数理解析研究所講究録
第 1861 巻 2013 年 146-155

Exact WKB analysis and multisummability — A case study —

By

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Abstract

The multisummability of WKB solutions of singularly perturbed linear ordinary differential equations is considered. We announce some results on the multisummability of WKB solutions of a concrete example of a perturbed Schrödinger equation and its third-order analogue.

§ 1. Introduction

The one-dimensional Schrödinger equation

$$\left(\frac{d^2}{dz^2}-\eta^2Q(z)\right)\psi(z,\eta)=0 \tag{1.1}$$

with a large parameter η admits formal solutions, often called WKB solutions, of the following form:

$$\widehat{\psi}_{\pm}(z,\eta)=\exp\left(\pm\eta\int^z\sqrt{Q(z)}\,dz\right)\sum_{n=0}^{\infty}\psi_{\pm,n}(z)\eta^{-n}. \tag{1.2}$$

In the exact WKB analysis the WKB solutions (1.2) are endowed with an analytic meaning through the Borel resummation technique with respect to η. Consequently global behavior of solutions of (1.1) (e.g., the monodromy group, Stokes multipliers around irregular singular points, etc.) can be explicitly analyzed by using Borel resummed WKB solutions. (See, for example, [KT].)

However, if some perturbative terms (with respect to η) are added to (1.1) like

$$\left(\frac{d^2}{dz^2}-\eta^2\big(Q_0(z)+\eta^{-1}Q_1(z)+\cdots\big)\right)\psi(z,\eta)=0, \tag{1.3}$$

2010 Mathematics Subject Classification(s): Primary 34M60; Secondary 34E20, 34M30, 40G10.
Key Words: Exact WKB analysis, WKB solution, multisummability.
Supported in part by JSPS grants-in-aid No.21340029.
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then, in general, we need the so-called multisummability, that is, the summability which is more refined than the Borel summability, to give an analytic meaning to WKB solutions

$$(1.4) \qquad \widehat{\psi}_\pm(z,\eta) = \exp\left(\pm\eta\int^z\sqrt{Q_0(z)}\,dz\right)\sum_{n=0}^{\infty}\psi_{\pm,n}(z)\eta^{-n}$$

of (1.3). The purpose of this note is to show this fact by considering some concrete examples.

Recently R. Schäfke ([Sc]) showed that the following first-order inhomogeneous ordinary differential equation

$$(1.5) \qquad \left(\epsilon\frac{d}{dz} - \left(z-\epsilon z^2\right)\right)\psi(z,\epsilon) = \epsilon^2,$$

where ϵ is a small parameter (i.e., $\epsilon = \eta^{-1}$), has a formal solution which is $(3,1)$-multisummable. Inspired by this result and discussions with him, we consider the following equation

$$(1.6) \qquad \left(\frac{d^2}{dz^2} - \eta^2\left(z-\eta^{-2}z^2\right)\right)\psi(z,\eta) = 0$$

and its third-order analogue in this note. Our main results (Theorems 3.1 and 3.2) claim that WKB solutions of these equations are also multisummable (with some appropriate indices). In this note we only announce the main results; their detailed proofs will be published elsewhere.

The plan of this note is as follows: First, following the lecture note of Balser [B], we review the definition of the multisummability in Section 2. Then in Section 3 we introduce concrete examples of differential equations to be considered and state our main results on the multisummability of their WKB solutions. In Section 4 we explain the core part of the proof of the main results. Finally in Section 5, we discuss the structure of the Borel transform of WKB solutions in question.

§ 2. Borel summability, k-summability and multisummability

In this section, following [B], we review the definition of the multisummability and some fundamental properties for it. We basically employ the same notation as [B] and use a small parameter $\epsilon = \eta^{-1}$ instead of a large parameter η as an asymptotic parameter in this section.

First, let us recall the definition of the k-summability.

Definition 2.1 (*k-summability*). Let $k > 0$ be a positive real number and $\widehat{f} = \sum_n f_n\epsilon^n$ be a formal power series of a small parameter ϵ. Then $\widehat{f}$ is said to be k-summable in the direction d if and only if $\mathcal{L}_k^d\widehat{\mathcal{B}}_k\widehat{f}$ is well-defined.

Here $\widehat{\mathcal{B}}_k \widehat{f}$ denotes the formal Borel transform with index k (or "formal k-Borel transform" for short) of $\widehat{f}$:

$$\left(\widehat{\mathcal{B}}_k \widehat{f}\right)(y) := \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+n/k)} y^n, \tag{2.1}$$

and $\mathcal{L}_k^d g$ denotes the Laplace transform with index k (or "k-Laplace transform" for short) of g in the direction d:

$$\left(\mathcal{L}_k^d g\right)(\epsilon) := \epsilon^{-k} \int_0^{\infty e^{id}} \exp\left(-\left(\frac{y}{\epsilon}\right)^k\right) g(y)\, d(y^k), \tag{2.2}$$

where the integration from 0 to ∞ is done along $\arg y = d$.

Note that the 1-summability exactly coincides with the Borel summability.

It is well-known that the k-summability of $\widehat{f}$ is equivalent to the existence of an analytic function whose Gevrey asymptotic expansion of order k is given by $\widehat{f}$ in a sector with sufficiently large opening. To be more specific, $\widehat{f}$ is k-summable in the direction d if and only if there exists an analytic function $f(\epsilon)$ in a sector S with bisecting direction d and opening larger than π/k such that the asymptotic expansion of Gevrey order k of $f(\epsilon)$ is given by $\widehat{f}$:

$$f(\epsilon) \cong_k \widehat{f} = \sum_{n=0}^{\infty} f_n \epsilon^n \quad \text{as } \epsilon \to 0 \quad \text{in } S, \tag{2.3}$$

that is, for every closed subsector $\overline{S_1}$ of S and every non-negative integer N

$$\left| f(\epsilon) - \sum_{n=0}^{N-1} f_n \epsilon^n \right| \leq C K^N \Gamma(1+N/k) \tag{2.4}$$

holds in $\epsilon \in \overline{S_1}$ with positive constants $C, K > 0$ independent of N.

In some cases, to define the summability of a given formal power series, we need to consider the k_j-summability with several different indices k_j simultaneously. Roughly speaking, the multisummability deals with such situations. (A typical example is a formal solution near an irregular singular point of a higher-order ordinary differential equation.) The precise definition of the multisummability is given as follows:

Definition 2.2 (*multisummability*). Let $k = (k_1, \ldots, k_q)$ be a q-tuple of positive real numbers $\{k_j\}$ $(1 \leq j \leq q)$ satisfying $k_1 > k_2 > \cdots > k_q > 0$ and $\widehat{f} = \sum_n f_n \epsilon^n$ be a formal power series of a small parameter ϵ. Then $\widehat{f}$ is said to be k-multisummable in the direction d if and only if the following functions $\{f_j\}$ $(0 \leq j \leq q)$ are successively well-defined:

$$(2.5)\qquad \begin{aligned} f_q &:= \widehat{\mathcal{B}}_{k_q}\widehat{f}, \\ f_{q-1} &:= \mathcal{A}^d_{k_{q-1},k_q} f_{k_q}, \\ &\cdots \\ f_1 &:= \mathcal{A}^d_{k_1,k_2} f_2, \\ f_0 &:= \mathcal{L}^d_{k_1} f_1. \end{aligned}$$

Here $\mathcal{A}^d_{\tilde{k},k} = \mathcal{B}_{\tilde{k}} \circ \mathcal{L}^d_k$ denotes the acceleration operator introduced by Ecalle, that is,

$$(2.6)\qquad \left(\mathcal{A}^d_{\tilde{k},k} g\right)(\epsilon) := \epsilon^{-k} \int_0^{\infty e^{id}} C_{\tilde{k}/k}\left(\left(\frac{y}{\epsilon}\right)^k\right) g(y)\, d(y^k),$$

where the integration is done along $\arg y = d$ from 0 to ∞ and the kernel function $C_\alpha(z)$ $(\alpha > 1)$ is given as follows:

$$(2.7)\qquad C_\alpha(z) := \frac{1}{2\pi i}\int_\gamma u^{1/\alpha - 1}\exp\left(u - z u^{1/\alpha}\right) du,$$

where γ is a path going from $-\infty$ to $-\delta$ $(\delta > 0)$ along the negative real axis, encircling the origin anti-clockwise once, and returning to $-\infty$ again along the negative real axis. When $\widehat{f}$ is k-summable, the function f_0 defined by (2.5) is called *the k-sum of $\widehat{f}$*.

The multisummability is usually defined in the multidirection $d = (d_1, \ldots, d_q)$, that is, in defining the function f_j in (2.5), we use different directions at each level (i.e., $f_{j-1} = \mathcal{A}^{d_j}_{k_{j-1},k_j} f_{k_j}$ for $2 \le j \le q$ and $f_0 = \mathcal{L}^{d_1}_{k_1} f_1$). In this paper, however, we only consider the multisummability in a fixed single direction d for the sake of simplicity.

The following proposition clearly shows that the multisummability of $\widehat{f}$ means the necessity of considering the k_j-summability with several different indices k_j simultaneously.

Proposition 2.3 ([B, §6.2 and §6.3]). *Suppose $k_q > 1/2$. Then a formal power series $\widehat{f}$ is $(k_1, \ldots, k_q)$-multisummable in the direction d if and only if $\widehat{f}$ can be decomposed into the sum of k_j-summable series $\widehat{f}_j$ in the direction d, that is,*

$$(2.8)\qquad \widehat{f} = \sum_{j=1}^{q} \widehat{f}_j \quad \textit{where } \widehat{f}_j\textit{: } k_j\textit{-summable in } d.$$

§ 3. Main results

From now on we discuss the multisummability of WKB solutions of some concrete examples of singularly perturbed linear ordinary differential equations.

First, let us consider the following perturbed Schrödinger equation:

$$\left(\frac{d^2}{dz^2} - \eta^2\left(z - \eta^{-2}z^2\right)\right)\psi(z,\eta) = 0. \tag{3.1}$$

If we ignore the term $\eta^{-2}z^2$, Equation (3.1) becomes the Airy equation. Otherwise stated, (3.1) is a perturbation of the Airy equation. On the other hand, by the scaling

$$z = \eta^2 x, \tag{3.2}$$

(3.1) is transformed into

$$\left(\frac{d^2}{dx^2} - (\eta^4)^2(x - x^2)\right)\psi = 0, \tag{3.3}$$

which is nothing but the Weber equation. Making use of the well-known fact that the Weber equation has an integral representation of solutions, we then find that (3.1) also has the following integral representation of solutions:

$$\psi(z,\eta) = \int \exp\left(-\eta^4 g(t; z, \eta)\right) t^{-1/2}\, dt, \tag{3.4}$$

where the phase function $g(t; z, \eta)$ is given by

$$g(t; z, \eta) = \frac{i}{8}\left(2t^2 - 4t(1 - 2\eta^{-2}z) + \log t + (1 - 2\eta^{-2}z)^2\right). \tag{3.5}$$

Let $t = t_\pm$ be a saddle point of $g(t; z, \eta)$, that is, $t = t_\pm$ are zeros of

$$\frac{\partial g}{\partial t} = \frac{i}{8}\left(4t - 4(1 - 2\eta^{-2}z) + \frac{1}{t}\right), \tag{3.6}$$

more explicitly,

$$t_\pm = \frac{(1 - 2\eta^{-2}z) \pm \sqrt{(1 - 2\eta^{-2}z)^2 - 1}}{2}. \tag{3.7}$$

Note that

$$t_+ - t_- = O(\eta^{-1}), \quad g(t_+; z, \eta) - g(t_-; z, \eta) = O(\eta^{-3}). \tag{3.8}$$

Let $\Gamma_\pm$ be a steepest descent path of $\Re(-\eta^4 g)$ passing through the saddle point $t_\pm$, respectively, and let $\psi_\pm(z,\eta)$ denote a solution of (3.1) defined by

$$\psi_\pm(z,\eta) = \int_{\Gamma_\pm} \exp\left(-\eta^4 g(t; z, \eta)\right) t^{-1/2}\, dt, \tag{3.9}$$

Then, by considering the asymptotic expansion of $\psi_\pm(z,\eta)$ with respect to η (for fixed z), we obtain a (suitably normalized) WKB solution $\widehat{\psi}_\pm(z,\eta)$ of (3.1):

$$\psi_\pm(z,\eta) \cong \widehat{\psi}_\pm(z,\eta) = \exp\left(\pm\eta\int^z \sqrt{z}\, dz\right)\sum_{n=0}^{\infty}\psi_{\pm,n}(z)\eta^{-n}. \tag{3.10}$$

We denote the formal power series part of $\widehat{\psi}_\pm(z,\eta)$ by $\widehat{\varphi}_\pm(z,\eta)$, that is,

$$\widehat{\psi}_\pm(z,\eta) = \exp\left(\pm\eta \int^z \sqrt{z}\,dz\right) \widehat{\varphi}_\pm(z,\eta). \tag{3.11}$$

Our first main result is the following:

Theorem 3.1. *The formal power series part $\widehat{\varphi}_\pm(z,\eta)$ of the WKB solution $\widehat{\psi}_\pm(z,\eta)$ of (3.1) is $(4,1)$-multisummable with respect to η. To be more precise, for each fixed z $\widehat{\varphi}_\pm(z,\eta)$ is $(4,1)$-multisummable with respect to η (or η^{-1}) except for a finite number of singular directions.*

As a second example, let us next consider the following third-order differential equation:

$$\left(\frac{d^3}{dz^3} + (z\eta^{-3})\eta\frac{d^2}{dz^2} + (3+2z\eta^{-1})\eta^2\frac{d}{dz} + 2i(z+1)\eta^3\right)\psi(z,\eta) = 0. \tag{3.12}$$

Similarly to (3.1), as Equation (3.12) is the so-called Laplace type equation, (3.12) also has the following integral representation of solutions:

$$\psi(z,\eta) = \int \exp\left(-\eta^8 h(t;z,\eta)\right)\,dt, \tag{3.13}$$

where

$$h(t;z,\eta) = tz\eta^{-5} - \int^t \frac{u^3 + (3\eta^{-4} - 2\eta^{-8})u - 2i\eta^{-6} + 2\eta^{-8}}{u^2 - 2u + 2i\eta^{-1}}\,du. \tag{3.14}$$

In this case the phase function $h(t;z,\eta)$ has three saddle points, which are denoted by $t = t_j$ $(j = 0,1,2)$. Let $\psi_j(z,\eta)$ $(j=0,1,2)$ be a solution of (3.12) defined by

$$\psi_j(z,\eta) = \int_{\Gamma_j} \exp\left(-\eta^8 h(t;z,\eta)\right)\,dt, \tag{3.15}$$

where Γ_j $(j=0,1,2)$ is a steepest descent path of $\Re(-\eta^8 h)$ passing through the saddle point $t = t_j$. Then, in parallel to the above discussion for (3.1), by considering the asymptotic expansion of $\psi_j(z,\eta)$ with respect to η, we obtain a WKB solution $\widehat{\psi}_j(z,\eta)$ of (3.12):

$$\psi_j(z,\eta) \cong \widehat{\psi}_j(z,\eta) = \exp\left(\pm\eta\int^z \zeta_j(z)\,dz\right)\sum_{n=0}^{\infty}\psi_{j,n}(z)\eta^{-n}, \tag{3.16}$$

where $\zeta_j(z)$ $(j=0,1,2)$ is a root of the cubic equation

$$\zeta^3 + 3\zeta + 2i(z+1) = 0. \tag{3.17}$$

Let $\widehat{\varphi}_j(z,\eta)$ denote the formal power series part of $\widehat{\psi}_j(z,\eta)$:

$$\widehat{\psi}_j(z,\eta) = \exp\left(\pm\eta\int^z \zeta_j(z)\,dz\right)\widehat{\varphi}_j(z,\eta). \tag{3.18}$$

Our second main result is then the following:

Theorem 3.2. *The formal power series part $\widehat{\varphi}_j(z,\eta)$ $(j=0,1,2)$ of the WKB solution $\widehat{\psi}_j(z,\eta)$ of* (3.12) *is* $(8,5,1)$*-multisummable with respect to η.*

Theorem 3.2 shows that, in addition to the index 1, two other different indices 8 and 5 appear in the description of the multisummability of WKB solutions of (3.12). Roughly speaking, this is a consequence of the fact that (3.12) admits the following two different scalings: Firstly, by the scaling $z=\eta^3x_1$ (3.12) is transformed into

$$\left(\frac{d^3}{dx_1^3}+(x_1\eta^{-1})\eta^5\frac{d^2}{dx_1^2}+(3\eta^{-2}+2x_1)\eta^{10}\frac{d}{dx_1}+2i(x_1+\eta^{-3})\eta^{15}\right)\psi=0 \tag{3.19}$$

and, secondly, by the scaling $z=\eta^5x_2$ (3.12) is transformed into

$$\left(\frac{d^3}{dx_2^3}+x_2\eta^8\frac{d^2}{dx_2^2}+(3\eta^{-4}+2x_2)\eta^{16}\frac{d}{dx_2}+2i(x_2\eta^{-1}+\eta^{-6})\eta^{24}\right)\psi=0. \tag{3.20}$$

§ 4. A sketch of the proof of the main results

In this section we explain the core part of the proof of the main results.

Since Theorem 3.2 is proved in a manner similar to Theorem 3.1, we only consider Theorem 3.1, that is, we only discuss (3.1). In the proof of Theorem 3.1, i.e., in the study of the multisummability of its WKB solutions $\widehat{\psi}_\pm(z,\eta)$ (or its formal power series part $\widehat{\varphi}_\pm(z,\eta)$), the most important step is to investigate what kind of Stokes phenomena occurs with $\widehat{\psi}_\pm(z,\eta)$ when $\arg\eta$ varies from 0 to 2π for fixed z. In view of the integral representation (3.9) of the analytic realization $\psi_\pm(z,\eta)$ of $\widehat{\psi}_\pm(z,\eta)$, we find that this can be explicitly done by analyzing the change of the configuration of the steepest descent paths $\Gamma_\pm$ when $\arg\eta$ varies from 0 to 2π. For example, for $z=1+i$ we can confirm that the following two different types of Stokes phenomena occur with the formal power series part $\widehat{\varphi}_-(z,\eta)$ of $\widehat{\psi}_-(z,\eta)$:

Proposition 4.1. *Let $z=1+i$ be fixed. Then, when* $\arg\eta$ *varies from* 0 *to* 2π*, the following two types of Stokes phenomena occur with $\widehat{\varphi}_-(z,\eta)$.*

(type A)

$$\varphi_-(z,\eta)-\widetilde{\varphi}_-(z,\eta)=O(\exp(-c\eta^4)), \tag{4.1}$$

where $\varphi_-(z,\eta)$ and $\widetilde{\varphi}_-(z,\eta)$ denote the analytic realizations of $\widehat{\varphi}_-(z,\eta)$ in neighboring two sectors, respectively, and c is a constant. This type of Stokes phenomena occurs at $\arg\eta = k\pi/4$ *with* $k = 0, 1, \ldots, 5$.

(type B)

$$\varphi_-(z,\eta) - \widetilde{\varphi}_-(z,\eta) = O(\exp(-c\eta)). \tag{4.2}$$

This type of Stokes phenomena occurs only at $\arg\eta = 5\pi/8$.

That is, the Stokes phenomenon of type A is that of exponential order 4 and the Stokes phenomenon of type B is that of exponential order 1. The indices 4 and 1 of the multisummability of $\widehat{\varphi}_-(z,\eta)$ described in Theorem 3.1 exactly corresponds to these exponential orders of the Stokes phenomena for $\widehat{\varphi}_-(z,\eta)$.

The proof of Theorem 3.1 is completed by combining Proposition 4.1 with an argument typical to the asymptotic analysis, i.e., a reasoning based on the use of the Cauchy-Heine transform. In [Su] the proof of Theorem 3.1 is given along this line when $z = 1 + i$. The complete proof of Theorems 3.1 and 3.2 will be provided elsewhere.

In the subsequent section, instead of giving the proof of the main results, we discuss the structure of the Borel transform of the WKB solutions $\widehat{\psi}_\pm(z,\eta)$ of (3.1) by using the integral representation (3.9).

§ 5. Structure of the Borel transform of WKB solutions

In the case of (3.1) the analytic realization $\psi_\pm(z,\eta)$ of the WKB solutions $\widehat{\psi}_\pm(z,\eta)$ has an integral representation (3.9), i.e.,

$$\psi_\pm(z,\eta) = \int_{\Gamma_\pm} \exp\left(-\eta^4 g(t;z,\eta)\right) t^{-1/2}\, dt, \tag{5.1}$$

where $g(t;z,\eta)$ is given by (3.5). On the other hand, as noted in Section 3, (3.1) is transformed into (3.3) by the scaling $z = \eta^2 x$. Corresponding to this scaling, we have another expression of the integral representation (3.9), that is, if we employ a change of integration variable

$$t = i\eta^{-1}s + 1/2, \tag{5.2}$$

(3.9) can be written also as

$$\psi_\pm(z,\eta) = \int_{\tilde{\Gamma}_\pm} \exp\left(-\eta f(s;z,\eta)\right) ds, \tag{5.3}$$

where

$$f(s;z,\eta) = \frac{1}{3}s^3 - zs - i\eta^{-1}\int^s \frac{2u^3 - \eta^{-1}}{1 + 2i\eta^{-1}u}\, du. \tag{5.4}$$

As discussed below, we expect that these two expressions of the integral representation may enable us to analyze the structure of Borel transforms of the WKB solutions $\widehat{\psi}_\pm(z,\eta)$ explicitly through an argument similar to the discussion employed in [T].

In what follows we omit the suffix $\pm$ and do not specify the path of integration for the sake of simplicity. First, using a change of integration variable

$$y = y(s;z) := \frac{1}{3}s^3 - zs, \tag{5.5}$$

we rewrite (5.3) as

$$\psi = \int \exp(-\eta y)\chi(y;z)\,dy \tag{5.6}$$

with

$$\chi(y;z) = \left[\exp\left(i\int^s \frac{2u^3-\eta^{-1}}{1+2i\eta^{-1}u}\,du\right)\frac{1}{\partial y/\partial s}\right]\Bigg|_{s=s(y;z)}. \tag{5.7}$$

Here $s = s(y;z)$ denotes the inverse function of $y = y(s;z)$ given by (5.5). Then, if higher order terms of $\chi(y;z)$ with respect to η^{-1} can be interpreted in an appropriate manner, $\chi(y;z)$ is considered to be the 1-Borel transform of the WKB solution $\widehat{\psi}$:

$$\widehat{\mathcal{B}}_1\widehat{\psi} = \chi(y;z) = \left[\exp\left(i\int^s \frac{2u^3-\eta^{-1}}{1+2i\eta^{-1}u}\,du\right)\frac{1}{\partial y/\partial s}\right]\Bigg|_{s=s(y;z)}. \tag{5.8}$$

Similarly, a change of integration variable

$$w = w(t;z) := g(t;z,\eta) \tag{5.9}$$

in (5.1) leads to

$$\psi = \int \exp(-\eta^4 w)\tilde{\chi}(w;z)\,dw, \tag{5.10}$$

where

$$\tilde{\chi}(w;z) = \left[t^{-1/2}\frac{1}{\partial w/\partial t}\right]\Bigg|_{t=t(w;z)} \tag{5.11}$$

with $t = t(w;z)$ being the inverse function of $w = w(t;z)$. Then $\tilde{\chi}(w;z)$ is also considered to describe the 4-Borel transform of $\widehat{\psi}$ or, to be more precise, the image of $\widehat{\mathcal{B}}_1\widehat{\psi}$ through the acceleration operator $\mathcal{A}_{4,1}$:

$$\mathcal{A}_{4,1}\left(\widehat{\mathcal{B}}_1\widehat{\psi}\right) = \tilde{\chi}(w;z) = \left[t^{-1/2}\frac{1}{\partial w/\partial t}\right]\Bigg|_{t=t(w;z)}. \tag{5.12}$$

It is expected that several properties of the Borel transforms of $\widehat{\psi}$ can be derived from these expressions (5.8) and (5.12). For example, the analysis of the top order part of (5.8) and (5.12) with respect to η^{-1} suggests that the following properties should hold for the 1-Borel transform $\widehat{\mathcal{B}}_1\widehat{\psi}$ and the 4-Borel transform $\mathcal{A}_{4,1}(\widehat{\mathcal{B}}_1\widehat{\psi})$:

(5.13) $(\widehat{\mathcal{B}}_1\widehat{\psi})(y;z)$ has singularities at $y = \mp(2/3)z^{3/2}$ (after a suitable translation in y-variable),

(5.14) $(\widehat{\mathcal{B}}_1\widehat{\psi})(y;z) \neq O\left(e^{c|y|}\right)$ (c : const) as $y \to \infty$,

(5.15) $(\mathcal{A}_{4,1}(\widehat{\mathcal{B}}_1\widehat{\psi}))(w;z)$ has singularities at $w = 2m\pi i$ ($m \in \mathbb{Z}$) (after a suitable translation in w-variable),

(5.16) $(\mathcal{A}_{4,1}(\widehat{\mathcal{B}}_1\widehat{\psi}))(w;z) = O\left(e^{c|w|}\right)$ (c : const) as $w \to \infty$.

Note that the singularities $y = \mp(2/3)z^{3/2}$ of $\widehat{\mathcal{B}}_1\widehat{\psi}$ come from zeros of $\partial y/\partial s = s^2 - z$ and that the periodic singularities $w = 2m\pi i$ ($m \in \mathbb{Z}$) of $\mathcal{A}_{4,1}(\widehat{\mathcal{B}}_1\widehat{\psi})$ originate from the term of $\log t$ in $g(t;z,\eta)$. The former singularities $y = \mp(2/3)z^{3/2}$ (resp., the latter singularities $w = 2m\pi i$) correspond to the so-called movable singularities (resp., fixed singularities) of the Borel transform of $\widehat{\psi}$. The Stokes phenomena of type A and type B discussed in the preceding section are induced by these singularities $w = 2m\pi i$ of $\mathcal{A}_{4,1}(\widehat{\mathcal{B}}_1\widehat{\psi})$ and $y = \mp(2/3)z^{3/2}$ of $\widehat{\mathcal{B}}_1\widehat{\psi}$, respectively. Also, the properties (5.14) and (5.16) for the exponential growth of the Borel transforms clearly explain why we need not only 1-summability but (4,1)-multisummability for the WKB solution $\widehat{\psi}$ of (3.1).

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