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Generalised Lichnerowicz lemma, black hole uniqueness and positive mass theorem

Tetsuya Shiromizu, *Department of Physics, Kyoto University*

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Abstract

In stationary/static spacetimes, the positive mass theorem(PMT) implies us the strong restriction on the spacetime configurations. The famous one is the Lichnerowicz lemma/theorem in 1955: contractible stationary vacuum spacetime manifolds are static. Since the vacuum spacetime has the zero mass, PMT tells us that the spacetime is Minkowski spacetime. But, we are often interested in non-vacuum cases. For such cases, the spacetime may have the non-trivial mass. However, we can show that the mass vanishes for some cases and then spacetime is Minkowski spacetime. this means that the non-trivial stationary configration of matters are not permitted(no-go!).

PMT also gives us powerful tool to show the uniqueness of static black hole spacetimes. This was done by Bunting and Masood-ul-Alam in 1987 for vacuum black holes. Now the main topics on fundamental problems are changed to be about black holes in higher dimensional black holes or string theory set-up. I will review the recent development on the uniqueness/classilication of higher dimensional black holes.

1 Introduction

The positive mass theorem(PMT) guarantees the classical stability of spacetimes. The Arnowitt-Deser-Misner(ADM) mass is shown to be non-negative [1]. Most striking fact is the fact that the ADM mass is zero if and only if the spacetime is the Minkowski spacetime. This property implies a stringent constraint on some spacetimes. In this report, we will discuss the constraints on the final fate of the spacetimes if they are.

As a final fate, we would expect that the spacetime will settle down to the stationary states. There are several possibilities. One may want to classify three cases, (i) strictly stationary spacetimes, (ii) stationary black hole spacetimes and (iii) others which may contain a naked singularity or be dynamical forever. By "strictly stationary" we mean that there are a timelike Killing vector whole spacetime and no horizons.

The rest of this report is organised as follows. In Sec. 2, we briefly review the positive mass theorem which will be powerful tool here. In Sec 3. we will discuss the strictly stationary spacetimes. In Sec. 4, we will focus on the static black holes and show the uniqueness. Finally we will give a short summary and discuss future issues.

2 Summary of notations

Let us summarise the notation adopted here. The spacetime metric is Lorentzian and expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{1}$$

The signature is $(-,+,+,\cdots,+)$. The Greek indeces μ,ν run over $0,1,2,...,n-1$. Here we suppose that the dimension of spacetime is n. 0 stands for the time component. The Latin indeces i,j appeard later soon indicate the spatial components.

We denote the covariant derivative with respect to $g_{\mu\nu}$ by ∇_μ. For example,

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha}X^\alpha, \tag{2}$$

where $\Gamma^\mu_{\alpha\beta}$ is the affine connection

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}). \tag{3}$$

The Riemann tensor is defined by

$$R_{\mu\nu\alpha}{}^\beta X_\beta = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)X_\alpha \tag{4}$$

and then we see that it is written in terms of the affine connection as

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha\Gamma^\mu_{\nu\beta} - \partial_\beta\Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\rho\alpha}\Gamma^\rho_{\nu\beta} - \Gamma^\mu_{\rho\beta}\Gamma^\rho_{\nu\alpha}. \tag{5}$$

The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{6}$$

where $R_{\mu\nu} = g^{\alpha\beta}R_{\mu\alpha\nu\beta}$, $R = g^{\mu\nu}R_{\mu\nu}$ and $T_{\mu\nu}$ is the energy-momentum tensor.

3 Positive mass theorem

Firstly we review the positive mass theorem [1]. In asymptotically flat spacetimes, we can naturally define the conserved mass at spatial infinity. This is so called the ADM(Arnowitt-Deser-Misner) mass. The spatial metric g_{ij} behaves like

$$g_{ij} = \delta_{ij}\Big(1 + \frac{2}{n-3}\frac{M}{r^{n-3}}\Big) + O(1/r^{n-2}), \tag{7}$$

where M is the ADM mass and n is the dimension of spacetimes. If $4 \le n \le 8$(or spacetime manifold with $n \ge 4$ is spin) and ${}^{(n-1)}R \ge 0$, the ADM mass is non-negative and the $(n-1)$-dimensional spacelike hypersurface Σ is Euclid space iff $M=0$. This is the Riemannian positive mass theorem. ${}^{(n-1)}R$ is the Ricci scalar of Σ. The condition on ${}^{(n-1)}R \geq 0$ corresponds to the non-negativity of energy density. This is because we see from the Hamiltonian constraint ${}^{(n-1)}R - K_{ij}K^{ij} + (K^i_i)^2 = 16\pi\rho$ that $\rho \geq 0$ implies ${}^{(n-1)}R \geq 0$ on maximal hypersurfaces of $K^i_i = 0$.

We also have the final version of the positive mass theorem, that is, if $4 \leq n \leq 8$(or spacetime manifold with $n \geq 4$ is spin), the Einstein field equation holds and the dominant energy condition is satisfied, the ADM mass is non-negative and the spacetime is the Minkowski spacetime iff the ADM mass vanishes. The dominant energy condition requires that $-T^{\mu}_{\ \nu}t^{\nu}$ is future directed causal vector for future directed timelike vector t^{ν}.

This theorem guarantees the classical stability of spacetime and that the ground state is the Minkowski spacetime. This statement gives us strong restriction on the stationary spacetimes. From now on we will see this.

4 Strictly stationaly spacetimes

One may be interested in the possible configurations of stationary spacetimes. Since the Einstein equation is non-linear, this is non-trivial issue. But, the positive mass theorem tells us that the striclty stationary and vacuum spacetimes should be the Minkowski spacetime [2]. We can also extend this result to the cases with gauge fields like the Maxwell field or anti-symmetric tensor(p-form fields) which often appears as a foundamental fields in string theory [3].

4.1 Vacuum cases

Let us consider the vacuum case first. The Einstein equation is

$$R_{\mu\nu} = 0. \tag{8}$$

For the stationary spacetimes, the ADM mass is written as

$$\begin{aligned} M &= -\frac{1}{8\pi}\int_{S_\infty} \nabla_\mu k_\nu dS^{\mu\nu} \\ &= \frac{1}{4\pi}\int_\Sigma R_{\mu\nu}k^\mu t^\nu d\Sigma \\ &= 2\int_\Sigma \Bigl(T_{\mu\nu} - \frac{1}{n-2}g_{\mu\nu}T\Bigr)k^\mu t^\nu d\Sigma, \end{aligned} \tag{9}$$

where k^μ is the timelike Killing vector and t^μ is the future directed unit normal vector of the spacelike hypersurface Σ and $dS^{\mu\nu}$ is the surface element. This is the Komar formula [4]. From the first to second line, we used the fact that the spacetime is strictly stationary. If not, there is an additional term from the event horizon. In the current case, it does not exist.

Since we consider the vacuum cases, the ADM mass vanishes. Note that the energy condition is satisfied. Then we can apply the positive mass theorem to the current system and then we realise that the spacetime should be the Minkowski spacetime. Therefore, a non-trivial configuration of the vacuum spacetime like Geon [5] should be dynamical if it is. Historically this result was obtained from the Lichnerowicz lemma which shows that the stationarity implies the staticity of the spacetime. It is easy to see that the strictly static spacetimes are the Minkowski spacetime.

4.2 With Maxwell and complex scalar fields

One may wonder if a non-trivial solution is in the Einstein-Maxwell system. In this subsection, we focus on the four dimensional cases($n = 4$). The Einstein equation is

$$R_{\mu\nu} = F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F^2 + \partial_\mu\pi\partial_\nu\pi^* + \partial_\mu\pi^*\partial_\nu\pi. \tag{10}$$

In the above we note that the Maxwell field does not couple with the complex scalar field.

We define V^2 as the norm of the timelike Killing vector k^μ, that is, $V^2 = -k^\mu k_\mu$. We assume that all fields are also stationary, $\pounds_k F_{\mu\nu} = \pounds_k \pi = 0$. In this set-up, we cannot show that the ADM mass vanishes using the Komar mass formula.

Let us define the twist vector ω^μ by

$$\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}k^\nu\nabla^\alpha k^\beta. \tag{11}$$

The field strength of the Maxwell field is decomposed into the electric and magnetic parts as usual

$$V^2 F_{\mu\nu} = -2k_{[\mu}E_{\nu]} + \epsilon_{\mu\nu\alpha\beta}k^\alpha B^\beta. \tag{12}$$

The source-free Maxwell equations are

$$dE = dB = 0, \tag{13}$$

$$\nabla_\mu(E^\mu V^{-2}) - 2\omega_\mu B^\mu V^{-4} = 0 \tag{14}$$

and

$$\nabla_\mu(B^\mu V^{-2}) + 2\omega_\mu E^\mu V^{-4} = 0. \tag{15}$$

From the definition of ω_μ, we see

$$\nabla_\mu(\omega^\mu V^{-4}) = 0 \tag{16}$$

holds regardless of the field equations.

The Einstein equation gives us

$$\frac{2}{V^2}R_{\mu\nu}k^\mu k^\nu = \nabla_\mu\Big(\frac{\nabla^\mu V^2}{V^2}\Big) + \frac{\omega^\mu\omega_\mu}{V^4} = \frac{E_\mu E^\mu + B_\mu B^\mu}{V^2} \tag{17}$$

and

$$d\omega = E \wedge B. \tag{18}$$

If the manifold is contractible, Eq. (13) implies that the electro and magnetic fields have the potential as

$$E = d\Phi \quad \text{and} \quad B = d\Psi. \tag{19}$$

Using the potentials, Eq. (18) is written as

$$d(\omega - \Psi E) = 0 \quad \text{or} \quad d(\omega + \Phi B) = 0. \tag{20}$$

Therefore, there are the functions U_E and U_B such that

$$\omega - \Psi E = dU_E \quad \text{and} \quad \omega + \Phi B = dU_B. \tag{21}$$

Using the remaining Maxwell equations and the aboves, we show

$$\nabla_\mu \Big(U_E \frac{\omega^\mu}{V^4} - \frac{\Psi B^\mu}{2V^2} \Big) = \frac{\omega_\mu \omega^\mu}{V^4} - \frac{B_\mu B^\mu}{2V^2} \tag{22}$$

and

$$\nabla_\mu \Big(U_B \frac{\omega^\mu}{V^4} + \frac{\Phi E^\mu}{2V^2} \Big) = \frac{\omega_\mu \omega^\mu}{V^4} - \frac{E_\mu E^\mu}{2V^2}. \tag{23}$$

Together with Eq. (17), the aboves lead us the divergence-free identity

$$\nabla_\mu \Big(\frac{\nabla^\mu V^2}{V^2} + W^\mu \Big), \tag{24}$$

where

$$W^\mu = 2(U_E + U_B) \frac{\omega^\mu}{V^4} - \frac{\Psi B^\mu + \Phi E^\mu}{V^2}. \tag{25}$$

Then its volume integral tells us that the ADM mass vanishes, because it is rewritten in the surface integral and W^μ does not contribute to it.

Since the energy condition is satisfied in the Einstein-Maxwell-complex scalar fields, the positive mass theorem holds. Then we see that the spacetime should be the Minkowski spacetime.

We can extend the current argument for the Maxwell field to the cases with p-form fields in higher dimensions. In the same way, we can show that the ADM mass vanishes and see that the energy condition is satisfied. Therefore, the spacetime is the Minkowski spacetime again. We call these statements as the generalised Lichnerowicz lemma.

The key point here is to have the divergence-free identitiy which show us the vanishing of the ADM mass.

5 Static black hole spacetimes

When gravitational collapse occurs, we expect that the black hole forms if matter will be concentrated in a compact space. After the black hole formation, the spacetime will settle down to a stationary state due to the emission of the gravitational wave and so on. In four dimensions, we know that the final stationary state of the black holes is unique to be the Kerr solution [6]. One may be also interested in the higher dimensional black holes inspired by superstring theory. Interestingly, we realised that the conventional uniqueness of the stationary black holes does not hold in higher dimensions [7]. Event if one specifies the ADM mass and the angular momentum, the spacetimes are not unique and there are several different stationary spacetimes with the same mass and angular momentum. Indeed, the exaxt solutions have been discovered. As a typical example, black ring solutions are [8]. Although we have seen such complehensive/complex structure of the higher dimensional stationary black holes, we know that the static black holes are unique to be the higher dimensional Schwarzschild solution [9, 10]. In this report, we will review this.

Before the details, we will give a comment on the stationary and higher dimensional black holes briefly. Although we cannot show the uniqueness, we can prove that the stationanrity implies the axisymmetricity [7]. In four dimensions, the system can be reduced to two dimensions by virtue of the symmetry and then we can show the uniqueness. But, in higher dimensions, we cannot do. However, if one specfies the rod structure, which determines the locations of the event horizon and rotational axies as well as the asymptotic conditions, we can show that the solution is unique in five dimensions. Note that we do not know the relation between the rod structure and the observational quantities at distant observer.

5.1 Static black hole uniqueness

We consider the static black hole spacetimes. The staticity of spacetime guarantees that the metric can be written as

$$ds^2 = -V^2(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j. \tag{26}$$

In the static spacetime, the timelike Killing vector $k = \partial_t$ is hypsersufrace orthogonal and we can choose the metric component which does not depend on the time coordinate t. In this coordinate, the event horizon(the boundary of black hole) is located at $V = 0$.

We will give the sketch of the proof. There are two steps. The first step has been developed by Bunting and Masood-ul-Alam [11]. We first introduce the conformal transformation $\tilde{g}^+_{ij} = \Omega^2_+ g_{ij}$ such that the ADM mass vanishes and the Ricci scalar of $\tilde{g}^+_{ij}$ is non-negative. Then we apply the positive mass theorem for the conformally transformed space $\tilde{\Sigma}_+$. But, the presence of the boundary $V = 0$ disturbes the using of it. So we also introduce the another conformal transformation $\tilde{g}^-_{ij}$ such that the spatial infinity is compactified to a point and the boundary $V = 0$ can be connected to that of $\tilde{\Sigma}_+$ with C^2(seen from the regularity condition on the event horizon). As a result we obtain the new manifold $\tilde{\Sigma} = \tilde{\Sigma}_+ \cup \tilde{\Sigma}_-$ which does not have the boundary excpet for the infinity. Now we can apply the positive mass theorem for $\tilde{\Sigma}$. Therefore, we see that $\tilde{\Sigma}$ is flat space. Here the concrete expression for the conformal transformation are given by $\Omega_\pm = [(1 \pm V)/2]^{2/(n-3)}$ and the Einstein equation tells us that the Ricci scalar $\tilde{R}_\pm$ of $\tilde{g}^\pm_{ij}$ vanishes. Next, we will show that we can see that the function $v = 2/(1+V)$ follows $\Delta v = 0$ and the boundary($v = 2$) is spherical symmmetric in $\tilde{\Sigma}_+$. The problem is reduced to that in the electrostatic fields. So we know that the solution is unique and $v =$ constant surfaces are spherical symmetric. This means that (Σ, g_{ij}) also has the spherical symmetry. Under the presence of such symmetry, it is easy to solve the Einstein equation and then we see that the spacetime should be the Schwarzschild solution. In the work of Bunting and Masood-ul-Alam, they empolyed the four dimensional speciality for this second step which is not directly applicable to higher dimensional cases.

We can extend this into the Einstein-Maxwell-dilaton system motivated by superstring theory [12]. Then, introducing a rather non-trivial form of the conformal transformation, we see that the static black hole should be spherical symmetric and the spacetime is uniquely described by the Gibbons-Maeda solution.

There is a technical remark to find the conformal transformation. When one knows the exact solution, we can rewrite the spatial metric in the conformally flat form. Then we can guess the concrete expression of the conformal transformation to show the conformally flatness.

5.2 On no-hair

Using the argument in the previous subsection, we can also show some no-hair properties of black holes.

As a simple example, one may want to consider the massless scalar hair(ϕ) [13]. In this case, we will employ the same conformal transformation with the vacuum cases. Then we see that the Ricci scalar of $\tilde{\Sigma}$ has a form ${}^{(n-1)}\tilde{R} \sim (D\phi)^2$, where D is the covariant derivative with respect to the spatial metric g_{ij}. Therefore, the Ricci scalar of $\tilde{g}_{ij}$ is non-negative. In the same way with the vaccum cases, we can construct the new manifold $\tilde{\Sigma}$ so that the ADM vanishes. Now we can apply the positive mass theorem for $\tilde{\Sigma}$ and then we can show that $\tilde{\Sigma}$ is fat space. This means that the Ricci scalar of $\tilde{g}_{ij}$ vanishes and then ϕ is trivially constant. If the scalar fields have the potential, following the Bekenstein's argument [14], one can show that the scalar hair does not exist using the field equation for the scalar fields.

As a next example, one may wonder if the black hole has anti-symmetric tensor hair in higher dimensions [15]. Let us consider the system which follows the Lagragian

$$\mathcal{L} = {}^{(n)}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{p!}e^{-\alpha\phi}H^2_{(p)}, \tag{27}$$

where $H_{(p)}$ is the p-form field and has the $(p-1)$-form potential, $H_{(p)} = dB_{(p-1)}$, and ϕ is the scalar field(dilaton). This theory is motivated by string theory. In stationary and higer dimensional spacetimes, there are black ring solutions with the topology $S^2 \times S$. Here we consider the static cases where the metric takes the form of Eq. (26). We assume that the form fields have the eletric part only, that is,

$$B_{(p-1)} = \varphi_{i_1 \cdots i_{p-2}}(x^j) dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-2}}. \tag{28}$$

Then the non-trivial component of $H_{(p)}$ is $H_{0i_1 \cdots i_{p-1}}$. Now we take the same conformal transformation for g_{ij} with the vacuum cases and construct $\tilde{\Sigma}$ with the vanishing ADM mass. After some computations, we see

$$\Omega_\pm^{2(n-1)} {}^{(n-1)}\tilde{R} = \frac{1}{(p-1)!} \frac{e^{-\alpha\phi}}{V^2} \frac{\lambda_\pm}{\omega_\pm} H_0{}^{i_1 \cdots i_{p-1}} H_{0i_1 \cdots i_{p-1}} + \frac{1}{2}(D\phi)^2, \tag{29}$$

where

$$\lambda_\pm = \frac{1 \mp \frac{3n-4p-1}{n-3} V}{2} \quad \text{and} \quad \omega_\pm = \frac{1 \pm V}{2}. \tag{30}$$

In general, $\lambda_\pm$ does not have the definite signature. So one can see that ${}^{(n-1)}\tilde{R} \geq 0$ if p satisfies $(n+1)/2 \leq p \leq n-1$. Under this condition, we can apply the positive mass theorem and then we can see that $\tilde{\Sigma}$ is flat, $H_{(p)} = 0$ and $\phi =$ constant. According to the perturbative analysis, we can show the non-existence of $H_{(p)}$-hair except for $3 \leq p \leq (n-1)/2$. In the above we employed the same conformal transformation with the vacuum cases which do not optimise to show the no-hair. If we do not know the exact solutions or we expect that there are no solutions, we cannot have a hint to find the appropriate conformal transformation from them. This is remaining issue.

6 Outlook

In this report we gave a review that the positive mass theorem constrains the spacetime structure strongly. This may be ragarded as a collapse of moduli space.

We focused on the asymptotically flat spacetimes here. But, one is interested in asymptotically anti-deSitter(AadS) spacetime too. The AadS spacetimes are one with the negative cosmological constant which is prefered by string theory. We can also show that the stationary and vaccum spacetimes with negative cosmological constant are the anti-deSitter spacetime [3, 16, 17, 18, 19].

About the generalised Lichnerowicz's lemma, we do not have the systematic way to show it. So it is nice to have such way.

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