A computer-assisted study of the Landau-Nakanishi geometry

By

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§ 1. Introduction

The purpose of this article is to call forth the interest of specialists in microlocal analysis in the computer-assisted study of the Landau-Nakanishi geometry by showing concrete examples which we have encountered in making the effort with Henry P. Stapp to elucidate the concrete contents of Sato's postulate ([2]) on the analytic structure of the $S$-matrix near the 3-particle threshold. For the convenience of the reader we first recall the definition of a Feynman graph $G$ and the Landau-Nakanishi variety (hereafter abbreviated as $\mathcal{LN}$ variety) $\mathcal{L}(G)$ associated with $G$.

**Definition 1.1.** A Feynman graph $G$ is a graph that consists of finitely many points $V_1, V_2, \ldots, V_{n'}$ (called vertices), finitely many line segments $L_1, L_2, \ldots, L_N$ (called internal lines) and finitely many half-lines $L^+_1, L^+_2, \ldots, L^+_n$ (called external lines), where each of the end-points $W^+_\ell$ and $W^-_\ell$ of $L_\ell$ ($\ell = 1, 2, \ldots, N$) coincides with some $V_j$ ($j = 1, 2, \ldots, n'$) satisfying the condition

\[ W^+_\ell \neq W^-_\ell, \]

and the (unique) end-point of $L^+_r$ ($r = 1, \ldots, n$) coincides with some $V_j$ ($j = 1, \ldots, n'$).

In this article we assume that each internal line and each external line are oriented (and specified with an arrow like $\rightarrow$ if necessary). Using this orientation we define the incidence number $[j : \ell]$ for a pair of a vertex $V_j$ and an internal line $L_\ell$ by the following rule:

\[ [j : \ell] = \begin{cases} +1 & \text{when the internal line } L_\ell \text{ ends at the vertex } V_j, \\ -1 & \text{when } L_\ell \text{ starts from } V_j, \\ 0 & \text{neither of the end-points of } L_\ell \text{ coincides with } V_j. \end{cases} \]

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The incidence number \([j : r]\) for a pair of a vertex \(V_j\) and an external line \(L_r^e\) is defined in a similar manner.

We also assume that a \(\nu\)-dimensional real (or complex if so specified) vector \(p_r = (p_{r,0}, \ldots, p_{r,\nu-1})\) \((r = 1, 2, \ldots, n)\) is assigned to each external line \(L_r^e\) and a strictly positive number \(m_{\ell}\) \((\ell = 1, 2, \ldots, N)\) is assigned to each internal line \(L_{\ell}\).

**Figure 1.** An example of a Feynman graph.

Remark 1.2. In this article we assume, for the sake of simplicity, that all constants \(m_{\ell}\) are the same and we denote it by the number \(m\). That is, we consider only the so-called equal mass case.

Remark 1.3. Unless otherwise stated, we assume \(\nu = 2\) in what follows.

Remark 1.4. In this article we do **not** assume

\[
(1.3) \quad p_r^2(= p_{r,0}^2 - p_{r,1}^2) = m^2.
\]

In passing we note that, here and in what follows, for \(\nu\)-dimensional vector \(k = (k_0, k_1, \ldots, k_{\nu-1})\) the scalar \(k^2\) stands for \(k_0^2 - \sum_{\rho=1}^{\nu-1} k_{\rho}^2\).

In order to write down the defining equation of the \(\mathcal{LN}\) variety, we introduce the following numbers \(j^{\pm}(\ell)\) and \(j(r)\) for an internal line \(L_{\ell}\) and an external line \(L_r^e\):

\[
(1.4) \quad [j^{\pm}(\ell) : \ell] = \pm 1,
\]

\[
(1.5) \quad [j(r) : r] \neq 0.
\]

**Definition 1.5.** (i) The Landau-Nakanishi variety \(\mathcal{L}(G)\) associated with a Feynman graph \(G\) is, by definition, the totality of \((p, \sqrt{-1}u)\) in \(\mathbb{R}^{\nu n} \times (\sqrt{-1} \mathbb{R}^{\nu n})\) that satisfies the following equations for some \((\alpha_1, \ldots, \alpha_N; k_1, \ldots, k_N; v_1, \ldots, v_{\nu';} a) \in \mathbb{R}^N \times \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu'}\):
$\mathbb{R}^{\nu n'} \times \mathbb{R}^{\nu}$:

\[
\begin{align*}
\left\{ \begin{array}{l}
\sum_{r=1}^{n}[j:r]p_{r} + \sum_{\ell=1}^{N}[j:\ell]k_{\ell} = 0 \quad (j = 1, 2, \ldots, n') , \\
\alpha_{\ell}(k_{\ell}^{2} - m^{2}) = 0, \quad k_{\ell,0} > 0 \quad (\ell = 1, 2, \ldots, N) , \\
v_{j+(\ell)} - v_{j-(\ell)} = \alpha_{\ell}k_{\ell} \quad (\ell = 1, 2, \ldots, N) , \\
u_{r} = -[j(r) : r](v_{j(r)} + a) \quad (r = 1, 2, \ldots, n) .
\end{array} \right.
\]

(i) If $\alpha_{\ell} \geq 0 (\ell = 1, 2, \ldots, N)$ in (1.6), then $\mathcal{L}(G)$ is designated as $\mathcal{L}^{+}(G)$ and called the positive-\(\alpha\) variety associated with $G$.

(ii) If $\alpha_{\ell} > 0 (\ell = 1, 2, \ldots, N)$, then $\mathcal{L}^{+}(G)$ is designated as $\mathcal{L}^{\oplus}(G)$.

Remark 1.6. (i) If we formally define the Feynman integral $F_{G}(p)$ associated with $G$ by

\[
\int \cdots \int \frac{\Pi_{j=1}^{n} \delta^{\nu}(\sum_{r=1}^{n}[j:r]p_{r})}{\Pi_{\ell=1}^{N}(k_{\ell}^{2} - m^{2} + \sqrt{-1}0)} \prod_{\ell=1}^{N}d^{\nu}k_{\ell},
\]

then it is known ([2]) that under some moderate conditions $F_{G}(p)$ is well-defined as a microfunction and that it is supported by $\mathcal{L}^{+}(G)$. Thus $\mathcal{L}^{+}(G)$ is a variety in $\sqrt{-1}S^{*}\mathbb{R}^{\nu n}$. Denoting by $\pi$ the canonical projection map from $\sqrt{-1}S^{*}\mathbb{R}^{\nu n}$ to $\mathbb{R}^{\nu n}$, we denote $\pi(\mathcal{L}^{+}(G))$ by $L^{+}(G)$. It is also called the positive-\(\alpha\) \(\mathcal{L}\mathcal{N}\) variety. When we want to emphasize that we are dealing with the object projected down to the base manifold, we sometimes use somewhat loose expression "(positive-\(\alpha\)) LN surface". As we will show in Section 2 and Section 3, some higher codimensional component of an LN "surface" is of particular interest.

(ii) When $F_{G}(p)$ is well-defined, it has the form

\[
f_{G}(p)\delta^{\nu}(\sum_{j,r}[j:r]p_{r}).
\]

The vector $a$ in the last equation of (1.6) is a counterpart of the factor $\delta^{\nu}(\sum[j : r]p_{r})$. The factor $f_{G}(p)$ is called a Feynman amplitude (or function).

Concerning the concrete figure of $L^{+}(G)$ the book of Eden et al. ([1]) is a good introduction. Thanks to the progress of computers, mathematicians can now make the figures in [1] much more precise so that they may give a fresh impetus to study the
Landau-Nakanishi geometry, if they put sufficiently enough energy and time into the study of the subject. Actually, as we show in Section 2, the detailed description of $L^+(G)$ gives rise to interesting mathematical problems even for a very simple graph $G$. Section 3 is devoted to showing what kind of anomalies is observed when $G$ contains what we call the non-external vertices. The study of such graphs is not only challenging but also important in our future study of the analytic structure of the $S$-matrix near the 3-particle threshold, which will make essential use of the Borel resummation.

§ 2. LN surface $L(G)$ and its positive-$\alpha$ part $L^+(G)$ when $G$ is an ice-cream cone graph

As one of the most basic graph that is relevant to the 3-particle threshold we consider the so-called ice-cream cone graph, that is,

\[
\begin{array}{c}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6
\end{array}
\begin{array}{c}
V_1 \\
V_2 \\
V_3
\end{array}
\]

Figure 2. The ice-cream cone graph $G_1$.

The reason of our interest in $L^+(G_1)$ is twofold. First, $L^+(G_1)$ touches the 3-particle threshold $3PT$, and we know ([2], [3])

(2.1) \[ f_{G_1}(p)|_{3PT} = a(p)f_{G_0}(p) + b(p) \]

holds at a generic point of $3PT$, where $a(p)$ and $b(p)$ are holomorphic functions and the graph $G_0$ is described in the figure below:

\[
\begin{array}{c}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6
\end{array}
\]

Figure 3. The Feynman graph $G_0$.

Second, if we consider a point $p$ where the following configuration of Fig. 4 is realized, that is, if all internal lines are parallel keeping each vertex distinct, then we find
Figure 4. The configuration of vectors $v_j$'s and $\alpha_\ell k_\ell$'s.

\begin{equation}
\tag{2.2}
p_4 + p_5 = 2p_6,
\end{equation}

\begin{equation}
\tag{2.3}
p_6^2 = m^2.
\end{equation}

The totality $N_-$ of such points covers only a tiny portion of $L^+(G_1)$, but as Fig. 5 shows\(^1\), $N_-$ is a crucially important part of the singularity that $L^+(G_1)$ presents; the singularity is commonly known as “Whitney’s umbrella”, and $N_-$ belongs to its most singular part. Thus explicitly writing down the holonomic system that $f_{G_1}(p)$ satisfies near $N_-$ is a charming problem in microlocal analysis.

Figure 5. The “non-zero $\alpha$” LN surface of $G_1$ with $\nu = 2$ and $m = 1$.

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\(^1\)The surface appearing in the figure is analytically isomorphic to the one defined by the following equations of parameters $s > 0$ and $t > 0$: $x = s + \frac{1}{s}$, $y = \frac{s^2 t + 3s}{st - 1}$ and $z = t$. It has only one pinch point singularity $N_-$ and also has a self-intersection curve corresponding to a shank of an umbrella.
§ 3. Truss-bridge graphs

As our eventual purpose is to understand the analytic structure of the $S$-matrix near the 3-particle threshold, it is natural to try to study the concrete figure of the positive-$\alpha$ LN surface $L^+(G)$ associated with Feynman graph $G$ when it touches 3-particle threshold. One such a graph is $G_1$ studied in Section 2. One can readily note that $L^+(T_2)$ contains $L^+(G_1)$ and also note that $L^+(T_1)$ touches 3-particle threshold, where the truss-bridge graph $T_1$ (resp. $T_2$) is given in Fig. 6 (resp. Fig. 7) below.

![Figure 6. The truss-bridge graph $T_1$.](image)

![Figure 7. The truss-bridge graph $T_2$.](image)

Thus it is natural to study $L^+(T_3)$, as the next target, where

![Figure 8. The truss-bridge graph $T_3$.](image)

Interestingly enough, there is no reference which concretely describes $L^+(T_3)$, as far as we know. And, the actual figure shown in Fig. 9 is highly intriguing; the LN surface in the figure consists of two irreducible components. One is isomorphic to the surface defined by the following equations of parameters $s > 0$ and $t > 0$:

$$
\begin{align*}
x &= s + 1/s, \\
y &= -\frac{((b^2 - ab) s^2 + (a - b) s + 1) t^2 + ((a - 2b) s^2 + s) t + s^2}{((b^2 - ab) s - b) t^2 + ((a - 2b) s + 1) t + s}, \\
z &= bt^2/(bt - 1),
\end{align*}
$$

(3.1)
where $a$ and $b$ are some positive constants. This surface has two pinch point singularities and two self-intersection curves which form a combination of two umbrellas. Another component is the curve, i.e., the higher codimensional component, defined by equations of $s > 0$:

\[(3.2) \quad x = s + 1/s, \quad y = -\frac{as^2 - 3s}{as + 1}, \quad z = -b/(s^2 - bs).\]

Figure 9. A generic slice of the “non-zero $\alpha$” LN surface of $T_3$ in a transversally intersecting 3-dimensional space ($\nu = 2$ and $m = 1$).
Among other things, the existence of a higher codimensional component of the LN surface that corresponds to the configuration described in Fig. 10 was what we had not anticipated before the actual computation.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure10}
\caption{The configuration of vectors $v_j$'s.}
\end{figure}

Note that the vertex $V_3$ may move freely from $V_2$ to $V_4$ in the configuration of Fig. 10 even if $(p, k)$ is fixed. This flexibility of the configuration is tied up with the higher codimensionality of the component in question.

We believe that several intriguing features of $L^+(T_3)$ should be tied up with the existence of non-external vertex $V_3$. Here, and in what follows, we say that a vertex is non-external if no external line is incident upon the vertex. It is probably worth noting the following fact.

Let us consider the following graph $\overline{T_3}$:

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure11}
\caption{The Feynman graph $\overline{T_3}$.}
\end{figure}

Then, for any point $p$ in $L^\oplus(\overline{T_3}) (\subset L^+(T_3))$, we find

\begin{equation}
(3.3) \quad p_6^2 = m^2;
\end{equation}

otherwise stated, although the external line $p_6$ is originally assumed not necessarily
to be on-shell, the current configuration forces it to be on-shell. We note that we encountered a similar situation in Section 2; at some particular points of $L^\oplus(G_1)$, $p_6$ lies on mass-shell. But this time at all points in $L^\oplus(\overline{T}_3)$, $p_6$ obeys the mass-shell constraint. The confirmation of (3.3) is straightforward. First we note that the energy-momentum conservation at $V_3$ (i.e., the first equation of (1.6) with $j = 3$)

\[
(3.4) \quad k_5 = k_6 = k_2 = k_3,
\]

because $\nu = 2$ and $\alpha_\ell \geq 0$ ($\ell = 2, 3, 5, 6$). Then it follows from the third equation of (1.6) that

\[
(3.5) \quad \alpha_4 k_4 = \alpha_3 k_3 + \alpha_5 k_5 = (\alpha_3 + \alpha_5)k_3,
\]

and hence

\[
(3.6) \quad k_4 = k_3.
\]

Similarly the third equation of (1.6) applied to the triangle formed by $V_3$, $V_4$ and $V_5$ entails

\[
(3.7) \quad \alpha_6 k_6 = \alpha_5 k_5 + \alpha_7 k_7.
\]

Hence (3.4) guarantees

\[
(3.8) \quad k_7 = k_5 = k_3.
\]

Thus the energy-momentum conservation at $V_4$ implies

\[
(3.9) \quad p_6 = k_3,
\]

proving (3.3). In passing, we note that in the course of the above reasoning we have also confirmed

\[
(3.10) \quad p_4 + p_5 = 2p_6.
\]

The degeneration of this sort is a universal one, and we can confirm that at a point $p$ in $L^\oplus(T_n)$ ($n \geq 4$) where $T_n$ is the truss-bridge graph given in Fig. 12 below, all the internal lines become parallel, and hence we find (in the labeling of external energy-momentum vectors as in Fig. 12)

\[
(3.11) \quad p_4 + p_5 = 2p_6, \quad p_6^2 = m^2 \quad \text{if } n \text{ is odd},
\]

and

\[
(3.12) \quad p_5 + p_6 = 2p_4, \quad p_4^2 = m^2 \quad \text{if } n \text{ is even}.
\]
We also note

\[(3.13) \quad p_1 + p_2 = 2p_3, \quad p_3^2 = m^2\]

holds. Hence, by setting

\[(3.14) \quad N = N_+ \cup N_-\]

where

\[(3.15) \quad N_+ = \bigcup_{p_3^2 = m^2} \{(p_1, p_2, p_3); p_1 + p_2 = 2p_3\}\]

and

\[(3.16) \quad N_- = \bigcup_{p_6^2 = m^2} \{(p_4, p_5, p_6); p_4 + p_5 = 2p_6\}\]

we find

\[(3.17) \quad L^\oplus(T_n) \subset N \quad (n \geq 4)\]

with some change of labeling of \((p_4, p_5, p_6)\) if necessary. Thus the micro-analytic structure of the S-matrix near \(N\) should be formidable difficult to study, but we believe the analysis of individual Feynman integrals \(F_{T_n}(p)\) should be within reach of us.

\section*{§ 4. Concluding remarks and future problems}

Having in mind the study of micro-analytic structure of the S-matrix near the 3-particle threshold, we have made a detailed study of the LN surfaces associated with an ice-cream cone graph and a truss-bridge graph \(T_n\) with \(n = 3\) near the 3-particle...
threshold. Thanks to the power of recent computers our results are precise enough to stimulate the interest of mathematicians in the geometry of LN surfaces near the 3-particle threshold. Among other things we note that a central role is played by the set $N$ given by (3.14) (or $N_-$ for the configuration of Fig. 4). Although the singularity structure of the $S$-matrix near $N$ should be too complicated to analyze, we believe the study of the holonomic structure of individual Feynman integrals near $N$ is an interesting problem in microlocal analysis. Another interesting feature of our results is that the existence of non-external vertices in a Feynman graph normally gives strong constraint on the shape of the associated $LN$ variety. (See [4] and [5] for some related topics.) The study of the holonomic structure of a Feynman integral associated with a Feynman graph containing non-external vertices is an important and challenging problem in microlocal analysis. One natural way to approach this problem is to introduce fictitiously an external vector $p_j$ at a non-external vertex $V_j$ and then set it to be 0. As one immediately realizes, this procedure normally leads to the restriction of a holonomic system to a submanifold which contains characteristic points. We believe concrete studies of Feynman integrals of this sort should contribute much to the progress of the theory of holonomic systems.

References


