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Multi-point connection problem

By

Kana ANDO*

Abstract

The connection problem concerns the linear relations between fundamental sets of solutions near singular points. In this paper, we will emphasize the two-point connection problem. In the first section, we will explain why the two-point connection problem is interesting in the analysis of the Stokes phenomenon. In the second section, we will introduce an associated fundamental function which was introduced by K. Okubo in the 1960's [O]. In the third section, we will give an example of the two-point connection problem. In the final section, we will give an useful result of a reduction problem for solving the multi-point connection problem.

§ 1. Introduction

The method of associated fundamental functions was first applied to the two-point connection problem for a differential system with an irregular singular point of rank unity by K. Okubo in 1963 [O]. In 1974, M. Kohno applied it to a single differential equation with a regular singular point and an irregular singular point of arbitrary rank [K1]. In 1999, he also sketched an argument that would allow one to apply the associated fundamental functions to the problem in the case where one has an arbitrary number of regular sinsular points and one irregular singular point [K2].

It seems that this last advance has gone largely unnoticed, and there have been no further developments. In the future work, we will work on applying this method to solve the multi-point connection problem.

In this section, we will explain how the two-point connection problem is useful for analyzing the Stokes phenomenon.

For the rest of this paper, we assume that t is a complex variable. We consider an n-th order single differential equation which has one irregular singular point of rank

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*Department of Mathematics, Chiba University, Chiba 263-8522, Japan.

unity at infinity and a regular singular point at the origin, with unknown function y, of the form:

$$t^n \frac{d^n y}{dt^n} = \sum_{\ell=1}^{n} a_{n-\ell}(t) t^{n-\ell} \frac{d^{n-\ell} y}{dt^{n-\ell}}, \tag{1.1}$$

where $a_\ell(t)(\ell = 0, 1, \ldots, n-1)$ are holomorphic functions at the origin. There exists a fundamental set of solutions expressed in terms of convergent power series:

$$y_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \quad (j = 1, 2, \ldots, n),$$

in a punctured disc around the regular singular point $t = 0$, where $\rho_i - \rho_j \notin \mathbb{Z}$ $(i \neq j)$. We can calculate formal solutions:

$$y^k(t) = e^{\lambda_k t} t^{\mu_k} \sum_{s=0}^{\infty} h^k(s) t^{-s} \quad (k = 1, 2, \ldots, n)$$

at infinity, where $\lambda_k, \mu_k \in \mathbb{C}$. On each sector S with vertex at the origin and central angle not exceeding π, there exists a fundamental set of solutions $y_S^k(t)$ $(k = 1, 2, \ldots, n)$, such that

$$y_S^k(t) \sim y^k(t) \quad (|t| \to \infty \ in \ S).$$

We write $Y_0(t)$ to denote a vector function whose components are given by a fundamental set of solutions $y_j(t)$ near the origin, and $Y_S(t)$ to denote a vector function whose components are given by a fundamental set of solutions $y_S^k(t)$ near infinity on S;

$$Y_0(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad Y_S(t) = \begin{pmatrix} y_S^1(t) \\ y_S^2(t) \\ \vdots \\ y_S^n(t) \end{pmatrix}.$$

Let us denote the analytic continuation of the $y_S^k(t)$ into a sector S' by the same notation $y_S^k(t)$. Then we have a linear relation between $y_S^k(t)$ and $y_{S'}^k(t)$:

$$Y_S(t) = T(S:S') Y_{S'}(t) \quad T(S:S') \in \mathcal{M}_n(\mathbb{C}) \quad in \quad S'. \tag{1.2}$$

We call this constant matrix $T(S:S')$ the *Stokes matrix* or the *lateral connection matrix.* If we can find the exact value of the matrix $T(S:S')$, then the asymptotic behavior of $y_S^k(t)$ as t tends to infinity in S' will be immediately understood.

On the other hand, a linear relation between two fundamental sets of solutions $y_j(t)$ and $y_S^k(t)$ in S clearly holds:

$$Y_0(t) = W(S) Y_S(t) \quad in \quad S, \quad W(S) \in GL_n(\mathbb{C}). \tag{1.3}$$

We call this coefficients matrix the *central connection matrix.* Its derivation is often called *the central connection problem.*

If we can solve such a central connection problem (1.3) for every sector S, then after the analytic continuation of the $y_S^k(t)$ across a domain near $t = 0$ and then into the sector S', we can directly obtain the lateral connection formula (1.2). That is, once the central connection problem is solved, the Stokes phenomenon will be completely understood.

§ 2. Associated fundamental function

We will give here a short sketch of a method for the establishment of the asymptotic expansion $y_j(t)$ as t tends to infinity, together with the determination of the lateral connection matrices $T(S : S')$ for every sector S.

Assume that the central connection problem were solved. There exists a fundamental set of solutions of (1.1) expanded in terms of convergent power series in a punctured disc around the regular singular point $t = 0$:

$$y_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m)t^m \quad (j = 1, 2, \dots, n)$$

where $\rho_i - \rho_j \notin \mathbb{Z}(i \neq j)$. The fundamental solutions $y_S^k(t)(k = 1, 2, \dots, n)$ of (1.1) are characterized by formal solutions at the irregular singular point:

$$y_S^k(t) \sim y^k(t) \quad (|t| \to \infty \ in\ S).$$

Then $y_j(t)$ can be expressed as:

$$y_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m)t^m = \sum_{k=1}^{n} W_j^k(S)y_S^k(t)$$

where $W_j^k(S)$ are entries of the matrix $W(S)$:

$$W(S) = \begin{pmatrix} W_1^1(S) & W_1^2(S) & \cdots & W_1^n(S) \\ W_2^1(S) & W_2^2(S) & \cdots & W_2^n(S) \\ \vdots & \vdots & & \vdots \\ W_n^1(S) & W_n^2(S) & \cdots & W_n^n(S) \end{pmatrix}.$$

We shall introduce a set of functions $x_j^k(s;t)$, distinguished by the property that they admit the same local behavior as $y_j(t)$ in a punctured disc around the origin and $y^k(t)$ near infinity. We call the functions $x_j^k(s;t)$ the associated fundamental functions and we will work out the expansion of $y_j(t)$ in terms of $x_j^k(s;t)$:

$$x_j^k(s;t) \sim \begin{cases} t^{\rho_j} & (|t| \to 0) \\ e^{\lambda_k t} t^{\mu_k} & (|t| \to \infty). \end{cases}$$

Now we consider a first order non homogeneous differential equation:

$$t\frac{dx_j^k(s;t)}{dt} = (\lambda_k t + \mu_k - s)x_j^k(s;t) + t^{\rho_j}\lambda_k g_j^k(s-1) \quad (s = 0, 1, 2, \ldots)$$

which has the particular solutions:

$$x_j^k(s;t) = t^{\rho_j}\sum_{m=0}^{\infty} g_j^k(m+s)t^m.$$

By quadrature, from the first order non homogeneous differential equation, we obtain the integral representation:

$$x_j^k(s;t) = \lambda_k g_j^k(s-1)t^{\rho_j}\int_0^1 e^{\lambda_k t(1-\tau)}\tau^{s+\rho_j-\mu_k-1}d\tau.$$

We remark that the integral is well-defined for all integers s satisfying $s+\rho-\mu>0$, and if $\rho-\mu\notin\mathbb{Z}$, it can be regularized by analytic continuation for all integers s.

It is known that asymptotic behavior of $x(s;t)$ is

$$x_j^k(s;t) \sim e^{2\pi i(\rho_j-\mu_k)\ell}e^{\lambda_k t}t^{\mu_k - s} + t^{\rho_j}\{g_j^k(s-1)t^{-1} + g_j^k(s-2)t^{-2} + \cdots\}$$

as $|t|\to\infty$ in $|\arg(\lambda_k t) - 2\pi\ell| < \frac{3}{2}\pi$, where ℓ is an integer. This concludes our introduction of the associated fundamental functions $x_j^k(s;t)(k,j = 1,2,\ldots,n)$, and our analysis of the asymptotic behavior of $x_j^k(s;t)(k,j = 1,2,\ldots,n)$.

Next, we shall define additional functions:

$$f_j^k(m) = \sum_{m=0}^{\infty} h^k(s)g_j^k(m+s) \quad (k = 1,2,\ldots,n).$$

We can show that $f_j^k(m)(k = 1,2,\ldots,n)$ satisfies the same reccurances which $G_j(m)$ satisfies, but the proof is omitted. From these facts, we can analyze the asymptotic expansion of $y_j(t)$:

$$\begin{aligned} y_j(t) &= t^{\rho_j}\sum_{m=0}^{\infty} G_j(m)t^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=1}^{n} W_j^k f_j^k(m)\right)t^{m+\rho_j} \\ &= \sum_{k=1}^{n} W_j^k \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} h^k(s)g_j^k(m+s)t^{m+\rho_j} \\ &= \sum_{k=1}^{n} W_j^k \sum_{s=0}^{\infty} h^k(s)x_j^k(s;t). \end{aligned}$$

The asymptotic behavior of the associated fundamental function $x_j^k(s;t)$ is the same as that of $y^k(t)$. We will see more detail in the next section, where we work out an example.

§ 3. Example

In this section, we apply the Okubo-Kohno method to describe the global behavior of solutions of Airy's differential equation:

$$t^2y'' + \frac{1}{3}ty' - t^2y = 0. \tag{3.1}$$

This equation has one regular singular point at the origin, and one irregular singular point at infinity in the complex projective line. In [K2], Kohno computes some entries of the central connection matrix of (3.1). Here, we shall compute the remaining entries, and furthermore, we shall determine the Stokes matrix.

To begin, we find a fundamental set of solutions of (3.1) in a punctured disc around the regular singular point $t = 0$. These solutions have the form

$$y(t) = t^\rho \sum_{m=0}^{\infty} G(m)t^m \qquad (G(0) \neq 0). \tag{3.2}$$

By substituting this expansion into (3.1), we obtain the linear difference equation

$$\begin{cases} (m+\rho)(m+\rho-\frac{2}{3})G(m) = G(m-2), \\ G(0) \neq 0, \qquad G(r) = 0 \quad (r < 0). \end{cases} \tag{3.3}$$

In order for negative terms to vanish, it is necessary that ρ is equal to 0 or 2/3, and that $G(1) = 0$. By induction, $G(2m+1) = 0$ for all $m \geq 0$. If we set $G(0) = 1$, we obtain

$$\begin{cases} G(2m) = \dfrac{\Gamma(\frac{\rho}{2})\Gamma(\frac{\rho}{2}+\frac{2}{3})}{4^m\Gamma(m+\frac{\rho}{2}+1)\Gamma(m+\frac{\rho}{2}+\frac{2}{3})}, \\ G(2m+1) = 0. \end{cases} \tag{3.4}$$

Consequently, the two values of ρ yield a fundamental set of solutions in a punctured disc around the regular singular point $t = 0$ as follows:

$$\begin{cases} y_1(t) = \displaystyle\sum_{m=0}^{\infty} \frac{\Gamma(\frac{2}{3})}{\Gamma(m+1)\Gamma(m+\frac{2}{3})}\left(\frac{t}{2}\right)^{2m}, \\ y_2(t) = \displaystyle\sum_{m=0}^{\infty} \frac{2^{2/3}\Gamma(\frac{4}{3})}{\Gamma(m+1)\Gamma(m+\frac{4}{3})}\left(\frac{t}{2}\right)^{2m+2/3}. \end{cases} \tag{3.5}$$

By the asymptotic properties of Γ, the first series has infinite radius of convergence, and the second series is $t^{2/3}$ times a series with infinite radius of convergence.

We now consider solutions of (3.1) near $t = \infty$. Because the singularity is irregular, the solutions do not have the convergent expansions of the form (3.2). However, there are formal power series solutions of the form

$$y(t) = e^{\lambda t} t^{\mu} \sum_{s=0}^{\infty} h(s) t^{-s} \quad (h(0) \neq 0). \tag{3.6}$$

In order to seek the value of the characteristic constant λ and the characteristic exponent μ, we follow the method in the paper [K1]. We define $y^{(\kappa)}(t)$ ($\kappa = 0, 1, 2$) to be the κth derivative of $y(t)$ with respect to t:

$$y^{(\kappa)}(t) = \frac{d^{\kappa} y(t)}{dt^{\kappa}},$$

and we shall write the coefficients of the formal series $h^{\kappa}(s)$, that is

$$y^{(\kappa)}(t) = e^{\lambda t} t^{\mu} \sum_{s=0}^{\infty} h^{\kappa}(s) t^{-s}, \tag{3.7}$$

with $h^0(s) := h(s)$ Then, we have the relation:

Lemma 3.1. *From $y^{(\kappa)}(t) = (y^{(\kappa-1)}(t))'(\kappa = 1, 2)$, the relation*

$$h^{\kappa}(s) = \lambda h^{\kappa-1}(s) + (\mu - s + 1) h^{\kappa-1}(s-1) \quad (s = 0, 1, \ldots) \tag{3.8}$$

holds.

Proof.

$$\begin{aligned}
&y^{(\kappa)}(t) = (y^{(\kappa-1)}(t))' \\
&\Leftrightarrow e^{\lambda t} t^{\mu} \sum_{s=0}^{\infty} h^{\kappa}(s) t^{-s} = e^{\lambda t} t^{\mu} \left\{ \lambda \sum_{s=0}^{\infty} h^{\kappa-1}(s) t^{-s} + \sum_{s=0}^{\infty} (\mu - s) h^{\kappa-1}(s) t^{-s-1} \right\} \\
&\Leftrightarrow \sum_{s=0}^{\infty} h^{\kappa}(s) t^{-s} = \left\{ \lambda \sum_{s=0}^{\infty} h^{\kappa-1}(s) t^{-s} + \sum_{s=0}^{\infty} (\mu - s) h^{\kappa-1}(s) t^{-s-1} \right\}.
\end{aligned}$$

Comparing the coefficients of t^{-s}, we have the above formula. □

We substitute (3.7) into (3.1) to find that our initial terms satisfy:

$$(\lambda^2 - 1) h(0) = 0, \tag{3.9}$$

$$(\lambda^2 - 1) h(1) + 2\lambda \left(\mu + \frac{1}{6} \right) h(0) = 0. \tag{3.10}$$

and the remaining terms satisfy the following recursion for $s \geq 0$:

$$(\lambda^2 - 1)h(s+2) + 2\lambda\left(-s - 1 + \mu + \frac{1}{6}\right)h(s+1) + (s-\mu)\left(s - \mu + \frac{2}{3}\right)h(s) = 0.$$

Because we assumed $h(0) \neq 0$, we see from the initial term equations that λ must be equal to ± 1 and μ must be equal to $-\frac{1}{6}$. Then, from the recursion, we obtain the linear difference equation in s:

$$h(s) = \frac{(s-\mu-1)(s-\mu-\frac{1}{3})}{2\lambda s} h(s-1)$$

Setting $h(0) = 1$, we obtain the explicit formula:

$$h(s) = \left(\frac{1}{2\lambda}\right)^s \frac{\Gamma(s-\mu)\Gamma(s-\mu+\frac{2}{3})}{\Gamma(s+1)\Gamma(-\mu)\Gamma(-\mu+\frac{2}{3})}.$$

Using the two possible values of λ, we obtain two formal solutions near $t = \infty$:

$$\text{(3.11)} \qquad \begin{cases} y^1(t) = e^t t^{-\frac{1}{6}} \displaystyle\sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{6})\Gamma(s+\frac{5}{6})}{\Gamma(s+1)\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})} \left(\frac{1}{2\lambda}\right)^s & (\lambda = 1), \\ y^2(t) = e^{-t} t^{-\frac{1}{6}} \displaystyle\sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{6})\Gamma(s+\frac{5}{6})}{\Gamma(s+1)\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})} \left(-\frac{1}{2\lambda}\right)^s & (\lambda = -1). \end{cases}$$

It is straightforward to see that these formal solutions diverge wildly, but they are useful because they are in fact asymptotic expansions of holomorphic solutions in sectors near infinity.

We shall now apply the Okubo-Kohno method.

Suppose that we are given a convergent power series solution of the form (3.2) near $t = 0$, and suppose we have an additional expansion as a combination of holomorphic functions $\{x(s;t) : s = 0, 1, \ldots\}$ as follows:

$$y(t) = \sum_{s=0}^{\infty} h(s)x(s;t).$$

The solution $y(t)$ behaves near infinity like

$$y(t) \sim Te^{\lambda t}t^{\mu}\left\{1 + O\left(\frac{1}{t}\right)\right\} \qquad (|t| \to \infty),$$

where T is a Stokes multiplier. If our functions $\{x(s;t) : s = 0, 1, \ldots\}$ admit the following asymptotic behavior

$$\text{(3.12)} \qquad \begin{cases} x(s;t) \sim t^{\rho} & (|t| \to 0), \\ x(s;t) \sim e^{\lambda t}t^{\mu - s} & (|t| \to \infty), \end{cases}$$

we can reasonably expect them to combine to form y, and satisfy convenient uniqueness properties.

We will construct functions $\{x(s;t) : s = 0, 1, \ldots\}$ of the form :

$$x(s;t) = t^\rho \sum_{m=0}^{\infty} g(m+s)t^m \tag{3.13}$$

that satisfy the first order non-homogeneous linear differential equations

$$tx'(s;t) = (\lambda t + \mu - s)x(s;t) + \lambda g(s-1)t^\rho \qquad (s = 0, 1, \ldots), \tag{3.14}$$

and the asymptotics given in (3.12). We will see that $x(s;t)$ is uniquely defined by these properties once we have chosen $g(0)$.

By substituting (3.13) into (3.14) and isolating powers of t, we see that the coefficient $g(m+s)$ satisfies the first order linear difference equation

$$(m+s+\rho-\mu)g(m+s) = \lambda g(m+s-1). \tag{3.15}$$

This linear difference equation therefore uniquely determines $x(s;t)$ once the initial term is specified. We set:

$$g(m+s) = \frac{\lambda^{m+s+\rho-\mu}}{\Gamma(m+s+\rho-\mu+1)} \tag{3.16}$$

as a particular solution of (3.15). By quadrature, the non-homogeneous equation (3.14) has solution given by the integral representation

$$x(s;t) = \lambda g(s-1)t^\rho \int_0^1 \exp\{\lambda t(1-\tau)\}\tau^{s+\rho-\mu-1}d\tau. \tag{3.17}$$

We therefore have our sequence of associated fundamental functions $\{x(s;t) : s = 0, 1, \ldots\}$, and they have the expected asymptotic behavior in sectors. Indeed, for arbitrarily small positive ε, and any integer ℓ, we have:

$$x(s;t) \sim e^{2\pi i(\rho-\mu)\ell}e^{\lambda t}t^{\mu-s} + t^\rho\{g(s-1)t^{-1} + g(s-2)t^{-2} + \cdots\} \tag{3.18}$$

as $t \longrightarrow \infty$ in $|\arg(\lambda t) - 2\pi\ell| \leq \frac{3}{2}\pi - \varepsilon$.

We return to our example, where our solutions were determined by the values of $\rho \in \{0, \frac{2}{3}\}$ and $\lambda = \pm 1$. Here, we consider the cases where $\rho = \frac{2}{3}$, $\lambda = \pm 1$ and $\mu = -\frac{1}{6}$. Then, the associated fundamental functions are defined by

$$\left(m+s+\frac{5}{6}\right) g_2^k(m+s) = \lambda_k g_2^k(m+s-1) \qquad (k = 1, 2; \lambda_1 = 1, \lambda_2 = e^{\pi i}), \tag{3.19}$$

and using the explicit formula for $g_2^k(m)$ from (3.16), we have

$$\begin{aligned} x_2^k(s;t) &= \sum_{m=0}^{\infty} g_2^k(m+s)t^{m+\frac{2}{3}}, \\ &= \sum_{m=0}^{\infty} \frac{(\lambda_k)^{m+s+\frac{5}{6}}}{\Gamma(m+s+\frac{11}{6})} t^m \qquad (k=1,2). \end{aligned} \tag{3.20}$$

If we write $h^k(s)(k=1,2)$ to denote the coefficients in the formal power series expansion (3.11) of $y^k(t)$, we may define the functions $f_2^k(m)(k=1,2)$ by

$$f_2^k(m) = \sum_{s=0}^{\infty} h^k(s) g_2^k(m+s) \qquad (k=1,2). \tag{3.21}$$

Because our explicit formula for $g_2^k(m)$ from (3.16) yields a holomorphic function on the right half m-plane, the same is true for $f_2^k(m)$. Indeed, we have the asymptotic relations:

$$f_2^k(m) \sim \frac{(\lambda_k)^{m+\frac{5}{6}}}{\Gamma(m+\frac{11}{6})} \left\{1 + O\left(\frac{1}{m}\right)\right\}. \tag{3.22}$$

Here the proof is omitted.

We claim that $f_2^k(m)(k=1,2)$ satisfies the same recurrence that defines $G_2(m)$, but we omit the proof. Therefore, $G_2(m)$ can be expressed as a linear combination of the $f_2^k(m)(k=1,2)$ as follows :

$$G_2(m) = W_2^1 f_1^1(m) + W_2^2 f_2^2(m) \tag{3.23}$$

where the $W_2^k(k=1,2)$ are, in general, periodic functions of m with period 1, however, they may be considered to be constants for integral values of m. From this, we consequently obtain the expansion of $y_2(t)$ in terms of sequences of associated fundamental functions $\{x_2^k(s;t) : s = 0, 1, \dots (k=1,2)\}$:

$$\begin{aligned} y_2(t) &= \sum_{m=0}^{\infty} G_2(m) t^{m+\frac{2}{3}} \\ &= W_2^1 \sum_{m=0}^{\infty} f_2^1(m) t^{m+\frac{2}{3}} + W_2^2 \sum_{m=0}^{\infty} f_2^2(m) t^{m+\frac{2}{3}} \\ &= W_2^1 \sum_{s=0}^{\infty} h^1(s) \left(\sum_{m=0}^{\infty} g_2^1(m+s) t^{m+\frac{2}{3}} \right) + W_2^2 \sum_{s=0}^{\infty} h^2(s) \left(\sum_{m=0}^{\infty} g_2^2(m+s) t^{m+\frac{2}{3}} \right) \\ &= W_2^1 \sum_{s=0}^{\infty} h^1(s) x_2^1(s;t) + W_2^2 \sum_{s=0}^{\infty} h^2(s) x_2^2(s;t). \end{aligned} \tag{3.24}$$

We conclude that for each nonnegative integer m, $f_2^k(m)(k=1,2)$ is the coefficient attached to $t^{m+\frac{2}{3}}$, when $y_2(t)$ is expanded as a power series. We may now use the asymptotic behavior (3.18) of the associated fundamental functions to analyze the asymptotic behavior of the original solutions. We derive from (3.24)

$$\begin{aligned} y_2(t) &\sim W_2^1 \sum_{m=0}^{\infty} h^1(s)\left\{e^t t^{-\frac{1}{6}-s} + \sum_{r=0}^{\infty} g_2^1(s-r)t^{-r}\right\} \\ &\quad + W_2^2 \sum_{s=0}^{\infty} h^s(s)\left\{e^{-t} t^{-\frac{1}{6}-s} + \sum_{r=0}^{\infty} g_2^2(s-r)t^{-r}\right\} \\ &\sim W_2^1 y^1(t) + W_2^2 y^2(t) \\ &\quad + \sum_{r=0}^{\infty}\left(W_2^1 f_2^1(-r) + W_2^2 f_2^2(-r)\right) t^{-r} \\ &\sim W_2^1 y^1(t) + W_2^2 y^2(t) + \sum_{r=0}^{\infty} G_2(-r)t^{-r} \\ &\sim W_2^1 y^1(t) + W_2^2 y^2(t) \end{aligned}$$

as $t \longrightarrow \infty$ in the sector

$$\widehat{S} = \bigcap_{k=1}^{2}\left\{|\arg(\lambda_k t)| < \frac{3}{2}\pi\right\} = \left\{-\frac{3}{2}\pi < \arg t < \frac{\pi}{2}\right\}.$$

Now that we have all of the necessary asymptotic information in hand, we can determine $W_2^k(k=1,2)$ by combining the fact that $G_j(m)(j=1,2)$ vanishes on odd inputs with our knowledge of the asymptotic behavior on even inputs. Explicitly, we combine (3.11) and (3.20) to get

$$f_2^2(m) = e^{\pi i(m+\frac{5}{6})} f_2^1(m)$$

for all $m \geq 0$, and from that, we apply

$$\begin{aligned} 0 = G_2(2m+1) &= W_2^1 f_2^1(2m+1) + f_2^2(2m+1) \\ &= (W_2^1 - W_2^2 e^{\frac{5}{6}\pi i}) f_2^1(2m+1). \end{aligned}$$

to deduce one relation:

$$W_2^1 = W_2^2 e^{\frac{5}{6}\pi i}. \tag{3.25}$$

For the second relation, we consider the formula:

$$G_2(2m) = W_2^1 f_2^1(2m) + W_2^2 f_2^2(2m).$$

From (3.4) and using the asymptotic behavior of $f_2^k(2m)$ given in (3.22), we may divide by $f_2^1(2m)$ to find that for sufficiently large m,

$$\begin{aligned} (3.26) \qquad & W_2^1 + W_2^2 e^{\frac{5}{6}\pi i} \\ &= \frac{\Gamma(\frac{4}{3})\Gamma(2m+\frac{11}{6})}{4^m\Gamma(m+1)\Gamma(m+\frac{4}{3})}\left\{1+O\left(\frac{1}{m}\right)\right\} \\ &= \frac{\Gamma(\frac{4}{3})2^{2m+\frac{11}{6}}\Gamma(m+\frac{11}{12})\Gamma(m+\frac{17}{12})}{\sqrt{2\pi}4^m\Gamma(m+1)\Gamma(m+\frac{4}{3})}\left\{1+O\left(\frac{1}{m}\right)\right\} \\ &= \frac{\Gamma(\frac{4}{3})2^{\frac{4}{3}}}{\sqrt{2\pi}}\left\{1+O\left(\frac{1}{m}\right)\right\}. \end{aligned}$$

However, $W_2^1 + W_2^2 e^{\frac{5}{6}\pi i}$ is constant, so the $O(1/m)$ terms vanish:

$$(3.27) \qquad W_2^1 + W_2^2 e^{\frac{5}{6}i} = \frac{2^{\frac{4}{3}}\Gamma(\frac{4}{3})}{\sqrt{2\pi}}.$$

By combining this with (3.25), we find that the connection coefficients $W_2^k (k = 1, 2)$ are:

$$W_2^1 = W_2^2 e^{\frac{5}{6}\pi i} = \frac{2^{\frac{7}{6}}\,\Gamma(\frac{5}{6})}{\sqrt{3}\,\Gamma(\frac{1}{3})}.$$

Therefore, we obtain the connection formula:

$$y_2(t) \sim \begin{cases} W_2^2 y^2(t) & (S_1 : -\frac{3}{2}\pi < \arg t < -\frac{\pi}{2}), \\ W_2^1 y^1(t) & (S_2 : -\frac{\pi}{2} < \arg t < \frac{\pi}{2}), \\ W_2^2 e^{\frac{5}{3}\pi i} y^2(t) & (S_3 : \frac{\pi}{2} < \arg t < \frac{3}{2}\pi). \end{cases}$$

For $y_1(t)$, in [K2], Kohno employed a similar calculation to find the following connection formula:

$$y_1(t) \sim \begin{cases} W_1^2 y^2(t) & (S_1 : -\frac{3}{2}\pi < \arg t < -\frac{\pi}{2}), \\ W_1^1 y^1(t) & (S_2 : -\frac{\pi}{2} < \arg t < \frac{\pi}{2}), \\ W_1^2 e^{\frac{\pi}{3} i} y^2(t) & (S_3 : \frac{\pi}{2} < \arg t < \frac{3}{2}\pi). \end{cases}$$

where $W_1^1 = W_1^2 e^{\frac{\pi}{6}i} = \left(\frac{1}{2}\right)^{\frac{1}{6}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})}$. Even without the exact value of W_1^1 and W_1^2, we can compute the Stokes coefficients. For example, the analytic continuation of Y_{S_2} from S_2 to S_3:

$$T(S_2 : S_3) = W^{-1}(S_2)W(S_3) = \begin{pmatrix} 0 & i \\ & \\ i & 1 \end{pmatrix},$$

with

$$Y_{S_2} = \begin{pmatrix} y^1_{S_2} \\ y^2_{S_2} \end{pmatrix}, W(S_1) = W(S_2) = \begin{pmatrix} W^1_1 & W^2_1 \\ \\ W^1_2 & W^2_2 \end{pmatrix} = \begin{pmatrix} W^2_1 e^{\frac{\pi}{6}i} & W^2_1 \\ \\ W^2_2 e^{\frac{5}{6}\pi i} & W^2_2 \end{pmatrix}$$

$$W(S_3) = \begin{pmatrix} W^1_1 e^{\frac{\pi}{3}i} & W^2_1 e^{\frac{\pi}{3}i} \\ \\ W^1_2 e^{\frac{5}{3}\pi i} & W^2_2 e^{\frac{5}{3}\pi i} \end{pmatrix} = \begin{pmatrix} W^2_1 e^{\frac{\pi}{2}i} & W^2_1 e^{\frac{\pi}{3}i} \\ \\ W^2_2 e^{\frac{\pi}{2}i} & W^2_2 e^{\frac{5}{3}\pi i} \end{pmatrix}.$$

§ 4. Our result

In [K2], Kohno outlines a method for solving a multi-point connection problem for a system of differential equation:

$$\frac{dX}{dt} = \left\{ \frac{A_0}{t} + \frac{A_1}{t-1} + A_2 \right\} X, \tag{4.1}$$

to which one can always reduce a single differential equation:

$$t^n(1-t)^n \frac{d^n y}{dt^n} = \sum_{\ell=0}^{n} \left(\sum_{r=0}^{2\ell} a_{\ell,r} t^r \right) t^{n-\ell}(1-t)^{n-\ell} \frac{d^{n-\ell} y}{dt^{n-\ell}}$$

where $A_i (i = 0, 1, 2)$ are n by n matrices.

The method explained in [K2] is likely to be useful for solving the connection problem for more general equations, with unknown function y, of the form:

$$P_n(t)\, y^{(n)} = P_{n-1}(t)\, y^{(n-1)} + \cdots + P_1(t)\, y' + P_0(t)\, y, \tag{4.2}$$

where

$$P_n(t) = \prod_{j=1}^{n} (t - \lambda_j)$$

and the coefficients $P_j(t)$ $(j = 0, 1, \ldots, n-1)$ are polynomials of degree at most n.

For the purpose of analyzing the multi-point connection problem, the following theorem is useful.

Theorem 4.1 (M.Kohno and K.Ando 2006, K.Ando 2012). *The differential equation (4.2) can be reduced to the system of linear differential equations, with unknown length n vector function X*

$$(tI - B)\frac{dX}{dt} = (A + Ct)X, \tag{4.3}$$

where I is the n by n identity matrix, A is an n by n constant matrix, C is an n by n constant lower triangular matrix, and

$$B = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),.$$

We can also apply this method to the more general reduction problem, in which $P_n(t)$ may have multiple roots.

Moreover, multiplying $(tI - B)^{-1}$ from the left side, we shall show that our system can be reduced to a generalized Schlesinger system:

$$\frac{dX}{dt} = \left(\sum_{i=1}^{q} \frac{\bar{A}_i}{t - \lambda_i} + C\right) X$$

where q is a number of regular singular points and $\bar{A}_i (i = 1, 2, \ldots, q)$ are n by n constant matrices.

Proof. (K.Ando 2012) The system (4.3) can be reduced to a generalized Schlesinger system:

$$\frac{dX}{dt} = \left(\sum_{i=1}^{q} \frac{\bar{A}_i}{t - \lambda_i} + C\right) X,$$

with $\bar{A}_i (i = 1, 2, \ldots, q)$ being n by n constant matrices. □

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