If, in addition,  $s(\infty) = \infty$ , we have  $ap\theta(\infty) = 0$ . In fact, if  $ap\theta(\infty) \neq 0$ , then there exist  $c > 0$  and  $N$  so that,  $|ap\theta(x)| \geq c > 0$  for  $x \geq N$ . Hence

$$|\theta(x)| \geq \frac{c}{ap}.$$

But the assumption  $\theta \in L^2(ap)$  implies

$$\infty > \int_x^\infty \theta^2 ap dt \geq \int_x^\infty \left( \frac{c}{ap} \right)^2 ap dt \geq \int_x^\infty \frac{c^2}{ap} dt = c^2(s(\infty) - s(x))$$

which contradicts to  $s(\infty) = \infty$ . Thus we have  $ap\theta(\infty) = 0$ .  $\square$

It may happen that  $a(0) = 0$  but this does not mean  $ap\theta(0+) = 0$ . So, to avoid misunderstanding, we do not use the notation  $ap\theta(0)$ . We use  $ap\theta(0+)$  instead.

We denote by  $C_0([0, \infty))$  the set of all continuous functions on  $[0, \infty)$  with compact support. Notice that  $u(0) \neq 0$  is possible. We denote it by  $C_0$  for simplicity. We have the following

**Proposition 2.4.**  $\text{Dom}(\mathcal{E}) \cap C_0$  is dense in  $\text{Dom}(\mathcal{E})$  and  $\text{Dom}(\mathcal{E}) \cap C_0 \cap \{u : u(0) = 0\}$  is dense in  $\text{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$ .

*Proof.* This is a deep result. See Fukushima-Oshima-Takeda [3] Example 1.2.2, for the proof.  $\square$

Further, we denote by  $C_\kappa([0, \infty))$  the set of all continuous functions on  $(0, \infty)$  with compact support. In this case, we have  $u(0) = 0$ . We denote it by  $C_\kappa$  for simplicity. We have the following

**Proposition 2.5.**  $\text{Dom}(\mathcal{E}) \cap C_\kappa$  is dense in  $\text{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$ .

*Proof.* This is rather well-known. For example, note that  $u(\varepsilon + \cdot)$  converges to  $u$  strongly in  $\mathcal{E}_1$  as  $\varepsilon \rightarrow 0$ .  $\square$

Now we can show the following integration by parts formula.

**Proposition 2.6.** Take any  $u \in \text{Dom}(\mathcal{E})$  and  $\theta \in L^2(ap)$ . Assume that  $ap\theta$  is absolutely continuous on  $(0, \infty)$  and  $\frac{(ap\theta)'}{p} \in L^2(p)$ . Then we have

$$(2.12) \quad \int_0^\infty u' \theta ap \, dt = -u(0)ap\theta(0+) - \int_0^\infty u(ap\theta)' \, dt.$$

Moreover we have  $uap\theta(\infty) = 0$ .

*Proof.* First we show that both hands of (2.12) are well-defined. We have

$$\int_0^\infty u' \theta ap \, dt \leq \left\{ \int_0^\infty (u')^2 ap \, dt \right\} \left\{ \int_0^\infty \theta^2 ap \, dt \right\}.$$

So the left hand side is well-defined.

In Proposition 2.1, Proposition 2.3, we have shown that  $u(0)ap\theta(0+)$  exists. Furthermore, noting that

$$\int_0^\infty u(ap\theta)' \, dt = \int_0^\infty u \frac{(ap\theta)'}{p} p \, dt \leq \left\{ \int_0^\infty u^2 p \, dt \right\} \left\{ \int_0^\infty \frac{(ap\theta)'^2}{p^2} p \, dt \right\},$$

we can see that the right hand side is well-defined.

Now assume that  $u, \theta$  satisfies the conditions. Since  $u, ap\theta$  are absolutely continuous on  $[0, l]$  for any  $l > 0$ , we have

$$(2.13) \quad \int_0^l u' \theta ap \, dt = u(l)ap\theta(l) - u(0)ap\theta(0+) - \int_0^l u(ap\theta)' \, dt.$$

If  $u \in \text{Dom}(\mathcal{E}) \cap C_0$ , then  $u(l)ap\theta(l) = 0$  for large  $l$ . Hence, by letting  $l \rightarrow \infty$ , we have for  $u \in \text{Dom}(\mathcal{E}) \cap C_0$ ,

$$\int_0^\infty u' \theta ap \, dt = -u(0)ap\theta(0+) - \int_0^\infty u(ap\theta)' \, dt.$$

Now, by Proposition 2.4,  $\text{Dom}(\mathcal{E}) \cap C_0$  is dense in  $\text{Dom}(\mathcal{E})$ . So taking limit, (2.12) follows.

On the other hand, by (2.13) we have

$$\lim_{l \rightarrow \infty} uap\theta(l) = u(0)ap\theta(0+) + \int_0^\infty u' \theta ap \, dt + \int_0^\infty u(ap\theta)' \, dt.$$

Since we have shown (2.12), the right hand side equals to 0, which means  $uap\theta(\infty) = 0$ .  $\square$

Now we can give a characterization of the dual operator  $V^*$ .

**Proposition 2.7.** The dual operator  $V^*: L^2(ap) \rightarrow L^2(p)$  of  $V: L^2(p) \rightarrow L^2(ap)$  is given by

$$(2.14) \quad V^*(\theta) = -\frac{(ap\theta)'}{p}.$$

Here, if  $\text{Dom}(V) = \text{Dom}(\mathcal{E})$ , then

$$(2.15) \quad \text{Dom}(V^*) = \{\theta \in L^2(ap); \frac{(ap\theta)'}{p} \in L^2(p), ap\theta(0+) = 0\},$$

and if  $\text{Dom}(V) = \text{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$ , then

$$(2.16) \quad \text{Dom}(V^*) = \{\theta \in L^2(ap); \frac{(ap\theta)'}{p} \in L^2(p)\}.$$

*Proof.* Take any  $\theta \in \text{Dom}(V^*)$  and set  $V^*\theta = u$ . Then, for any  $v \in C_0^\infty((0, \infty))$ , we have

$$\int_0^\infty uvp \, dx = (u, v)_{dm} = (V^*\theta, v)_{dm} = (\theta, Vv)_\nu = (\theta, v')_\nu = \int_0^\infty \theta v' ap \, dx.$$

This means that  $(\theta ap)' = -up$  in the distribution sense. Since  $\frac{(\theta ap)'}{p} = -u \in L^2(p)$ , we have that  $\theta ap$  is absolutely continuous on  $[0, \infty)$  and  $ap\theta(0+)$  exists by virtue of Proposition 2.3.

Using these, we first deal with the Neumann boundary condition case.

$$\begin{aligned} (v, V^*\theta)_{dm} &= (Vv, \theta)_\nu \\ &= (v', \theta)_\nu \\ &= \int_0^\infty a(x)p(x)\theta(x)v'(x) \, dx \\ &= -ap\theta(0+)v(0) - \int_0^\infty (ap\theta)'(t)v(t) \, dt \quad (\because (2.12)) \\ &= -ap\theta(0+)v(0) - \int_0^\infty \frac{(ap\theta)'(t)}{p(t)}v(t)p(t) \, dt. \end{aligned}$$

This identity holds for  $v \in \text{Dom}(V)$  and the mapping  $v \mapsto (v, V^*\theta)_{dm}$  is a continuous linear functional on  $L^2(p)$ . But the mapping  $v \mapsto v(0)$  is not continuous on  $L^2(p)$ . Thus  $ap\theta(0+) = 0$  must hold.

Conversely, if  $\theta \in L^2(ap)$  satisfies  $\frac{(ap\theta)'}{p} \in L^2(p)$  and  $ap\theta(0+) = 0$ , then, repeating the previous computation, we have

$$\begin{aligned} (\theta, Vv)_\nu &= (\theta, v')_\nu \\ &= \int_0^\infty \theta v' ap \, dt \\ &= -v(0)ap\theta(0+) - \int_0^\infty (ap\theta)'(t)v(t) \, dt \quad (\because (2.12)) \end{aligned}$$

$$= - \int_0^{\infty} \frac{(ap\theta)'(t)}{p(t)} v(t)p(t) dt,$$

which means  $\theta \in \text{Dom}(V^*)$  and  $V^*\theta = -\frac{(ap\theta)'}{p}$ .

Next we assume that the Dirichlet boundary condition on  $V$  is imposed. If  $\theta$  satisfies  $\frac{(\theta ap)'}{p} \in L^2(p)$ , then we have for  $v \in \text{Dom}(V)$

$$\begin{aligned} (\theta, Vv)_\nu &= (\theta, v')_\nu \\ &= \int_0^{\infty} \theta v' ap dt \\ &= -v(0)(ap\theta)(0+) - \int_0^{\infty} (ap\theta)'(t)v(t) dt \quad (\because (2.12)) \\ &= - \int_0^{\infty} \frac{(ap\theta)'(t)}{p(t)} v(t)p(t) dt. \end{aligned}$$

This means that  $\theta \in \text{Dom}(V^*)$  and  $V^*\theta = -\frac{(ap\theta)'}{p}$ , which completes the proof.  $\square$

The self-adjoint operator associated with the bilinear form  $b$  is  $-V^*V$ . We set  $\mathfrak{A} = -V^*V$ , which is characterized as follows:

**Theorem 2.8.**  $u \in \text{Dom}(\mathfrak{A})$  if and only if the following three conditions are fulfilled:

1.  $u$  is absolutely continuous on  $(0, \infty)$  and  $u' \in L^2(ap)$ ,
2.  $apu'$  is absolutely continuous on  $(0, \infty)$  and  $\frac{(apu')'}{p} \in L^2(p)$ ,
3.  $apu'(0+) = 0$  in the case of Neumann boundary condition and  $u(0) = 0$  in the case of Dirichlet boundary condition.

In this case we have  $\mathfrak{A}u = \frac{(apu')'}{p}$ .

*Proof.* This follows from the definition (2.8) of  $V$  and the characterization of  $V^*$  given in Proposition 2.7. In the case of Neumann boundary condition, we have

$$apu'(0+) = 0$$

since  $\text{Dom}(V^*)$  is restricted. Similarly we have  $u(0) = 0$  in the case of Dirichlet boundary condition since  $\text{Dom}(V) = \text{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$ .  $\square$

By this theorem, we have completely characterized the operator given by (2.1). The operator  $VV^*$  is also characterized as follows:

**Theorem 2.9.**  $\theta \in \text{Dom}(VV^*)$  if and only if the following three conditions are fulfilled:

1.  $ap\theta$  is absolutely continuous on  $(0, \infty)$  and  $\frac{(ap\theta)'}{p} \in L^2(p)$ ,
2.  $\frac{(ap\theta)'}{p}$  is absolutely continuous on  $(0, \infty)$  and  $\left(\frac{(ap\theta)'}{p}\right)' \in L^2(ap)$ ,

3.  $ap\theta(0+) = 0$  in the case of Neumann boundary condition and  $\frac{(ap\theta)'}{p}(0+) = 0$  in the case of Dirichlet boundary condition.

In this case we have

$$(2.17) \quad VV^*\theta = \left( \frac{(ap\theta)'}{p} \right)'$$

*Proof.* The proof is the same as Theorem 2.8.  $\square$

So far, we have assumed only the continuity of  $a$  and  $p$ . If we assume differentiability of them, we can give simpler expressions of  $V^*$ ,  $-V^*V$  and  $-VV^*$ . We set  $\hat{\mathfrak{A}} = -VV^*$ .

**Corollary 2.10.** Assume that  $a$  and  $p$  are  $C^2$  functions on  $(0, \infty)$ . Then, we have

$$(2.18) \quad -V^*\theta = a\theta' + b\theta.$$

Here we set

$$(2.19) \quad b = a' + a(\log p)'$$

This leads to

$$(2.20) \quad \mathfrak{A}u = -V^*Vu = au'' + bu',$$

$$(2.21) \quad \hat{\mathfrak{A}}\theta = -VV^*\theta = a\theta'' + (a' + b)\theta' + b'\theta.$$

*Proof.* First we have

$$\begin{aligned} -V^*\theta &= -\frac{(ap\theta)'}{p} \\ &= \frac{a'p\theta + ap'\theta + ap\theta'}{p} \\ &= a'\theta + a(p'/p)\theta + a\theta' \\ &= a\theta' + (a' + a(\log p)')\theta \\ &= a\theta' + b\theta, \end{aligned}$$

which shows (2.20). Further we have

$$-VV^*\theta = (a\theta' + b\theta)' = a'\theta' + a\theta'' + b'\theta + b\theta' = a\theta'' + (a' + b)\theta' + b'\theta,$$

which is (2.21).  $\square$

## 2.2 The Neumann boundary condition

From the expression in (2.21), it seems that  $\hat{\mathfrak{A}}$  is a sum of two operators  $a\theta'' + (a' + b)\theta'$  and  $b'\theta$ . But, in general, it is a subtle problem to define a sum of operators. So we give a characterization in terms of bilinear form. To do this, we restrict ourselves to the case of Neumann boundary condition, i.e., we assume that  $\text{Dom}(V) = \text{Dom}(\mathcal{E})$ .

The Dirichlet form associated with  $a\theta'' + (a' + b)\theta'$  in  $L^2(ap)$  is

$$(2.22) \quad \mathcal{E}^{(1)}(\theta, \eta) = \int_0^\infty \theta' \eta' a^2 p dx.$$

In fact, in our formulation with functions  $a$  and  $p$ , we take  $a$  and  $ap$ . The speed measure and scale function are given by

$$(2.23) \quad m^{(1)}(x) = \int_1^x a(y)p(y) dy,$$

$$(2.24) \quad s^{(1)}(x) = \int_1^x \frac{1}{a^2(y)p(y)} dy.$$

The classification of the boundaries is different from that with  $a$  and  $p$ . In this case, we impose *the Dirichlet boundary condition* if the boundary is exit and entrance. Therefore  $\text{Dom}(\mathcal{E}^{(1)})$  is the closure of  $\text{Dom}(\mathcal{E}^{(1)}) \cap C_\kappa$ . For  $\theta, \eta \in \text{Dom}(\mathcal{E}^{(1)}) \cap C_\kappa$ , we have

$$\begin{aligned} \int_0^\infty \theta' \eta' a^2 p dx &= - \int_0^\infty (a^2 p \theta')' \eta dx \\ &= - \int_0^\infty (a^2 p \theta'' + 2a a' p \theta' + a^2 p' \theta') \eta dx \\ &= - \int_0^\infty (a\theta'' + 2a'\theta' + a(p'/p)\theta') \eta ap dx \\ &= - \int_0^\infty (a\theta'' + 2a'\theta' + a(\log p)'\theta') \eta ap dx \\ &= - \int_0^\infty (a\theta'' + (a' + b)\theta') \eta ap dx. \end{aligned}$$

In the last line, we used  $b = a' + a(\log p)'$ . Since  $\text{Dom}(\mathcal{E}^{(1)}) \cap C_\kappa$  is dense in  $\text{Dom}(\mathcal{E}^{(1)})$ , the identity above is valid for  $\theta, \eta \in \text{Dom}(\mathcal{E}^{(1)})$ . Thus the bilinear form  $\mathcal{E}^{(1)}$  is associated with  $a\theta'' + (a' + b)\theta'$ .

The potential term  $b'\theta$  corresponds to the following bilinear form:

$$(2.25) \quad \mathcal{E}^{(2)}(\theta, \eta) = - \int_0^\infty b' \theta \eta ap dx.$$

To ensure the closedness, we assume that  $b'$  is bounded from above. We will prove that the self-adjoint operator  $-VV^*$  is associated with  $\mathcal{E}^{(1)} + \mathcal{E}^{(2)}$ .

We have considered the operator  $V: L^2(p) \rightarrow L^2(ap)$  but it is convenient to consider the operator  $\hat{V}: L^2(p) \rightarrow L^2(1/(ap))$  defined by

$$(2.26) \quad \hat{V}u = apu'.$$

This operator reflects the symmetry of  $dm$  and  $ds$ . Of course  $\hat{V}^*$  corresponds to  $V^*$ . Usually, they are denoted by  $\hat{V}u = \frac{du}{ds}$  and  $\hat{V}^*u = \frac{du}{dm}$ . Here  $m$  and  $s$  are defined in (2.3)

and (2.4). The following diagram is commutative:

$$\begin{array}{ccccc}
 & & L^2(ap) & & \\
 & \nearrow & \downarrow & \searrow & \\
 & Vu = u' & ap\theta & -V^*\theta = \frac{(ap\theta)'}{p} & \\
 L^2(dm) & \xrightarrow{\quad} & L^2(ds) & \xrightarrow{\quad} & L^2(dm) \\
 & \hat{V}u = apu' & & -\hat{V}^*\theta = \frac{\theta'}{p} & 
 \end{array}$$

The mapping  $\theta \mapsto ap\theta$  is a unitary operator from  $L^2(ap)$  onto  $L^2(ds)$ .

Since we have imposed the Neumann boundary condition on  $V$ , the boundary condition  $ap\theta(0+) = 0$  is imposed on  $V^*$ . Under the isomorphism  $\theta \mapsto ap\theta$ , this means that  $\theta \in \text{Dom}(\hat{V}^*)$  satisfies  $\frac{\theta'}{p} \in L^2(p)$  and the boundary condition is  $\theta(0) = 0$ . Note that if we replace  $ds$  and  $dm$  in the definition of  $V$ , we get  $\hat{V}^*$ . The boundary condition of  $\hat{V}^*$  is  $\theta(0) = 0$ , i.e., the Dirichlet boundary condition. The properties  $s(0) + m(0) > -\infty$  and  $s(\infty) + m(\infty) = \infty$  do not change even if we exchange  $ds$  for  $dm$ . Hence we can use Proposition 2.5 and obtain that  $\text{Dom}(\hat{V}^*) \cap C_\kappa$  is dense in  $\text{Dom}(\hat{V}^*)$ . This brings that  $\text{Dom}(V^*) \cap C_\kappa$  is dense in  $\text{Dom}(V^*)$  by the isomorphism  $\theta \mapsto ap\theta$ . In fact, the mapping  $\theta \mapsto ap\theta$  preserves the property that  $\theta$  has a compact support. Now we can have the following theorem.

**Theorem 2.11.** Impose the Neumann boundary condition and assume that  $b'$  is bounded from above. Then the bilinear form associated with  $-VV^*$  is  $\mathcal{E}^{(1)} + \mathcal{E}^{(2)}$ .

*Proof.* The bilinear form associated with  $-V^*V$  is  $(V^*\theta, V^*\eta)_\nu$ . Take any  $\theta, \eta \in \text{Dom}(V^*) \cap C_\kappa$ . Then, by (2.18), we have

$$\begin{aligned}
 (V^*\theta, V^*\eta)_{dm} &= \int_0^\infty (a\theta' + b\theta)(a\eta' + b\eta)p \, dx \\
 &= \int_0^\infty (a^2\theta' + ab(\theta'\eta + \theta\eta') + b^2\theta\eta)p \, dx \\
 &= \int_0^\infty (a^2\theta' + ab(\theta\eta)' + b^2\theta\eta)p \, dx \\
 &= \int_0^\infty \{(a^2\theta' + b^2\theta\eta)p + abp(\theta\eta)'\} \, dx \\
 &= \int_0^\infty \{(a^2\theta' + b^2\theta\eta)p - (abp)'\theta\eta\} \, dx \\
 &= \int_0^\infty \{(a^2\theta'\eta' + b^2\theta\eta)p - (a'bp + ab'p + abp')\theta\eta\} \, dx \\
 &= \int_0^\infty \{(a^2\theta'\eta' + b^2\theta\eta)p - b(a'p + ap')\theta\eta - ab'p\theta\eta\} \, dx \\
 &= \int_0^\infty \{(a^2\theta'\eta' + b^2\theta\eta)p - bp(a'p + a(p'/p))\theta\eta - ab'p\theta\eta\} \, dx
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \{(a^2\theta'\eta' + b^2\theta\eta)p - bp(a'p + a(\log p)')\theta\eta - ab'p\theta\eta\} dx \\
&= \int_0^\infty \{(a^2\theta'\eta' + b^2\theta\eta)p - b^2p\theta\eta - ab'p\theta\eta\} dx \\
&= \int_0^\infty (a\theta'\eta' - b'\theta\eta)ap dx \\
&= \mathcal{E}^{(1)}(\theta, \eta) + \mathcal{E}^{(2)}(\theta, \eta).
\end{aligned}$$

Since  $\text{Dom}(V^*) \cap C_\kappa$  is dense in  $\text{Dom}(V^*)$ , the identity above holds for any  $\theta, \eta \in \text{Dom}(V^*)$ . This completes the proof.  $\square$

### 3. Supersymmetry and one dimensional diffusion operator

We first give a quick review of the supersymmetry. See e.g., Simon et al. [2] §6.3 for details.

**Proposition 3.1.** Let  $T$  be a closed operator from a Hilbert space  $H_1$  to a Hilbert space  $H_2$ . Then operators  $T^*T$  and  $TT^*$  has the same spectral structure except for 0.

This can be easily seen by noting the mapping  $\sqrt{T^*T}u \mapsto Tu$  is an isometric isomorphism from  $\text{Ran}(\sqrt{T^*T})$  onto  $\text{Ran}(T)$ . In particular,  $T$  gives rise to a correspondence between eigenvectors as follows:

**Proposition 3.2.** Take any  $\lambda > 0$ . If  $x$  is an eigenvector for an eigenvalue  $\lambda$  of  $T^*T$ , then  $Tx$  is an eigenvector for an eigenvalue  $\lambda$  of  $TT^*$ . Conversely, if  $Tx$  is an eigenvector for an eigenvalue  $\lambda$  of  $TT^*$  and  $x \perp \text{Ker}(T)$ , then  $x$  is an eigenvector for an eigenvalue  $\lambda$  of  $T^*T$ .

Applying these results to the operator  $V$  defined by (2.8), we can get the following

**Theorem 3.3.** Two operators  $V^*V$  and  $VV^*$  have the same spectrum except for 0. Here if the Neumann boundary condition is imposed on  $V$ , then the condition  $ap\theta(0+) = 0$  is attached to  $VV^*$  and if the Dirichlet boundary condition is imposed on  $V$ , then the condition  $\frac{(ap\theta)'}{p}(0+) = 0$  is attached to  $VV^*$ . Moreover the correspondence between eigenfunctions is given by the mapping  $u \mapsto u'$ .

If, in addition, we assume that functions  $a$  and  $p$  are of class  $C^2$ , we can give more explicite expression. We recall that the bilinear forms  $\mathcal{E}^{(1)}$  of (2.22) and  $\mathcal{E}^{(2)}$  of (2.25) are closures of their restrictions to functions with compact support.

**Theorem 3.4.** Assume the same conditions as in Theorem 2.11. Then the operator  $\mathfrak{A}u = au'' + bu'$  and the operator  $\hat{\mathfrak{A}}\theta = a\theta'' + (a' + b)\theta' + b'\theta$  have the same spectrum except for 0. Here the Neumann boundary condition is imposed on  $\mathfrak{A}$  and  $\hat{\mathfrak{A}}$  is the self-adjoint operator associated to the bilinear form  $\mathcal{E}^{(1)} + \mathcal{E}^{(2)}$ . Moreover the mapping  $u \mapsto u'$  gives rise to a correspondence between eigenfunctions of  $\mathfrak{A}$  and  $\hat{\mathfrak{A}}$ .

In summing up, we can say that if we differentiate the system of eigenfunctions of one dimensional diffusion operator, then we get another system of eigenfunctions. This can be seen, at least formally, from the following computation. Assume  $au'' + bu' = \lambda u$ . Then

$$a'u'' + a'u''' + b'u' + bu'' = \lambda u'.$$

Hence

$$a(u')'' + (a' + b)(u') + b'u' = \lambda u'.$$

This means that  $u'$  is an eigenfunction of  $\hat{\mathfrak{A}}$ .

From Theorem 3.4, we also have the following

**Corollary 3.5.** Assume that  $b(x) \leq -c < 0$ , then  $-\mathfrak{A}$  has a spectral gap  $\geq c$ .

## 4. Examples

Applying the results of Section 3 to typical examples in one dimensional diffusion operators, we can get rather well-known results. But our aim here is to give a unified explanation.

### 4.1 Hermite polynomials

Take  $a = 1$ ,  $p = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$  on  $\mathbb{R}$ . Then

$$b = a' + a(\log p)' = (-\log \sqrt{2\pi} - x^2/2)' = -x$$

and so

$$\mathfrak{A}u = u'' - xu'$$

$$\hat{\mathfrak{A}}u = u'' - xu' - u.$$

By the result of Section 3,  $u'' - xu'$  and  $u'' - xu' - u$  have the same spectrum except for 0. From this, we can say that the spectrum of  $\mathfrak{A}$  is  $0, -1, -2, \dots$ . Here we use the fact that the constant function is an eigenfunction. Of course,  $\mathfrak{A}$  is the Ornstein-Uhlenbeck operator and all eigenfunctions are known, i.e., the Hermite polynomials defined by

$$(4.1) \quad H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

As is well-known, we have

$$H_n'(x) = H_{n-1}(x),$$

which gives the following correspondence as a system:

$$\begin{array}{ccc} \text{eigenvalue} & u'' - xu' & u'' - xu' - u \\ -1 & \left( \begin{array}{c} H_1(x) \\ H_2(x) \\ \vdots \\ H_n(x) \\ \vdots \end{array} \right) & \left( \begin{array}{c} H_0(x) \\ H_1(x) \\ \vdots \\ H_{n-1}(x) \\ \vdots \end{array} \right) \\ -2 & & \\ \vdots & & \\ -n & & \\ \vdots & & \end{array} \xrightarrow{\frac{d}{dx}}$$

## 4.2 Laguerre polynomials

Take  $a = x$ ,  $p = x^{\alpha-1}e^{-x}$  on  $[0, \infty)$ . Here we assume  $\alpha > 0$ . Then

$$b = a' + a(\log p)' = 1 + x((\alpha - 1)\log x - x)' = 1 + x\left(\frac{\alpha - 1}{x} - 1\right)' = \alpha - x$$

and so

$$\begin{aligned}\mathfrak{A}u &= xu'' + (\alpha - x)u', \\ \hat{\mathfrak{A}}u &= xu'' + (\alpha + 1 - x)u' - u.\end{aligned}$$

Let us call the operator  $xu'' + (\alpha - x)u'$  as the Laguerre operator since their eigenfunctions are Laguerre polynomials  $L_n^{(\alpha-1)}$ . Here the Laguerre polynomial  $L_n^{(c)}$ ,  $c > -1$  is defined by

$$(4.2) \quad L_n^{(c)}(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}), \quad n = 0, 1, 2, \dots$$

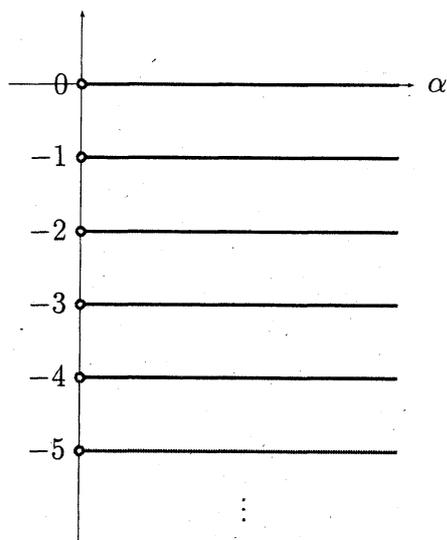
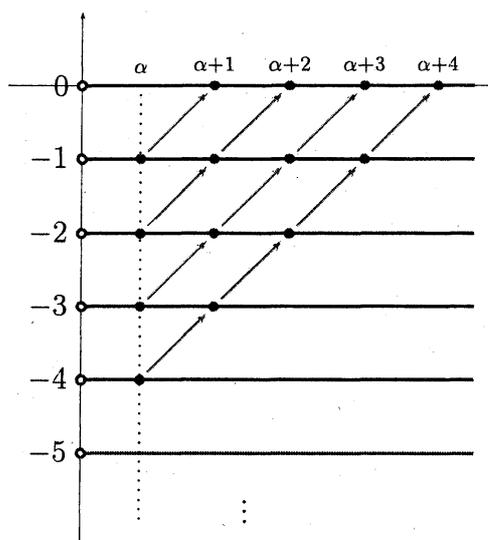
See, e.g., Beals-Wong [1] or Lebedev [6]. We mainly follow the notations in [1]. It is known that if we differentiate a Laguerre polynomial, we get a Laguerre polynomial with a different parameter:

$$(4.3) \quad \frac{d}{dx} L_n^{(\alpha-1)}(x) = -L_{n-1}^{(\alpha)}(x),$$

Thus the differentiation gives rise to the following correspondence between eigenfunctions:

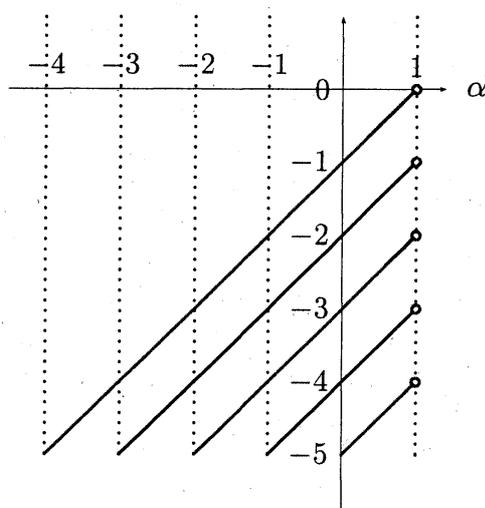
eigenvalue	$xu'' + (\alpha - x)u'$	$\xrightarrow{\frac{d}{dx}}$	$xu'' + (\alpha + 1 - x)u' - u$
-1	$L_1^{(\alpha-1)}(x)$	$\rightarrow$	$-L_0^{(\alpha)}(x)$
-2	$L_2^{(\alpha-1)}(x)$	$\rightarrow$	$-L_1^{(\alpha)}(x)$
$\vdots$	$\vdots$	$\rightarrow$	$\vdots$
-n	$L_n^{(\alpha-1)}(x)$	$\rightarrow$	$-L_{n-1}^{(\alpha)}(x)$
$\vdots$	$\vdots$	$\rightarrow$	$\vdots$

Let us draw a picture of the spectrum.

the spectrum of  $\mathfrak{Q}$ 

Correspondence between eigenfunctions

Here horizontal axis  $\alpha$  indicates the parameter of the Laguerre polynomial. For  $0 < \alpha < 1$ , we need the boundary condition because 0 is exit and entrance. We have to choose the Neumann boundary condition. We can also think of the Dirichlet boundary condition. In this case, the spectrum behaves differently. Moreover we can think of the case  $\alpha \leq 0$ . We only give a picture of the spectrum as follows:

the spectrum of  $\mathfrak{Q}$ 

### 4.3 Jacobi polynomials

Take  $a = (1 - x^2)$  and  $p = (1 - x)^\alpha(1 + x)^\beta$  on  $(-1, 1)$ . Here we assume  $\alpha > -1$ ,  $\beta > -1$ . Then

$$b = a' + a(\log p)'$$

$$\begin{aligned}
&= -2x + (1-x^2)(\alpha \log(1-x) + \beta \log(1+x))' \\
&= -2x + (1-x^2)\left(-\alpha \frac{1}{1-x} + \beta \frac{1}{1+x}\right) \\
&= -2x - \alpha(1+x) + \beta(1-x) \\
&= \beta - \alpha - (\alpha + \beta + 2)x
\end{aligned}$$

and hence

$$\begin{aligned}
\mathfrak{A}u &= (1-x^2)u'' + (\beta - \alpha - (\alpha + \beta + 2)x)u', \\
\hat{\mathfrak{A}}u &= (1-x^2)u'' + (-2x + \beta - \alpha - (\alpha + \beta + 2)x)u' - (\alpha + \beta + 2)u \\
&= (1-x^2)u'' + (\beta + 1 - (\alpha + 1) - (\alpha + 1 + \beta + 1 + 2)x)u' - (\alpha + \beta + 2)u.
\end{aligned}$$

Eigenfunctions of  $\mathfrak{A}$  are Jacobi polynomials defined by

$$(4.4) \quad P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{n!2^n} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \{(1-x)^{n+\alpha} (1+x)^{n+\beta}\}.$$

$P_n^{(\alpha, \beta)}(x)$  satisfies the following differential equation:

$$(4.5) \quad (1-x^2)[P_n^{(\alpha, \beta)}]'' + (\beta - \alpha - (\alpha + \beta + 2)x)[P_n^{(\alpha, \beta)}]' = -n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)},$$

which means that  $P_n^{(\alpha, \beta)}(x)$  is an eigenfunction for an eigenvalue  $-n(n + \alpha + \beta + 1)$ . By differentiation, we have

$$(4.6) \quad [P_n^{(\alpha, \beta)}]' = \frac{1}{2}(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1, \beta+1)}.$$

This gives a correspondence between eigenfunctions of  $\mathfrak{A}$  and  $\hat{\mathfrak{A}}$  as follows

eigenvalue	$\mathfrak{A}$	$\hat{\mathfrak{A}}$
$\lambda_1$	$\begin{pmatrix} P_1^{(\alpha, \beta)} \\ P_2^{(\alpha, \beta)} \\ \vdots \\ P_n^{(\alpha, \beta)} \\ \vdots \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(0 + \alpha + \beta + 1)P_0^{(\alpha+1, \beta+1)} \\ \frac{1}{2}(1 + \alpha + \beta + 1)P_1^{(\alpha+1, \beta+1)} \\ \vdots \\ \frac{1}{2}(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1, \beta+1)} \\ \vdots \end{pmatrix}$
$\lambda_2$		
$\vdots$		
$\lambda_n$		
$\vdots$		

Here we set

$$\lambda_n = -n(n + \alpha + \beta + 1).$$

#### 4.4 Laplacian

Take  $a = 1$  and  $p = 1$  on  $(0, 2\pi)$ . This is the simplest case. Eigenfunctions are  $\cos nx$   $n = 0, 1, 2, \dots$  under the Neumann boundary condition, and  $\sin nx$   $n = 1, 2, 3, \dots$  under the Dirichlet boundary condition. It is well-known that  $(\cos nx)' = -n \sin nx$ .

## References

- [1] R. Beals and R. Wong, "*Special functions*," Cambridge University Press, Cambridge, 2010.
- [2] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, "*Schrödinger operators with application to quantum mechanics and global geometry*," Springer-Verlag, Berlin, 1987.
- [3] M. Fukushima, T. Oshima and M. Takeda, "*Dirichlet forms and symmetric Markov processes*," Walter de Gruyter, Berlin-New York, 1994.
- [4] K. Itô and H. P. McKean, "*Diffusion processes and their sample paths*," Second printing, corrected, Springer-Verlag, Berlin-New York, 1974.
- [5] T. Kato, "*Perturbation theory for linear operators. Second edition*," Second edition, Springer-Verlag, Berlin-New York, 1976.
- [6] N. N. Lebedev, "*Special functions and their applications*," Revised edition, Dover, New York, 1972.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO  
606-8502, JAPAN

*E-mail address:* [ichiro@math.kyoto-u.ac.jp](mailto:ichiro@math.kyoto-u.ac.jp)

*URL:* <http://www.math.kyoto-u.ac.jp/~ichiro/>