数理解析研究所講究録
第 1861 巻 2013 年 132-139

A relation between instanton-type solutions of P_{J} $(\mathrm{J}=\mathrm{I}, \mathrm{II}, 34, \mathrm{IV})$-hierarchies with a large parameter

By

Yoko UMETA*

Abstract

We report a relation between instanton-type solutions for equations of P_{J} $(\mathrm{J}=\mathrm{I}, \mathrm{II}, 34, \mathrm{IV})$-hierarchies with a large parameter. The content of these notes is a short summary of our forthcoming papers [2], [16] and [17].

§ 1. Definitions of P_{J} $(\mathrm{J}=\mathrm{I}, \mathrm{II}, 34, \mathrm{IV})$-hierarchies with a large parameter η

We recall the definitions of equations of P_{J} $(\mathrm{J}=\mathrm{I}, \mathrm{II}, 34, \mathrm{IV})$-hierarchies with a large parameter η given in [15], [8] and [9].

(i) The m-th members $(P_{\mathrm{I}})_m$ and $(P_{34})_m$ of P_{I}, P_{34}-hierarchies with η

Let u_k and v_k $(k=1,2,\dots)$ be unknown functions of t and c_k's are constants. In what follows, δ_{jm} stands for Kronecker's delta.

- For $m=1,2,\dots$, $(P_{\mathrm{I}})_m$ has the following form (see [15]):

$$
(1.1) \qquad \begin{cases} \eta^{-1}\dfrac{du_j}{dt}=2v_j, & j=1,2,\dots,m, \\ \eta^{-1}\dfrac{dv_j}{dt}=2(u_{j+1}+u_1u_j+w_j), & j=1,2,\dots,m, \end{cases}
$$

with the assumption $u_{m+1}=0$. Here w_j is recursively defined by

$$
(1.2) \qquad w_j=\frac{1}{2}\sum_{k=1}^{j}u_ku_{j+1-k}+\sum_{k=1}^{j-1}u_kw_{j-k}-\frac{1}{2}\sum_{k=1}^{j-1}v_kv_{j-k}+c_j+\delta_{jm}t.
$$

2010 Mathematics Subject Classification(s): Primary 34E20; Secondary 34M40.
Key Words: exact WKB analysis, Painlevé hierarchies, instanton-type solutions
*Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.

- For $m=1,2,\ldots,$ $(P_{34})_m$ has the following form (see [9]):

$$(1.3)\quad \begin{cases} \eta^{-1}\dfrac{du_j}{dt}=2v_j, & j=1,2,\ldots,m,\\ \eta^{-1}\dfrac{dv_j}{dt}=2(u_{j+1}+u_1u_j+w_j), & j=1,2,\ldots,m,\end{cases}$$

with

$$(1.4)\quad u_{m+1}=-w_m+c_0u_m+\frac{v_m^2-\kappa^2}{2u_m}.$$

Here $\gamma(\neq 0)$, κ and $\{c_j\}_{j=0}^m$ are constants, and w_j is recursively defined by

$$(1.5)\quad \begin{aligned} w_j={}&\frac{1}{2}\sum_{k=1}^{j}u_ku_{j+1-k}+\sum_{k=1}^{j-1}u_kw_{j-k}-\frac{1}{2}\sum_{k=1}^{j-1}v_kv_{j-k}+c_j\\ &+c_0\Big(2u_j-\sum_{k=1}^{j-1}u_ku_{j-k}\Big)+\delta_{j,m-1}\gamma t+2\delta_{jm}\gamma tc_0.\end{aligned}$$

Note that the form above has been slightly modified from the original form given by [9]. If we replace u_m (resp. v_m) in (1.3) and (1.4) with $u_m-\gamma t$ (resp. $v_m-\eta^{-1}\frac{\gamma}{2}$), then we have the original form of $(P_{34})_m$.

(ii) The m-th members $(P_{\text{II}})_m$ and $(P_{\text{IV}})_m$ of P_{II}, P_{IV}-hierarchies with η

Let u_k and v_k $(k=1,2,\ldots)$ be unknown functions of t and c_k's are constants.

- For $m=1,2,\ldots,$ $(P_{\text{II}})_m$ has the following form (see [8]):

$$(1.6)\quad \begin{cases} \eta^{-1}\dfrac{du_j}{dt}=-2(u_1u_j+v_j+u_{j+1})+2c_ju_1, & j=1,2,\ldots,m,\\ \eta^{-1}\dfrac{dv_j}{dt}=2(v_1u_j+v_{j+1}+w_j)-2c_jv_1, & j=1,2,\ldots,m,\end{cases}$$

with $u_{m+1}=\gamma t$ and $v_{m+1}=\kappa$. Here $\gamma(\neq 0)$, κ and c_j's are constants, and w_j is recursively defined by

$$(1.7)\quad w_j=\sum_{k=1}^{j-1}u_{j-k}w_k+\sum_{k=1}^{j}u_{j-k+1}v_k+\frac{1}{2}\sum_{k=1}^{j-1}v_{j-k}v_k-\sum_{k=1}^{j-1}c_{j-k}w_k.$$

- For $m=1,2,\ldots,$ $(P_{\text{IV}})_m$ has the following form (see [8]):

$$(1.8)\quad \begin{cases} \eta^{-1}\dfrac{du_j}{dt}=-2(u_1u_j+v_j+u_{j+1})+2c_ju_1-2\delta_{j,m-1}\gamma t,\\ \eta^{-1}\dfrac{dv_j}{dt}=2(v_1u_j+v_{j+1}+w_j)-2c_jv_1, \quad j=1,2,\ldots,m,\end{cases}$$

with

$$u_{m+1} = -\alpha_1, \quad v_{m+1} = -w_m - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - c_m)}. \tag{1.9}$$

Here $\gamma(\neq 0)$, α_1, α_2 and c_j's are constants, and w_j is defined by

$$w_j = \sum_{k=1}^{j-1} u_{j-k}w_k + \sum_{k=1}^{j} u_{j-k+1}v_k + \frac{1}{2}\sum_{k=1}^{j-1} v_{j-k}v_k - \sum_{k=1}^{j-1} c_{j-k}w_k + \delta_{jm}\gamma t v_1. \tag{1.10}$$

Note that the form above has been slightly modified from the original form given by [8]. If we replace u_m in (1.8) and (1.9) with $u_m - \gamma t$, then we have the original form of $(P_{\mathrm{IV}})_m$.

§ 2. P_{J} (J = I, II, 34, IV)-hierarchies with η in terms of generating functions

In this note, we consider the represented forms with generating functions of unknown functions. Let θ denotes an independent variable.

(i) The m-th members $(P_{\mathrm{I}})_m$ and $(P_{34})_m$ of P_{I}, P_{34}-hierarchies with η

We define generating functions U, V and C of $(P_{\mathrm{I}})_m$ (resp. $(P_{34})_m$) by

$$\begin{aligned} &U(\theta) = \sum_{k=1}^{\infty} u_k\theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k\theta^k \quad \text{and} \\ &C(\theta) = \sum_{k=1}^{\infty}(c_k + \delta_{km}t)\theta^{k+1} \quad (\text{resp. } C(\theta) = \sum_{k=1}^{\infty} c_k\theta^{k+1}), \end{aligned} \tag{2.1}$$

respectively. Here u_k, v_k, c_k $(k = 1, 2, \dots)$ denote unknown functions and constants of $(P_{\mathrm{I}})_m$ (resp. $(P_{34})_m$). In what follows, by $A \equiv B$ we mean that $A - B$ is zero modulo θ^{m+2}.

- $(P_{\mathrm{I}})_m$ is rewritten in the following form

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} 2V\theta \\ -(1 + 2u_1\theta)(1 - U) + \dfrac{1 + 2C - \theta V^2}{1 - U} \end{pmatrix} \tag{2.2}$$

with the condition that the coefficients of θ^{m+1} of U and V are zero.

- $(P_{34})_m$ is rewritten in the following form

$$\begin{aligned} \eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv &\begin{pmatrix} 2V\theta \\ -(1 + 2(u_1 + c_0)\theta)(1 - U) + \dfrac{1 + 2C - \theta(V^2 - 2c_0)}{1 - U} \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 2\gamma t\theta^m(1 + (u_1 + 2c_0)\theta) \end{pmatrix} \end{aligned} \tag{2.3}$$

with the condition that the coefficient of θ^{m+1} of U (resp. V) is equal to the right hand side of (1.4) (resp. zero).

Remark that, if we compare the coefficients with respect to θ^j $(2 \leq j \leq m+1)$ on the both sides of (2.2) (resp. (2.3)), we obtain (1.1) (resp. (1.3)).

(ii) The m-th members $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ of P_{II}, P_{IV}-hierarchies with η

We define generating functions U, V and C of $(P_{\mathrm{II}})_m$ (resp. $(P_{\mathrm{IV}})_m$) by

$$U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \quad C(\theta) = \sum_{k=1}^{\infty} c_k \theta^k, \tag{2.4}$$

respectively. Here u_k, v_k, c_k $(k = 1, 2, \dots)$ denote unknown functions and constants of $(P_{\mathrm{II}})_m$ (resp. $(P_{\mathrm{IV}})_m$).

- $(P_{\mathrm{II}})_m$ is rewritten in the following form

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv 2 \begin{pmatrix} u_1(1-U+C)\theta - U - V\theta \\ -v_1(1-U+C)\theta + \dfrac{2UV + V^2\theta}{2(1-U+C)} + V \end{pmatrix} \tag{2.5}$$

with the condition that the coefficients of θ^{m+1} of U and V are γt and κ, respectively.

- $(P_{\mathrm{IV}})_m$ is rewritten in the following form

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv 2 \begin{pmatrix} u_1(1-U+C)\theta - U - V\theta - \gamma t\theta^m \\ -v_1(1-U+C)\theta + \dfrac{2UV + V^2\theta}{2(1-U+C)} + V + \gamma t v_1 \theta^{m+1} \end{pmatrix} \tag{2.6}$$

with the condition that the coefficients of θ^{m+1} of U and V are equal to the right hand sides of (1.9), respectively.

Remark that, if we compare the coefficients with respect to θ^j $(2 \leq j \leq m+1)$ on the both sides of (2.5) (resp. (2.6)), we obtain (1.6) (resp. (1.8)).

§ 3. The generating functions of the leading terms of 0-parameter solutions

As is shown in [4], each P_{J}-hierarchy has a formal power series of η^{-1} in the form

$$u_k(t) = \sum_{j=0}^{\infty} \eta^{-j}\hat{u}_{k,j}(t), \quad v_k(t) = \sum_{j=0}^{\infty} \eta^{-j}\hat{v}_{k,j}(t), \quad j = 1, \dots, m. \tag{3.1}$$

The solution taking the form of (3.1) is often called a 0-parameter solution, as the form does not have any free parameters. Let us define the generating functions of the leading terms $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ of their 0-parameter solutions of $(P_{\mathrm{J}})_m$ (J $=$ I, II, 34, IV) by

$$\hat{u}_0(\theta) = \sum_{i=1}^{\infty} \hat{u}_{i,0}\theta^i, \qquad \hat{v}_0(\theta) = \sum_{i=1}^{\infty} \hat{v}_{i,0}\theta^i. \tag{3.2}$$

Each explicit form of (3.2) for $(P_{\mathrm{J}})_m$ $(\mathrm{J} = \mathrm{I}, \mathrm{II}, 34, \mathrm{IV})$ is given as follows.

$(P_{\mathrm{I}})_m$	$\hat{u}_0 = 1 - \sqrt{\dfrac{1+2C}{1+2\hat{u}_{1,0}\theta}}, \qquad \hat{v}_0 = 0.$ Here $\hat{u}_{1,0}$ is taken so that the coefficient of θ^{m+1} in $\hat{u}_0$ is zero.
$(P_{34})_m$	$\hat{u}_0 \equiv 1 - \sqrt{\dfrac{1+2(C+c_0\theta)}{(1+2(\hat{u}_{1,0}+c_0)\theta)(1-2\gamma t\theta^m)}},$ $\hat{v}_0 = 0,$ where $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ are taken so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ are equal to $-\hat{w}_{m,0} + c_0\hat{u}_{m,0} + \dfrac{(\hat{v}_{m,0}^2 - \kappa^2)}{2\hat{u}_{m,0}}$ and 0, respectively. Here $\hat{w}_{m,0}$ is defined by (1.5) with u_k, v_k and w_k being replaced by $\hat{u}_{k,0}$, $\hat{v}_{k,0}$ and $\hat{w}_{k,0}$.
$(P_{\mathrm{II}})_m$	$\hat{u}_0 = (1+C)\Big(1 - \sqrt{\dfrac{1}{(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\Big),$ $\hat{v}_0\theta = (1+C)\Big(-1 + (1+\hat{u}_{1,0}\theta)\sqrt{\dfrac{1}{(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\Big).$ Here $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ are taken so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ are γt and κ, respectively.
$(P_{\mathrm{IV}})_m$	$\hat{u}_0 \equiv (1+C)(1 - \sqrt{1/f(t,\theta)}),$ $\hat{v}_0\theta \equiv (1+C)(-1 + (1+\hat{u}_{1,0}\theta)\sqrt{1/f(t,\theta)}) - \gamma t\theta^m,$ $f(t,\, \theta) := (1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2 - 2\gamma t\theta^m(1 + 2\hat{u}_{1,0}\theta - c_1\theta),$ where $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ are taken so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ are equal to $-\alpha_1$ and $-\hat{w}_{m,0} - \dfrac{(\hat{v}_{m,0}-\alpha_1)^2 - \alpha_2^2}{2(\hat{u}_{m,0} - c_m)},$ respectively. Here $\hat{w}_{m,0}$ is defined by (1.10) with u_k, v_k and w_k being replaced by $\hat{u}_{k,0}$, $\hat{v}_{k,0}$ and $\hat{w}_{k,0}$.

§ 4. Instanton-type solutions of $(P_{\mathrm{J}})_m$ $(\mathrm{J}=\mathrm{I},\mathrm{II},34,\mathrm{IV})$

We prepare some notation. Let $\nu_{\pm 1}(t),\dots,\nu_{\pm m}(t)$ of $(P_{\mathrm{J}})_m$ denote the roots of the following algebraic equation $\Lambda_{\mathrm{J}}(\lambda,\,t)=0$ of λ with $\nu_k=-\nu_{|k|}$ if $k<0$:

- If $\mathrm{J}=\mathrm{I}$ (resp. 34), then $\Lambda_{\mathrm{J}}(\lambda,\,t)$ is defined by

$$g_{\mathrm{J}}(\lambda)^m-\sum_{k=1}^{m}\hat{u}_{k,0}\,g_{\mathrm{J}}(\lambda)^{m-k}$$

with

$$g_{\mathrm{J}}(\lambda)=\frac{\lambda^2-8\hat{u}_{1,0}}{4}\quad(\text{resp. } g_{34}(\lambda)=\frac{\lambda^2-8(\hat{u}_{1,0}+c_0)}{4}).$$

Here $\hat{u}_{k,0}$'s are defined by (3.2) of $(P_{\mathrm{J}})_m$ $(\mathrm{J}=\mathrm{I},34)$ respectively.

- If $\mathrm{J}=\mathrm{II}$ or IV, then $\Lambda_{\mathrm{J}}(\lambda,t)$ is defined by

$$g_{\mathrm{J}}(\lambda)^m-\sum_{k=1}^{m}(\hat{u}_{k,0}-c_k)g_{\mathrm{J}}(\lambda)^{m-k}$$

with

$$g_{\mathrm{J}}(\lambda)=-\hat{u}_{1,0}-\sqrt{\frac{\lambda^2}{4}+2\hat{v}_{1,0}}\ .$$

Here $\hat{u}_{k,0}$, $\hat{v}_{1,0}$ are defined by (3.2) of $(P_{\mathrm{J}})_m$ $(\mathrm{J}=\mathrm{II},\mathrm{IV})$ respectively and so are c_k.

Let Ω be an open subset in $\mathbb{C}_t$ and the two conditions are always assumed:

(A1) The roots $\nu_i(t)$'s $(1\le|i|\le m)$ are mutually distinct for each $t\in\Omega$.

(A2) The function $p_1\nu_1(t)+\cdots+p_m\nu_m(t)$ does not vanish identically on Ω for any $(p_1,\dots,p_m)\in\mathbb{Z}^m\setminus\{0\}$.

Then we have the following theorem.

Theorem 4.1 ([2], [17]). *We have instanton-type solutions of* $(P_{\mathrm{J}})_m$ $(\mathrm{J}=\mathrm{I},34)$ *with free 2m-parameters* $(\beta_{-m},\dots,\beta_m)\in\mathbb{C}^{2m}[\![\eta^{-1}]\!]$ *of the form*

$$U=\hat{u}_0+(1-\hat{u}_0)u,\quad V=\hat{v}_0+(1-\hat{u}_0)v,$$

$$\begin{pmatrix}u\\v\end{pmatrix}=\sum_{1\le|k|\le m}f_k^{\mathrm{J}}(\tau,t;\eta)A(\nu_k),\quad A(\nu_k)=\begin{pmatrix}a(\nu_k)\\ \frac{\nu_k}{2}a(\nu_k)\end{pmatrix},\tag{4.1}$$

$$f_k^{\mathrm{J}}(\tau,t;\eta)=\sum_{j=1}^{\infty}\Big(\sum_{\substack{\ell\ge0,\,p\in\mathbb{Z}^m\\2\ell+|p|=j}}f_{k,p,\ell}^{\mathrm{J}}(t)e^{p\cdot\tau}\Big)\eta^{-j/2},\tag{4.2}$$

with $a(\nu_k)=\dfrac{\theta}{1-\theta g_{\mathrm{J}}(\nu_k)}=\sum_{j=0}^{\infty}g_{\mathrm{J}}(\nu_k)^j\theta^{j+1}$. *Here* $(\hat{u}_0,\hat{v}_0)$, ν_k *and* g_{J} *of* $(P_{\mathrm{J}})_m$ $(\mathrm{J}=\mathrm{I},34)$ *have been defined in the previous section respectively. For the explicit forms of* $f_{k,p,l}^{\mathrm{J}}(t)$ $(\mathrm{J}=\mathrm{I},34)$, *see* [2] *and* [17].

The following describes the relation between instanton-type solutions of $(P_{\mathrm{I}})_m$ and $(P_{34})_m$.

Theorem 4.2 ([17]). *An instanton-type solution of $(P_{34})_m$ is transformed algebraically to that of $(P_{\mathrm{I}})_m$ by the replacements of all terms which are depending on their 0-parameter solutions.*

Theorem 4.3 ([16]). *We have instanton-type solutions of $(P_{\mathrm{J}})_m$ (J = II, IV) with free $2m$-parameters $(\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m}[\![\eta^{-1}]\!]$ of the form*

$$U = \hat{u}_0 + (1 - \hat{u}_0 + C)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0 + C)v,$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \le |k| \le m} f_k^{\mathrm{J}}(\tau, t; \eta) A(\nu_k), \quad A(\nu_k) := \begin{pmatrix} a(\nu_k) \\ \varrho_{\mathrm{J}}(\nu_k) a(\nu_k) \end{pmatrix} \tag{4.3}$$

$$f_k^{\mathrm{J}}(\tau, t; \eta) = \sum_{j=1}^{\infty} \Big(\sum_{\substack{\ell \ge 0,\, p \in \mathbb{Z}^m \\ 2\ell + |p| = j}} f_{k,p,\ell}^{\mathrm{J}}(t) e^{p \cdot \tau} \Big) \eta^{-j/2}, \tag{4.4}$$

where $a(\nu_k) = \dfrac{\theta}{1 - \theta g_J(\nu_k)} = \sum_{j=0}^{\infty} g_{\mathrm{J}}(\nu_k)^j \theta^{j+1}$ *and* $\varrho_{\mathrm{J}}(\nu_k) := -\dfrac{\nu_k}{2} + \sqrt{\dfrac{\nu_k^2}{4} + 2\hat{v}_{1,0}}$. *Here $(\hat{u}_0, \hat{v}_0)$, ν_k and g_{J} of $(P_{\mathrm{J}})_m$ (J = II, IV) have been defined in the previous section respectively. For the explicit forms of $f_{k,p,l}^{\mathrm{J}}(t)$ (J = II, IV), see [16].*

The following describes the relations between instanton-type solutions of $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$.

Theorem 4.4 ([16]). *An instanton-type solution of $(P_{\mathrm{II}})_m$ is transformed algebraically to that of $(P_{\mathrm{IV}})_m$ by the replacements of all terms which are depending on their 0-parameter solutions.*

Acknowledgments

The author would like to express her sincere gratitude to Prof. Yasunori Okada, Prof. Naofumi Honda and Prof. Susumu Yamazaki for giving the opportunity to her to take part in the conference.

References

[1] Aoki, T., Multiple-scale analysis for higher-order Painlevé equations, *Algebraic Analysis and the Exact WKB Analysis for Systems of Differential Equations* (T. Aoki, N. Kumano-go and S. Yamazaki, eds), *RIMS Kôkyûroku Bessatsu* **B5**, 2008, pp. 89–98.

[2] Aoki, T., Honda, N. and Umeta, Y., On a construction of general formal solutions for equations of the first Painlevé hierarchy I, *Adv. Math.* **235** (2013), 496–524.

[3] Iwaki, K., Parametric Stokes Phenomenon for the second Painlevé equation with a large parameter, `arXiv:1106.0612v2 [math.CA]`

[4] Kawai, T., Koike, T., Nishikawa, Y. and Takei, Y., On the Stokes geometry of higher order Painlevé equations, *Analyse complexe, Systèmes Dynamiques, Sommabilité des Séries Divergentes et Théories Galoisiennes* (II) — *Volume en l'honneur de Jean-Pierre Ramis* (M. Loday-Richaud, ed.), *Astérisque* **297**, 2004, pp. 117–166.

[5] Kawai, T. and Takei, Y., *Algebraic Analysis of Singular Perturbation Theory*, Transl. Math. Monogr. **227**. Amer. Math. Soc., 2005, Original Japanese edition was published by Iwanami, 1998.

[6] ———, WKB analysis of higher order Painlevé equations with a large parameter — Local reduction of 0-parameter solutions for Painlevé hierarchies (P_J) $(J = \mathrm{I}, \mathrm{II}\text{-}1 \text{ or } \mathrm{II}\text{-}2)$, *Adv. Math.* **203** (2006), 636–672.

[7] ———, WKB analysis of higher order Painlevé equations with a large parameter. II. Structure theorem for instanton-type solutions of $(P_J)_m$ $(J = \mathrm{I}, 34, \mathrm{II}\text{-}2 \text{ or } \mathrm{IV})$ near a simple P-turning point of the first kind, *Publ. Res. Inst. Math. Sci.* **47** (2011), 153–219.

[8] Koike, T., On the Hamiltonian structures of the second and the forth Painlevé hierarchies and degenerate Garnier systems, *Algebraic, Analytic and Geometric Aspects of Complex Differential Equations and their Deformations. Painlevé hierarchies* (Y. Takei, ed.), *RIMS Kôkyûroku Bessatsu* **B2**, 2007, pp. 99–127.

[9] ———, On new expressions of the Painlevé hierarchies, *Algebraic Analysis and the Exact WKB Analysis for Systems of Differential Equations* (T. Aoki, N. Kumano-go and S. Yamazaki, eds), *RIMS Kôkyûroku Bessatsu* **B5**, 2008, pp. 153–198.

[10] Kudryashov, N. A., The first and second Painlevé equations of higher order and some relations between them, *Phys. Lett. A* **224** (1997), 353–360.

[11] Kudryashov, N. A. and Soukharev, M. B., Uniformization and transcendence of solutions for the first and second Painlevé hierarchies, *Phys. Lett. A* **237** (1998), 206–216.

[12] Takei, Y., Singular-perturbative reduction to Birkhoff normal form and instanton-type formal solutions of Hamiltonian systems, *Publ. Res. Inst. Math. Sci.* **34** (1998), 601–627.

[13] ———, An explicit description of the connection formula for the first Painlevé equation, *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto University Press, 2000, pp. 271–296.

[14] ———, Toward the exact WKB analysis for instanton-type solutions of Painlevé hierarchies, *Algebraic, Analytic and Geometric Aspects of Complex Differential Equations and their Deformations. Painlevé hierarchies* (Y. Takei, ed.), *RIMS Kôkyûroku Bessatsu* **B2**, 2007, pp. 247–260.

[15] ———, Instanton-type formal solutions for the first Painlevé hierarchy, *Algebraic Analysis of Differential Equations, Festschrift in Honor of Takahiro Kawai* (T. Aoki, H. Majima, Y. Takei and N. Tose, eds), Springer-Verlag, 2008, pp. 307–319.

[16] Umeta, Y., Construction of instanton-type solutions for the second and the fourth Painlevé hierarchies, in preparation.

[17] ———, Instanton-type solutions of P_{34}-hierarchy with a large parameter, in preparation.