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# Connection problem for first integrals of nonintegrable Hamiltonian system

By

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## Abstract

We study the connection problem for a system of first integrals of a nonitegrable Hamiltonian system. We will show several new properties of the connection functions.For the proof we construct a formal first integral and then we use the moment Borel sum of the first integrals. Indeed, this method is convenient in order to avoid the small denominator difficultiy in constructing formal first integrals.

## § 1. Introduction

Let $n\geq 2$ and $\sigma\geq 1$ be an integer and let $q=(q_{2}, \ldots, q_{n})$ and $p=(p_{2}, \ldots, p_{n})$ be the variables in $\mathbb{R}^{2(n-1)}$ or in $\mathbb{C}^{2(n-1)}$. We consider a Hamiltonian system

$$(1.1)\qquad z^{2\sigma}\frac{dq}{dz}=\nabla_{p}\mathcal{H}(z, q, p), \quad z^{2\sigma}\frac{dp}{dz}=-\nabla_{q}\mathcal{H}(z, q, p),$$

where $\mathcal{H}=\mathcal{H}(z, q, p)$ is a Hamiltonian function in $(z, q, p)\in \mathbb{C}\times \mathbb{C}^{n-1}\times \mathbb{C}^{n-1}.$

We take $q_{1}=z$ as a unknown function and define the Hamiltonian function by

$$(1.2)\qquad H(z, q_{1}, p_{1}, q, p):=p_{1}q_{1}^{2\sigma}+\mathcal{H}(q_{1}, q, p).$$

Eq. (1.1) can be written in the equivalent autonomous form

$$(1.3)\qquad \begin{aligned}&\dot{q}_{1}=q_{1}^{2\sigma}, \quad \dot{p}_{1}=-2\sigma p_{1}q_{1}^{2\sigma-1}-\frac{\partial}{\partial q_{1}}\mathcal{H}(q_{1}, q, p),\\ &\dot{q}=\nabla_{p}H(z, q, p), \quad \dot{p}=-\nabla_{q}H(z, q, p).\end{aligned}$$

The main subject in this note is to study the connection problem or the nonlinear Stokes functions for (1.1). We say that a function $\psi(q_{1}, p_{1}, q, p)$ is the first integral of (1.3) if

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for every solution $(q_1(t), q(q), p_1(t), p(t))$ of (1.3) the function $\psi(q_1(t), p_1(t), q(t), p(t))$ is constant in $t$. We will construct a (divergent) formal first integral and use the moment Borel sum in order to construct functionally independent first integrals. We then study the connection problem for first integrals by the moment Laplace transform. The proofs of the theorems in this note will be published elsewhere.

## § 2. Construction of formal first integrals

Consider the Hamiltonian system

$$\dot{q}_j = \partial_{p_j} H, \quad \dot{p}_j = -\partial_{q_j} H, \; j = 1, 2, \ldots, n, \tag{2.1}$$

with the Hamiltonian funtion $H := H_0 + H_1$ given by

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \; H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q) \tag{2.2}$$

where we assume the nonresonance condition

$$\lambda_j \in \mathbb{C} \; (j = 2, 3, \ldots, n) \text{ are linearly independent over } \mathbb{Z}. \tag{2.3}$$

We suppose that $B_j \equiv B_j(q_1, s, q)$ is holomorphic in some neighborhood of $(q_1, s, q) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$ and

$$B_j(q_1, q_1^{2\sigma} p_1, q) = B_{j,0}(q_1, q) + q_1^{2\sigma} p_1 B_{j,1}(q_1, q), \tag{2.4}$$

where $B_{j,0}$ and $B_{j,1}$ are holomorphic at $q_1 = 0$, $q = 0$.

*Construction of formal first integral.* We continue to assume the conditions in the preceeding paragraph. Set $E^\alpha = E_{\lambda_2}^{\alpha_2} \cdots E_{\lambda_n}^{\alpha_n}$, where $E_c(q_1) := \exp\left(c q_1^{-2\sigma+1}/(2\sigma - 1)\right)$. We fix $\alpha \geq 0$. We look for the solution $v = v^{(\alpha)} E^\alpha$, where

$$v^{(\alpha)} = \sum_{\nu,k,\ell} v_{\nu,k,\ell}^{(\alpha)}(q_1)(q_1^{2\sigma} p_1)^\nu p^k q^\ell.$$

Indeed,for $m = 2, \ldots, n$, the lowerest order term with respect to the expansion of $q$ is given by $p_m q_m q^\alpha$. Next, one can show that the coefficients of $q^\ell$ for $\ell \not\geq e_m + \alpha$ vanish. On the other hand, as for $\ell \geq e_m + \alpha$ the coefficients of $q^\ell$ are calculated inductively. We set $\alpha = 0$ or $\alpha = e_m$, where $m = 2, \ldots, n$. Then we obtain functionally independent $2n - 1$ formal first integrals because $H$ is also a first integral. We can also show that the first integrals are linear with respect to $p$ and $p_1$.

## § 3. Moment Borel and Laplace transforms

We begin with the definition of a Gevrey asymptotic expansion. We say that the formal power series $\hat{f}(z) = \sum_{n=0}^{\infty} \hat{f}_n z^n$ belongs to $\tau$-Gevrey class $G^\tau$ $(\tau > 0)$ if there exist $C_1 > 0$ and $C_2 > 0$ such that for every integer $n \geq 0$ we have $|\hat{f}_n| \leq C_1 C_2^n n!^\tau$.

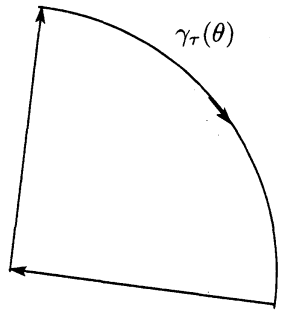


Figure 1. Path of Borel Transform

Let $\tau = 1/(2\sigma - 1)$ and the direction $\xi \in \mathbb{C} \setminus 0$ be given. A formal power series $\hat{f} \in G^\tau$ is said to be $(2\sigma - 1)$-Borel summable in the direction $\xi$ if there exist a sector $\Sigma$ with direction $\xi$ and opening greator than $\pi/(2\sigma - 1)$ and the holomorphic function $f$ in $\Sigma$ such that $f$ has a $\tau$- Gevrey expansion, $\hat{f}$ in $\Sigma$, namely $f \sim_\tau \hat{f}$ in $\Sigma$.

*Moment Borel and Laplace transforms.* The moment sum is defined in terms of the pair of the so-called kernel functions. Let $\tau > 1/2$ and $\nu \in \mathbb{Z}_+$ be given. We define kernel functions of order $\tau$, $e(x)$ and $E(x)$ $(x \in \mathbb{C})$, respectively by

$$e(x) := \tau x^{-2\sigma\nu} \exp\left(-x^\tau\right), \quad E(x) := \sum_{j > 2\sigma\nu} \frac{x^j}{\Gamma(\frac{j-2\sigma\nu}{\tau})}. \tag{3.1}$$

Note that we use kernel functions which is not integrable at the origin. In the usual Borel summation we use exponential functions for the kernel functions. (cf. [1]).

Let $\theta \in \mathbb{R}$, $r > 0$, and $0 < \varepsilon < \pi$ be given. Let $\gamma_\tau(\theta)$ denote the path from the origin along $\arg z = \theta + (\varepsilon + \pi)/(2\tau)$ to some $z_1$ of modulus $r$, then along the circle $|z| = r$ to the ray $\arg z = \theta - (\varepsilon + \pi)/(2\tau)$, and back to the origin along the ray. (cf. Figure 1). Then the moment Borel transform and the moment Laplace transform are defined, respectively, by

$$\mathcal{B}_M(f)(z) := -\frac{1}{2\pi i} \int_{\gamma_\tau(\theta)} E(z/t) f(t) \frac{dt}{t}, \tag{3.2}$$

$$\mathcal{L}_M(g)(t) := \int_0^{\infty(d)} e(z/t) g(z) \frac{dz}{z}, \tag{3.3}$$

where the path of integration in (3.3) is the straight line in the direction $d$. We also assume that $f(t)$ in (3.2) satisfies $f(t) = O(t^{2\sigma\nu+1})$, as $t \to 0$ which implies the convergence of the integral (3.2). Indeed, the convergence of (3.2) is clear except for the case when $t$ tends to zero on the two lines of $\gamma_\tau(\theta)$. We have

$$E(z/t) f(t) t^{-1} = z^{2\sigma\nu} t^{-2\sigma\nu-1} f(t) \frac{z}{t} \tilde{E}\left(\frac{z}{t}\right).$$

If $t \in \gamma_\tau(\theta)$ and arg $z$ is sufficiently small, then with $x = z/t$ the function $x\tilde{E}(x)$ is bounded when $x$ tends to infinity as $t \to 0$, $t \in \gamma_\tau(\theta)$. This yields the convergence of (3.2) and the desired estimate. We have that $\mathcal{B}_M(f)(z) = O(z^{2\sigma\nu+1})$ as $z \to \infty$. Hence, it is natural to assume $g(z) = O(z^{2\sigma\nu+1})$, from which the integral (3.3) converges.

*Moment summability.* Let $v(q_1, p_1, q, p) = O(q_1^{2\sigma\nu+1})$ be the formal power series of $q_1$ analytic in $q$ and polynomial in $p_1$ and $p$. We say that $v(q_1, p_1, q, p)$ is $\tau$- Borel moment summable in the direction $\theta$ if there exists a cone $\Omega_0$, $\Omega_0 := \{z \in \mathbb{C}; |\arg z - \theta| < \varepsilon_1/2\}$ such that the formal Borel transform $\hat{\mathcal{B}}_M v$ is analytic at $z = 0$, $q = 0$ and it can be extended as an analytic function of $z$ in $\Omega_0$ with exponential growth

$$\sup_{z\in\Omega_0} |\hat{\mathcal{B}}_M v(z, q, p_1, p)e^{-cz^\tau}| < \infty,$$

for some $c > 0$ where $q$ is in some neighborhood of the origin and $p_1$, $p$ in a bounded set. Then the moment Borel sum is defined by $\mathcal{L}_M\hat{\mathcal{B}}_M v$. For the general $v(q_1, p_1, q, p)$, write $v(q_1, p_1, q, p) = v_0(q_1, p_1, q, p) + \tilde{v}(q_1, p_1, q, p)$ with $v_0$ being the polynomial of $q_1$ and $\tilde{v} = O(q_1^{2\sigma\nu+1})$. Then the moment Borel sum is defined by

$$\mathcal{L}_M\hat{\mathcal{B}}_M v := v_0(q_1, p_1, q, p) + \mathcal{L}_M\hat{\mathcal{B}}_M\tilde{v}. \tag{3.4}$$

We note that the summability and the sum of a formal power series does not depend on the choice of $v_0$ and the moments. (cf. [1]). Hence if there is no fear of confusion we say Borel summable instead of moment Borel summable. In order to study global behaviors of summed integrals we need to study fundamental properties of the moment Borel and the moment Laplace transforms.

## § 4. Borel summability of formal first integrals

For the neighborhood of the origin $\Omega_0 \subset \mathbb{C}$ and the convex cone with vertex at the orign $\Omega_1 \subset \mathbb{C}$ we set $\Sigma_0 := \Omega_0 \cup \Omega_1$.

*Singular directions.* Let $\alpha \geq 0$ be given. For $v^{(\alpha)}$ we define the set of singular directions $S_0$ by

$$\begin{aligned} S_0 := \{z \in \mathbb{C}; \exists\nu \geq 0,\ k \geq 0,\ \ell \geq 0,\ \alpha \geq 0 \\ (2\sigma - 1)z^{2\sigma-1} + \lambda \cdot (\ell - \alpha - k) = 0;\ v_{\nu,k,\ell}^{(\alpha)} \neq 0, \ell - \alpha - k \geq 0\} \setminus 0. \end{aligned} \tag{4.1}$$

Then we have

**Theorem 4.1** (Borel summability). *Assume that* (2.4) *and* (2.3) *are satisfied. Suppose that there exists $\Sigma_0$ such that $\overline{S_0} \cap \Sigma_0 = \emptyset$. Then, for every $\xi \in \Omega_1$, there exists an neighborhood of the orign of $q$, $V_0$ for which $v^{(\alpha)}$ is analytic in $q \in V_0$, and it is* $(2\sigma - 1)$*-Borel summable in the direction $\xi$ with respect to $q_1$.*

*Especially, if there exists a polynomial $B_{j,0}$ of $q$ with coefficients analytic at $q_1 = 0$ such that $B_j = B_{j,0}(q_1, q)$, $2 \le j \le n$, then $S_0$ is a finite set. Hence $v^{(\alpha)}$ is $(2\sigma-1)$-Borel summable with respect to $q_1$.*

The $(2\sigma - 1)$-sum in the above theorem can be constructed as the Borel sum. By this theorem one can construct $(2n - 1)$-functionally independent first integrals. The first integrals have the form of the so-called transseries or log-exponential series. Note that it gives the alternative expression of analytic continuation of the solution of an initial value problem.

## § 5. Semi formal solution and Stokes function

We begin with the alternative definition of a semi formal solution introduced by Balser in [2]. Given functionally independent first integrals $H(q_1, p_1, q, p)$, $F_j(q_1, p_1, q, p)$ $(j = 1, 2, \ldots, 2n - 2)$ of (1.3), where the functional independentness means that the vectors

$$\nabla_{q,p,p_1} H, \quad \nabla_{q,p,p_1} F_j, \quad (j = 1, 2, \ldots, 2n - 2) \tag{5.1}$$

have full rank, $2n - 1$ on some open dense set. In case $F_j$'s are formal power series of $p_1$, $q$ and $p$, then we understand that the linear part of the Taylor expansions of $H$ and $F_j$ in $p_1, q, p$ is invertible. Then, for $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{2n-2}) \in \mathbb{C}^{2n-2}$ sufficiently small we can solve $p_1, q$ and $p$ from the system of equations

$$H(q_1, p_1, q, p) = 0, \quad F_j(q_1, p_1, q, p) = \tilde{c}_j + c_j^0 = c_j, \ j = 1, 2, \ldots, 2n - 2, \tag{5.2}$$

where

$$H(q_1^0, p_1^0, q^0, p^0) = 0, \quad F_j(q_1^0, p_1^0, q^0, p^0) = c_j^0, \ j = 1, 2, \ldots, 2n - 2, \tag{5.3}$$

and

$$q_1^0 \neq 0, \ q_k^0 \neq 0, \ q_k^0 \neq 0 \quad (k = 2, 3, \ldots, n), q^0 = (q_2^0, \ldots, q_n^0), p^0 = (p_2^0, \ldots, p_n^0). \tag{5.4}$$

We write the solution of (5.2) by $q = q(q_1, c)$, $p = p(q_1, c)$ and $p_1 = p_1(q_1, c)$. If the first integrals are formal series, then we call them a semi formal solution of (1.3).

*Remark.* In the category of formal power series , one can give the alternative definition of $q = q(q_1, c)$, $p = p(q_1, c)$ and $p_1 = p_1(q_1, c)$. (cf. [2]). Let $\tilde{S}_0$ be the universal covering space of the punctured disk $\{z; |z| < r\} \setminus 0$ for some $r > 0$ and $\mathcal{O}(\tilde{S}_0)$ be the set of holomorphic functions on $\tilde{S}_0$. The vector $\check{x}(q_1, c)$ of formal power series of $c$

$$\check{x}(q_1, c) := \sum_{|\nu| \ge 0} \check{x}_\nu(q_1) c^\nu = \check{x}_0(q_1) + X(q_1)c + \sum_{|\nu| \ge 2} \check{x}_\nu(q_1) c^\nu \tag{5.5}$$

is said to be a *semi formal solution* of (1.1) if $\check{x}_\nu \in \mathcal{O}(\tilde{S}_0)$. Here $X(q_1)$ is a $2n-2$ square matrix with component belonging to $\mathcal{O}(\tilde{S}_0)$. If $X(q_1)$ is invertible, then we call $\check{x}(q_1, c)$ a *complete semi formal solution* of (1.3). We can construct a complete semi formal solution by solving an initial value problem.

*Stokes function.* Suppose that $\tilde{F}_j$ $(j = 1, 2, \ldots, 2n-2)$ satisfy (5.1). Moreover, assume that we have the relations

$$F_j(q_1, p_1, q, p) = \tilde{F}_j(q_1, p_1, q, p) + m_j(q_1, p_1, q, p), \quad j = 1, 2, \ldots, 2n-2. \tag{5.6}$$

For example (5.4) holds for $m_j = F_j - \tilde{F}_j$ in the category of formal power series. Clearly, $m_j$'s are first integral of (1.3). Let $(p_1, q, p)(q_1, c)$ be the (formal) solution of (5.2). Because $m_j$ is a first integral we define $\tilde{v}_j(c) := m_j(q_1, p_1, q, p)$ for some constant $\tilde{v}_j(c)$ and $\tilde{v} := (\tilde{v}_j(c))$. Hence we have $\tilde{F}_j(q_1, p_1, q, p) = c_j - \tilde{v}_j(c)$. Therefore, by (5.1) we have

$$q(q_1, c) = \tilde{q}(q_1, c - \tilde{v}(c)), \quad p(q_1, c) = \tilde{p}(q_1, c - \tilde{v}(c)). \tag{5.7}$$

We call $v(c) := c - \tilde{v}(c)$ the Stokes function. Let $X(q_1)$ and $\tilde{X}(q_1)$ be the linear part of $(q, p)$ and $(\tilde{q}, \tilde{p})$,respectively. Let $V$ be the linear part in the Taylor expansion of $v(c)$. Then we have $X(q_1) = \tilde{X}(q_1)V$. Hence $V$ is the Stokes multiplier in a wider sense.

One can deduce a property of the Stokes function from that of the corresponding connection problem for first integrals. The details will be published in the forthcoming paper.

## § 6. Connection problem for Borel summed first integrals

Let $\theta_0$ be any singular direction which is not an accumulation point of the set of singular directions. Let $\Omega_1$ and $\Omega_2$ be the adjacent sectors in the Borel plane whose common boundary is $\theta_0$.(Figure 2). Let $\Sigma_1$ and $\Sigma_2$ be the sectors in the $q_1$ plane which correspond to $\Omega_1$ and $\Omega_2$ by the Laplace transform, respectively. Let $F := (F_1, F_2, \ldots, F_{2n-1})$ and $\tilde{F} := (\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_{2n-1})$ be the Borel sum of functionally independent formal first integrals in the sectors $\Sigma_1$ and $\Sigma_2$, respectively. We study the connection relation (5.6) in $\Sigma_1 \cap \Sigma_2$.

**Theorem 6.1** (robustness). *Suppose that the equation*

$$q_1^{2\sigma} \frac{dv}{dq_1} - 2\lambda_k v = B_k(q_1, 0, 0) \tag{6.1}$$

*has no analytic solution $v$ in some neighborhood of the origin for $k = 2, 3, \ldots, n$. Then, if $m(q_1, p_1, q, p)$ is analytic at the origin, then there exists an analytic vector function of one variable $\phi$ in some neighborhood of the origin such that $m(q_1, p_1, q, p) = \phi(H)$.*

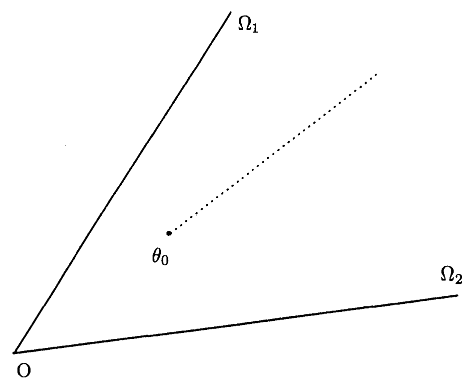


Figure 2. Choice of sectors

We note that the condition of the non existence of an analytic solution of (6.1) is a generic condition.

**Theorem 6.2** (monodromy vanishing theorem). *Suppose that*

$$B_j(q_1, t, q) = \tilde{B}_j(t, q) \qquad (j = 2, \ldots, n)$$

*holds for some function $\tilde{B}_j$ being analytic in $q$ and a polynomial in $t$. Moreover, assume* (2.3) *and the Poincaré condition, namely the convex hull of $\{\lambda_j; j = 2, 3, \ldots, n\}$ does not contain the origin. Then we have*

(i) *$m_j(q_1, p_1, q, p)$ in* (5.6) *vanishes.*

(ii) *Let $V_m$ $(m = 2, \ldots, n)$ be the first integral constructed in Section2 for $\alpha = 0$. Then $V_m$'s are analytic at the origin $q_1 = 0$, $p_1 = 0$, $q = 0$, $p = 0$. Moreover, if $W$ is a unique analytic solution of the equation $q_m \frac{\partial}{\partial q_m} W = q_m p_m - V_m$, then $W$ is independent of $m$, $2 \leq m \leq n$.If we define $\tilde{W}$ by $\tilde{W} := \sum_{j=2}^{n} q_j y_j - W(q)$,then the (partial) symplectic transformation $(q, p) \mapsto (y, -x)$*

$$q_1 = x_1, p_1 = y_1, x_j = \tilde{W}_{y_j} = q_j, p_j = \tilde{W}_{q_j} = y_j - W_{q_j}, (j = 2, \ldots, n) \tag{6.2}$$

*maps $\chi_H$ to $\chi_{\tilde{H}_0}$. Namely it gives the generating function of a resonant Birkoff transformation. Here $\tilde{H}_0 := x_1^{2\sigma} y_1 + \sum_{j=2}^{n} \lambda_j x_j y_j$,and $\chi_H$ and $\chi_{\tilde{H}_0}$ are the corresponding Hamiltonian vector fields.*

*Single-valuedness of connecting functions.* Let $\Omega(\lambda_2, \ldots, \lambda_n) \equiv \Omega(\lambda)$ be the convex positive cone generated by $\lambda_j$ $(j = 2, 3, \ldots, n)$. Then we have

**Theorem 6.3.** *Suppose* (2.3) *and the conditions $B_j = B_{j,0}(q_1, q)$, $2 \leq j \leq n$ are satisfied for some $B_{j,0}$ being a polynomial in $q$ with coefficients analytic at $q_1 = 0$. Then the connecting function $m$ in* (5.6) *exists and is holomorphic in $q_1$, $p_1$, $q$ and $p$*

*when $q_1 \neq 0$. Moreover, $m$ is not analytic at $q_1 = 0$ provided the equation* (6.1) *has no analytic solution $v$ at the origin for $k = 2, 3, \ldots, n$. There exists a neighborhood of the origin $U$ such that $m$ is a single-valued function of $q_1$ in $\{q_1 \in \mathbb{C} \cap U; q_1 \neq 0\}$.*

*Exponential series expansion of a connecting function.* Next we study the connection problem with dense singular directions in some proper cone. In such a case, an exponential series expansion naturally appears for a connection function. For the detailed study of such a series we refer [5] and [6]. To be more precise, let $z_j \equiv z_j(\ell, \alpha, k)$ $(j = 1, \ldots, 2\sigma - 1)$ be the solution of the equation $(2\sigma - 1)z^{2\sigma-1} + \lambda \cdot (\ell - \alpha - k) = 0$. Let $C_j(S_0)$ be the closed convex positive cone containing $z_j(\ell, \alpha, k)$ for $\ell$, $k$ such that $v_{0,k,\ell}^{(\alpha)} \neq 0$ and $\ell - \alpha - k \geq 0$, $\ell - \alpha - k \neq 0$. Let $C(S_0) := \bigcup_{j=1}^{2\sigma-1} C_j(S_0)$. Note that $C(S_0) = -\Omega(\lambda)$ if $\sigma = 1$, where $\Omega(\lambda)$ is the convex positive cone generated by $\lambda_j$ $(j = 2, 3, \ldots, n)$. The opening of every $C_j(S_0)$ is smaller than $\pi/(2\sigma - 1)$ if we assume the Poincaré condition for $\lambda_j$. We remark that the singular directions may be dense in $C(S_0)$. Take $C_j(S_0)$ arbitrarily and define $\tilde{C}(S_0) := C_j(S_0)$. Take the adjacent sectors $\Omega_1$ and $\Omega_2$ to $\tilde{C}(S_0)$ so that $\Omega_j \cap \tilde{C}(S_0) = \emptyset$. (cf. Figure 3.) We define $\Sigma_k$ for $k = 1, 2$ by $\Sigma_k := \{q_1; \arg(q_1 - z) < \pi/(2(2\sigma - 1)), z \in \Omega_k\}$. For the sake of simplicity we assume that $\tilde{C}(S_0)$ lies in the direction of positive real axis. Then we have

**Theorem 6.4.** *Assume that* (2.3) *and the condition*

$$B_j(q_1, q_1^{2\sigma} p_1, q) = B_{j,0}(q_1, q) + q_1^{2\sigma} p_1 \tilde{B}_{j,1}(q), \quad 2 \leq j \leq n, \tag{6.3}$$

*are satisfied for some analytic $\tilde{B}_{j,1}(q)$ independent of $q_1$. Assume that the opening of $\Omega(\lambda)$ is smaller than $\pi$. Then we have $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ and there exist a neighborhood of the origin $V$ of $(q, p_1, p)$ and the connecting function $m \equiv m(q_1, q, p_1, p)$ in* (5.6) *which is holomorphic in $(q_1, q, p_1, p) \in \Sigma_1 \cap \Sigma_2 \times V$.*

*There exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon_1 < 1$ and every $N \geq 0$ satisfying $\Re\lambda \cdot (\ell - k - \alpha) \neq N\tau$ for any $\ell$ and $k$ we have the asymptotic expansion*

$$m(q_1, p_1, q, p) = \sum_{k, \ell, \Re\lambda\cdot(\ell-k-\alpha) < N\tau} c_{\ell,k}(q, p, p_1) \exp\left(\frac{\lambda \cdot (\ell - k - \alpha)}{q_1^\tau \tau}\right) + O(e^{-\varepsilon_1 N q_1^{-\tau}}) \tag{6.4}$$

*when $q_1 \to 0$, $q_1 \in \{q_1; \Re(z/q_1)^\tau > 0, \forall z^\tau \in -\Omega(\lambda)\} \cap \{q_1; |\arg q_1| < \varepsilon_0\}$, where $c_{\ell,k}(q, p, p_1)$'s are holomorphic at the origin.*

*Multi-valuedness when there are dense singular directions.* We show multi valuedness of a connecting function in the case of dense singular directions. In the following we continue to use the same notation as in Theorem 6.4.

**Theorem 6.5.** *Assume (2.4) and (2.3). Suppose that the opening of $\Omega(\lambda)$ is smaller than $\pi$. Then we have $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ and there exist a neighborhood of the origin $V$*

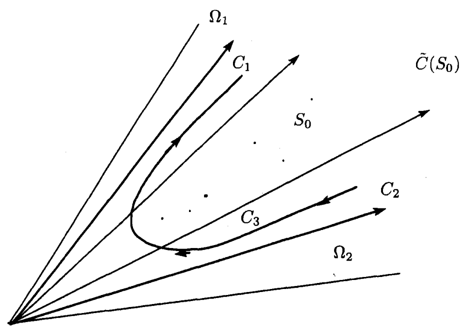


Figure 3. Deform of Path

*of* $(q, p_1, p)$ *and a connecting function* $m(q_1, q, p_1, p)$ *in (5.6) which is holomorphic in* $(q_1, q, p_1, p) \in \Sigma_1 \cap \Sigma_2 \times V$.

## § 7. Proof of Theorem 6.2

*Proof.* We look for the formal first integral $v = \phi^{(\alpha)} E^\alpha$ with

$$\phi^{(\alpha)} = \sum_{\nu, k, \ell, \ell \geq \alpha} \phi_{\nu,k,\ell}(q_1)(q_1^{2\sigma} p_1)^\nu p^k q^\ell. \tag{7.1}$$

We substitute the expansion into $\chi_H v = 0$ and compare the coefficients of $(q_1^{2\sigma} p_1)^\nu p^k q^\ell$. Then we have a recurrence relation

$$(q_1^{2\sigma} \partial_{q_1} + \lambda \cdot (\ell - k - \alpha)) \phi_{\nu,k,\ell} = F_\ell(\phi_\gamma, \gamma < \ell), \tag{7.2}$$

where $\phi_\gamma$ denotes the terms $\phi_{\nu,k,\gamma}$ for some $\nu$ and $k$, and $\ell - \alpha \neq 0$. Here we regard $t := q_1^{2\sigma} p_1$ as an independent variable. Indeed, the right-hand side follows from the assumption on $B_j$ and the use of expansion of $t = q_1^{2\sigma} p_1$ instead of that of $p_1$.

In order to determine the form of $F_\ell$ we first note that the term $\partial_{p_1} B_j \frac{\partial}{\partial q_1} - \partial_{q_1} B_j \frac{\partial}{\partial p_1}$ in the right-hand side vanishes if it is applied to the function of $t = q_1^{2\sigma} p_1$. On the other hand we have

$$(\partial_{p_1} B_j) \partial_{q_1} E^\alpha = (\partial_t B_j) q_1^{2\sigma} \partial_{q_1} E^\alpha = -\langle \lambda, \alpha \rangle (\partial_t B_j) E^\alpha.$$

Therefore, by simple calculations of $\{H_1, \cdot\}$, the terms in $F_\ell$ are calculated by subtituting the expansion of $\phi^{(\alpha)}$, (7.1) into the following

$$\sum_{j=2}^{n} \nabla_q (q_j^2 B_j) \cdot \nabla_p \phi^{(\alpha)} - \langle \lambda, \alpha \rangle \sum_{j=2}^{n} q_j^2 (\partial_t B_j) \phi^{(\alpha)}. \tag{7.3}$$

We note that $F_\ell$ does not contain the function of $q_1$ by assumption. By the same argument as in the construction of formal integral one can determine the formal series $\phi_{\nu,k,\ell}$ from (7.2). Indeed we have $\phi_{\nu,k,\ell} = F_\ell/\lambda\cdot(\ell-k-\alpha)$. By the Poincaré condition we see that the sum (7.1) with respect to $\ell$ converges when $q$ is in some neighborhood of the origin because $k$ moves on a finite set by definition. On the other hand the sum with respect to $\nu$ also converges because the coefficients are analytic function of $t = q_1^{2\sigma}p_1$ and $\lambda\cdot(\ell-k-\alpha)$ does not contain $\nu$. Therefore the moment Borel sum of $\phi_{\nu,k,\ell}$ with respect to $q_1$ coincides with itself. This proves that connection function $m(q_1,p_1,q,p)$ vanishes for every $\Sigma_1$ and $\Sigma_2$.

We will show the latter half. If we expand $\sum_j q_j^2\tilde{B}_j = \sum_\mu c_\mu(t)q^\mu$, then we have $F_\ell(v_\gamma^{(0)}) = \ell_m c_\ell$ and $v_\ell^{(0)} = -\ell_m c_\ell/\lambda\cdot\ell$. Let $W$ be the analytic function whose coefficient of $q^\ell$ is given by $c_\ell/\lambda\cdot\ell$ if $|\ell|\geq 2$, and 0 if otherwise. Clearly $W$ is independent of $m$, $2\leq m\leq n$. Then the unique solution of $q_m\frac{\partial}{\partial q_m}W = q_mp_m - V_m$ is given by $W$.

Moreover, the Hamiltonian $\tilde{H}_0$ is transformed to the one

$$q_1^{2\sigma}p_1 + \sum\lambda_jp_jq_j + \sum\lambda_mq_mW_{q_m} = H_0 + \sum_m\lambda_m(q_mp_m - V_m^{(0)})$$

$$= H_0 + \sum_m\lambda_m\Big(\sum_{|\ell|\geq 2}\frac{\ell_mc_\ell}{\lambda\cdot\ell}q^\ell\Big) = H_0 + \sum q_j^2\tilde{B}_j = H.$$

Hence we see that (6.2) transforms $\chi_H$ to $\chi_{\tilde{H}_0}$. □

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