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Inverse scattering and the long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation

By

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Abstract

The integrable discrete nonlinear Schrödinger equation was introduced by Ablowitz-Ladik. It can be solved by the inverse scattering transform based on the Riemann-Hilbert technique. By combining it with the nonlinear steepest descent method of Deift-Zhou, we can calculate the long-time asymptotic behavior of a solution to the defocusing version of the equation.

§ 1. Introduction

The (focusing) nonlinear Schrödinger equation $ir_t + r_{xx} + 2|r|^2 r = 0$ can be solved by the inverse scattering transform (IST) method as was proved by Zakhalov-Shabat ([11]). It was later extended to other equations by Manakov ([7]) and Ablowitz-Kaup-Newell-Segur ([1]). The latter general result includes the IST scheme for the *defocusing* integrable nonlinear Schrödinger equation

$$ir_t + r_{xx} - 2|r|^2 r = 0.$$

A way of discretization of the nonlinear Schrödinger equation was proposed in [2]. The point here is the choice of the nonlinear term. The trivial choice $\pm 2|R_n|^2 R_n$ messes up integrability, while $\pm|R_n|^2(R_{n+1}+R_{n-1})$ preserves it. The integrable discrete nonlinear Schrödinger equation

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) \pm |R_n|^2(R_{n+1} + R_{n-1}) = 0. \tag{1.1}$$

admits a Lax pair (an AKNS pair) representation and can be solved by the IST method.

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An interesting topic about integrable equations is the long-time behavior of solutions. There are a lot of results in this direction. Some are formal and are based on some *ansatz* about the leading terms. A rigorous approach, called the nonlinear steepest descent method, was established by Deift-Zhou ([6]) and has been applied in studying a lot of problems[1]. In particular, according to Deift-Its-Zhou ([5]), the long-time asymptotics of a solution of the defocusing nonlinear Schrödinger equation is decaying oscillation of order $O(t^{-1/2})$. For (1.1) (the focusing version, under the assumption that there are no solitons), a formal calculation was performed by [8]. The aim of the present article is to review our recent result about the long-time behavior of solutions of the defocusing integrable discrete nonlinear Schrödinger equation

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2(R_{n+1} + R_{n-1}) = 0. \tag{1.2}$$

The result is as follows. If $|n/t| < 2$, there exist $C_j = C_j(n/t) \in \mathbb{C}$ and $p_j = p_j(n/t), q_j = q_j(n/t) \in \mathbb{R}$ $(j = 1, 2)$ depending only on the ratio n/t such that

$$R_n(t) = \sum_{j=1}^{2} C_j t^{-1/2} e^{-i(p_j t + q_j \log t)} + O(t^{-1}\log t) \quad \text{as } t \to \infty. \tag{1.3}$$

A more precise statement will be given in §3. The behavior of each term in the sum is *decaying oscillation* of order $t^{-1/2}$.

§ 2. Inverse scattering

In this section we explain the inverse scattering transform for (1.2) following [3, Chap. 3]. The Lax pair for (1.2) consists of a recurrence relation in n (the n-part) and an ordinary differential equation in t (the t-part).

The n-part, called the Ablowitz-Ladik scattering problem, is given by

$$X_{n+1} = \begin{bmatrix} z & \bar{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n. \tag{2.1}$$

The t-part is

$$\frac{d}{dt}X_n = \begin{bmatrix} iR_{n-1}\bar{R}_n - \frac{i}{2}(z - z^{-1})^2 & -i(z\bar{R}_n - z^{-1}\bar{R}_{n-1}) \\ i(z^{-1}R_n - zR_{n-1}) & -iR_n\bar{R}_{n-1} + \frac{i}{2}(z - z^{-1})^2 \end{bmatrix} X_n \tag{2.2}$$

and (1.2) is the compatibility condition of (2.1) and (2.2).

[1] An easy-to-read account of the method is given in [4].

We can construct eigenfunctions satisfying the n-part (2.1) for any fixed t. Following [3], one can construct the eigenfunctions $\phi_n(z,t), \psi_n(z,t) \in \mathcal{O}(|z|>1)\cap C^0(|z|\geq 1)$ and $\psi_n^*(z,t) \in \mathcal{O}(|z|<1)\cap C^0(|z|\leq 1)$ such that

$$\phi_n(z,t) \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{as } n \to -\infty, \tag{2.3}$$

$$\psi_n(z,t) \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \psi_n^*(z,t) \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{as } n \to \infty. \tag{2.4}$$

On the circle C: $|z|=1$, there exist unique functions $a(z,t)$ and $b(z,t)$ such that

$$\phi_n(z,t) = b(z,t)\psi_n(z,t) + a(z,t)\psi_n^*(z,t) \tag{2.5}$$

holds. It is known that $a(z,t)$ never vanishes. One can define the *reflection coefficient*

$$r(z,t) = \frac{b(z,t)}{a(z,t)}. \tag{2.6}$$

It has the property $r(-z,t) = -r(z,t), 0 \leq |r(z,t)| < 1$.

Remark. If $\{n; R_n(t) \neq 0\}$ is finite, the reflection coefficient can be calculated concretely with ease.

The time evolution of $r(z,t)$ according to the t-part (2.2) is given by

$$r(z,t) = r(z)\exp\left(it(z-z^{-1})^2\right), \tag{2.7}$$

where $r(z) = r(z,0)$. Let us introduce the following Riemann-Hilbert problem[2]:

$$m_+(z) = m_-(z)v(z) \text{ on } C\colon |z|=1, \tag{2.8}$$

$$m(z) \to I \text{ as } z \to \infty, \tag{2.9}$$

$$\begin{aligned} v(z) = v(z,t) &= \begin{bmatrix} 1-|r(z,t)|^2 & -z^{2n}\bar{r}(z,t) \\ z^{-2n}r(z,t) & 1 \end{bmatrix} \\ &= e^{-\frac{it}{2}(z-z^{-1})^2 \operatorname{ad}\sigma_3} \begin{bmatrix} 1-|r(z)|^2 & -z^{2n}\bar{r}(z) \\ z^{-2n}r(z) & 1 \end{bmatrix}. \end{aligned} \tag{2.10}$$

Here m_+ and m_- are the boundary values from the *outside* and *inside* of C respectively of the unknown matrix-valued analytic function $m(z) = m(z;n,t)$ in $|z| \neq 1$. As is customary, $\sigma_3 = \operatorname{diag}(1,-1)$, $e^{\operatorname{ad}\sigma_3}Q = e^{\sigma_3}Qe^{-\sigma_3}$ (Q: a 2×2 matrix).

Set

$$\varphi = \varphi(z) = \varphi(z;n,t) = \frac{1}{2}it(z-z^{-1})^2 - n\log z$$

[2]It is an alternative to the Gelfand-Levitan-Marchenko equation.

so that the jump matrix $v(z)$ in (2.8) is given by

$$v = v(z) = \mathrm{e}^{-\varphi \mathrm{ad}\sigma_3} \begin{bmatrix} 1 - |r(z)|^2 & -\bar{r}(z) \\ r(z) & 1 \end{bmatrix}. \tag{2.11}$$

The "phase" φ has four saddle points, all on the circle C, and they play important roles in the method of nonlinear steepest descent. The four points are actually two pairs of antipodal points. Each pair contributes to one of the terms in the sum in (1.3).

The solution $\{R_n\} = \{R_n(t)\}$ to (1.2) can be reconstructed from the $(2,1)$ component of $m(z)$ by a formula on [3, p.69]. One has $m(z)_{21} = -zR_n(t) + O(z^2)$ $(z \to 0)$, namely,

$$R_n(t) = -\lim_{z\to 0} \frac{1}{z}\, m(z)_{21} = -\left.\frac{d}{dz} m(z)_{21}\right|_{z=0}. \tag{2.12}$$

Summing up, the initial value problem for (1.2) can be solved by the following algorithm:

1. the initial value $\{R_n(0)\}$ and the n-part of the Lax pair determine $r(z) = r(z,0)$.
2. $r(z,t)$ $(t > 0)$ is determined by the t-part of the Lax pair.
3. $m(z) = m(x,t;z)$ is obtained from the Riemann-Hilbert problem involving $r(z,t)$.
4. $R_n(t)$ $(t > 0)$ is obtained from $m(x,t;z)$.

§ 3. Statement of the result

The function φ has four saddle points. They are $S_1 = e^{-\pi i/4}A$, $S_2 = e^{-\pi i/4}\bar{A}$, $S_3 = -S_1$, $S_4 = -S_2$,, where $A = 2^{-1}\left(\sqrt{2+n/t} - i\sqrt{2-n/t}\,\right)$. Notice that $|A| = |S_j| = 1$ for $j = 1,2,3,4$. Set

$$\beta_1 = \frac{-e^{\pi i/4}A}{2(4t^2-n^2)^{1/4}}, \qquad \beta_2 = \frac{e^{\pi i/4}\bar{A}}{2(4t^2-n^2)^{1/4}}$$

$$D_1 = \frac{-iA}{2(4t^2-n^2)^{1/4}(A-1)}, \quad D_2 = \frac{i\bar{A}}{2(4t^2-n^2)^{1/4}(\bar{A}-1)}.$$

We need to introduce several quantities involving S_j and $r(z) = r(z,0)$. We set

$$\delta(0) = \exp\left(\frac{-1}{\pi i}\int_{S_1}^{S_2} \log(1-|r(\tau)|^2)\frac{d\tau}{\tau}\right),$$

$$\chi_j(S_j) = \frac{1}{2\pi i}\int_{\exp(-\pi i/4)}^{S_j} \log\frac{1-|r(\tau)|^2}{1-|r(S_j)|^2}\frac{d\tau}{\tau - S_j},$$

$$\nu_j = -\frac{1}{2\pi}\log(1-|r(S_j)|^2),$$

$$\widehat{\delta}_j(S_j) = \exp\left(\frac{1}{2\pi}\left[(-1)^j\int_{e^{-\pi i/4}}^{S_{3-j}} - \int_{-S_1}^{-S_2}\right]\frac{\log(1-|r(\tau)|^2)}{\tau - S_j}\,d\tau\right),$$

$$\delta_j^0 = S_j^n e^{-it(S_j - S_j^{-1})^2/2} D_j^{(-1)^{j-1} i\nu_j} e^{(-1)^{j-1}\chi_j(S_j)} \widehat{\delta}_j(S_j)$$

for $j = 1, 2$. Here the integrals are taken along minor arcs included in C. We have $\operatorname{Re} D_j > 0$ and $z^{(-1)^{j-1} i\nu_j}$ has a cut along the negative real axis. It follows from $|r(z)| < 1$ that $\delta(0) \geq 1$, $\nu_j \geq 0$. Notice that $A, S_j, \delta(0), \chi_j(S_j), \nu_j$ and $\hat{\delta}_j(S_j)$ are functions in n/t and that β_j and D_j are of the form $t^{-1/2} \times$ (a function in n/t). As $t \to \infty$, β_j is decaying and δ_j^0 is oscillatory if n/t is fixed.

Theorem. Assume $\sum n^{10} |R_n(0)| < \infty$ and $\sup |R_n(0)| < 1$. Then on $|n| \leq 2t$, we have

$$R_n(t) = -\frac{\delta(0)}{\pi i} \sum_{j=1}^{2} \beta_j (\delta_j^0)^{-2} S_j^{-2} M_j + O(t^{-1} \log t) \text{ as } t \to \infty.$$

Here we set

$$M_j = \frac{\sqrt{2\pi} \exp\left((-1)^j 3\pi i/4 - \pi\nu_j/2\right)}{\bar{r}(S_j) \Gamma((-1)^{j-1} i\nu_j)}$$

if $r(S_j) \neq 0$, and $M_j = 0$ if $r(S_j) = 0$.

Proof. The asymptotic behavior is proved by using the nonlinear steepest descent method. We deform the contour in the Riemann-Hilbert problem (2.8)–(2.10) by adding crosses near the saddle points and some other curves. The crosses are steepest descent paths of $\pm\varphi$. The details will be given in [9]. □

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