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Left-orderable fundamental groups and Dehn surgery on two-bridge knots

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1 Introduction

In Heegaard Floer homology theory, L-spaces introduced in [17] have an important role. A rational homology 3-sphere Y is called an *L-space* if $\widehat{HF}(Y)$ is a free abelian group whose rank is equal to the order of $H_1(Y)$. Lens spaces are typical L-spaces, and several other families of L-spaces are known so far. However, it is still an open problem to give a characterization of L-spaces without involving Heegaard Floer homology.

In [4], Boyer, Gordon and Watson conjecture that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. This would be an algebraic characterization of L-spaces. Here, a non-trivial group G is said to be *left-orderable* if it admits a strict total ordering "<" which is invariant under left-multiplication. That is, if $g < h$ then $fg < fh$ for any $f, g, h \in G$. As a convention, the trivial group is defined to be not left-orderable. It is easy to see that G is left-orderable if and only if G is right-orderable, which is defined similarly. The history of research on orderable groups is long, and many groups which appear in topology are left-orderable. For example, free groups, free abelian groups, knot or link groups, braid groups are left-orderable. Also, the fundamental groups of surfaces but the projective plane are left-orderable. Since left-orderable groups are torsion-free, the fundamental groups of lens spaces, elliptic manifolds are not left-orderable. It is natural to ask which 3-manifolds have left-orderable fundamental groups. As a classical fact, the free products of left-orderable groups are left-orderable. Hence we may restrict ourselves to prime 3-manifolds. Boyer, Rolfsen and Wiest [5] prove that if a compact connected orientable prime 3-manifold has non-zero first betti number, then its fundamental group is left-orderable. Thus irreducible rational homology 3-spheres remain to be done.

Dehn surgery might be the easiest way to create rational homology 3-spheres. For a given knot K in the 3-sphere S^3, r-surgery yields a rational homology sphere whenever $r \neq 0$. By considering the cabling conjecture, the resulting rational homology sphere would be irreducible if K is not cabled. On the other hand, there are some strong constraints for knots which admit Dehn surgery yielding L-spaces. For example, such knots are fibered ([16]), and their Alexander polynomials have a specified form ([17]). Thus the above conjecture by Boyer, Gordon and Watson suggests that any non-trivial Dehn surgery on K yields a 3-manifold with left-orderable fundamental group, unless K passes such criteria.

Any knot group is left-orderable. The fundamental group of the resulting manifold by Dehn surgery on a knot is a quotient of the knot group. Although any subgroup of a left-orderable group is left-orderable, a quotient may not be left-orderable. For torus knots, the resulting manifold by Dehn surgery is either a Seifert fibered manifold or the connected sum of two lens spaces. Since Boyer, Gordon and Watson [4] solved the conjecture affirmatively for Seifert fibered manifolds, the left-orderability of the fundamental groups of the resulting manifolds by Dehn surgery is completely understandable for torus knots.

The simplest hyperbolic knot is the figure-eight knot. By [17], it does not admit Dehn surgery yielding an L-space. Hence we may expect that any non-trivial Dehn surgery yields a 3-manifold whose fundamental group is left-orderable. Toward this direction, Boyer, Gordon and Watson [4] showed if the surgery slope r lies in the interval $(-4,4)$, then r-surgery yields a manifold with left-orderable fundamental group. Later, Clay, Watson and Lidman [6] confirmed the same conclusion for $r=\pm 4$. (We remark that as noted in [4], this is also true for any integral surgery by [9].) These two arguments are quite different. The former builds a non-trivial representation of the fundamental group of the resulting manifold by r-surgery into $\widetilde{SL_2(\mathbb{R})}$, which is known to be left-orderable ([2]). But the latter makes use of the torus decomposition of the resulting (graph) manifold into two Seifert fibered pieces and some gluing technique of left-orderings ([3]). The argument of [6] was generalized to all hyperbolic twist knots in [19]. We showed that 4-surgery on a hyperbolic twist knot yields a manifold with left-orderable fundamental group. (Here, the hook of a twist knot is assumed to be left-handed.) Furthermore, we extended the argument for any exceptional Dehn surgery on hyperbolic two-bridge knots in [7].

In this note, we report a generalization of the argument of [4] from the figure-eight knot to hyperbolic genus one two-bridge knots. Details are found in [11]. Let $K=K(m,n)$ be a hyperbolic genus one two-bridge knot $S(4mn+1,2m)$ as shown in Figure 1. Here, the twists in the vertical box is left-handed (resp. right-handed) if $m>0$ (resp. $m<0$), but those in the horizontal box is right-handed (resp. left-handed) if $n>0$ (resp. $n<0$). By symmetry, $K(m,n)$ is equivalent to $K(-n,-m)$. Also, $K(-m,-n)$ is the mirror image of $K(m,n)$. Hence we may assume that $m>0$. Thus $K(1,1)$ is the figure-eight knot, and $K(1,-1)$ is the right-handed trefoil.

For a knot K, a slope r is said to be *left-orderable* if the resulting manifold $K(r)$ by

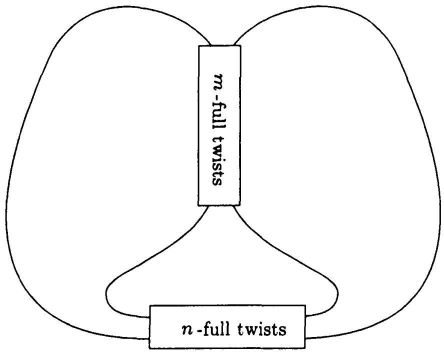

図 1: A genus one two-bridge knot $K(m,n)$

r-surgery has a left-orderable fundamental group.

Theorem 1.1 (**[11]**) *Let $K(m,n)$ be a hyperbolic genus one two-bridge knot $S(4mn+1,2m)$ in the 3-sphere S^3. Let I be the interval defined by*

$$I=\begin{cases}(-4n,4m) & \text{if } n>0,\\ [0,\max\{4m,-4n\}) & \text{if } m>1 \text{ and } n<-1,\\ [0,4] & \text{otherwise.}\end{cases}$$

Then any slope in I is left-orderable. That is, the fundamental group of the resulting manifold by r-surgery on $K(m,n)$ is left-orderable if $r\in I$.

Among $K(m,n)$, $K(1,n)$ and $K(m,\pm1)$ are twist knots. Moreover, $K(m,-1)$ is equivalent to $K(1,-m)$, and $K(m,1)$ is the mirror image of $K(1,m)$.

Corollary 1.2 *Let $K(1,n)$ be the n-twist knot with $n\neq -1$. If $n>0$, then any slope in the interval $(-4n,4]$ is left-orderable. If $n<-1$, then then any slope in $[0,4]$ is left-orderable.*

Our argument works for the figure-eight knot, and it is much simpler than one in [4], which involves character varieties. The fact that a knot has genus one is crucial in our argument as well as that of [4]. In general, the longitude of a knot group is a product of commutators. If a knot has genus one, then the longitude is a single commutator. For a representation of a knot group into the universal covering group $\widetilde{SL_2}(\mathbb{R})$, we need to control the image of the longitude, by using Wood's inequality [21]. See Lemma 2.7.

Anh Tran [20] obtained independently a similar result to Theorem 1.1.

2 Outline

Let $K = K(m, n)$ and let $G = \pi_1(S^3 - K)$ be its knot group. We always assume that $m > 0$ and $n \neq 0$, unless specified otherwise.

Proposition 2.1 *The knot group G admits a presentation*

$$G = \langle x, y \mid w^n x = y w^n \rangle,$$

where x and y are meridians and $w = (xy^{-1})^m (x^{-1}y)^m$. Furthermore, the longitude $\mathcal{L}$ is given as $\mathcal{L} = w_*^n w^n$, where $w_* = (yx^{-1})^m (y^{-1}x)^m$ is obtained from w by reversing the order of letters.

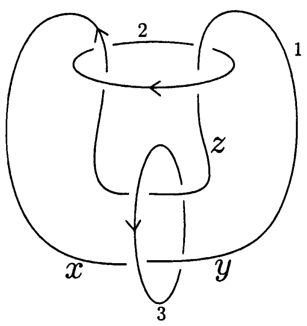

図 2: A surgery diagram of $K(m, n)$

This is slightly different from that in [13, Proposition 1], but both are isomorphic. It is derived from a surgery diagram of K as illustrated in Figure 2, where $1/m$-surgery and $-1/n$-surgery are performed along the second and third components, respectively.

Let s and t be real numbers such that $s > 0$ and $t > 1$. Let $\rho : G \to SL_2(\mathbb{R})$ be a representation of G defined by

$$\rho(x) = \begin{pmatrix} \sqrt{t} & 1/\sqrt{t} \\ 0 & 1/\sqrt{t} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \sqrt{t} & 0 \\ -s\sqrt{t} & 1/\sqrt{t} \end{pmatrix}.$$

By [18], ρ gives a non-abelian representation if s and t are a pair of solutions of the Riley polynomial. Let $W = \rho(w)$ and $z_{i,j}$ be the (i, j)-entry of W^n. Then the Riley polynomial of K is given by $\phi_K(s, t) = z_{1,1} + (1 - t) z_{1,2}$. (See also [8].) Since s and t are limited to be positive real numbers in our setting, it is not obvious that there exist solutions for Riley's equation $\phi_K(s, t) = 0$. However, this will be verified in Proposition 2.3 under some condition.

To describe the Riley polynomial of K explicitly, we need two sequences of polynomials with a single variable s.

For non-negative integer m, let $f_m \in \mathbb{Z}[s]$ be defined by the recursion

$$f_{m+2} - (s+2)f_{m+1} + f_m = 0 \tag{2.1}$$

with initial conditions $f_0 = 1$ and $f_1 = s+1$. Also, let $g_m \in \mathbb{Z}[s]$ be defined by the same recursion

$$g_{m+2} - (s+2)g_{m+1} + g_m = 0 \tag{2.2}$$

with slightly different initial conditions $g_0 = 1$ and $g_1 = s+2$. We remark that g_m is equivalent to the Chebyshev polynomial of the second kind.

The closed formulas for f_m and g_m are

$$f_m = \sum_{i=0}^{m} \binom{m+i}{m-i} s^i, \quad g_m = \sum_{i=0}^{m} \binom{m+1+i}{m-i} s^i.$$

In particular, all coefficients of f_m and g_m are positive integers, and the degree of f_m and g_m is m. Also, f_m and g_m are monic.

Let $\lambda_\pm \in \mathbb{C}$ be the eigenvalues of $W = \rho(w)$. For any integer k, set $\tau_k = (\lambda_+^k - \lambda_-^k)/(\lambda_+ - \lambda_-)$.

Proposition 2.2 *The Riley polynomial of K is*

$$\phi_K(s,t) = (\tau_{n+1} - \tau_n) + (s+2-t-1/t)f_{m-1}g_{m-1}\tau_n.$$

For convenience, we introduce a variable $T = t + 1/t$. Then the Riley polynomial of K is $\phi_K(s,T) = (\tau_{n+1} - \tau_n) + (s+2-T)f_{m-1}g_{m-1}\tau_n$.

For example, if $n = 1$ then

$$\begin{aligned}
\phi_K(s,T) &= (\tau_2 - \tau_1) + (s+2-T)f_{m-1}g_{m-1}\tau_1 \\
&= (\mathrm{tr}W - 1) + (s+2-T)f_{m-1}g_{m-1} \\
&= s(s+2-T)g_{m-1}^2 + 1 + (s+2-T)f_{m-1}g_{m-1} \\
&= (s+2-T)g_{m-1}(sg_{m-1} + f_{m-1}) + 1 \\
&= (s+2-T)g_{m-1}f_m + 1.
\end{aligned}$$

Thus Riley's equation $\phi_K(s,T) = 0$ has the unique solution $T = s + 2 + 1/(f_m g_{m-1})$ for any $s > 0$. Then $T > s+2 > 2$, because $f_m > 0$ and $g_{m-1} > 0$. Hence we have a real solution $t = (T + \sqrt{T^2-4})/2 > 1$. In fact, we have $s + 2 < T < s + 2 + 4/(sg_{m-1}^2)$.

Proposition 2.3 *Suppose $n \neq \pm 1$. For any $s > 0$, Riley's equation $\phi_K(s,T) = 0$ has a solution T satisfying $s + 2 + c/(sg_{m-1}^2) < T < s + 2 + d/(sg_{m-1}^2)$, where c and d are constants in $(0,4)$ depending only on n. In particular, $\phi_K(s,t) = 0$ has a solution $t > 1$ for any $s > 0$.*

Now, we introduce a continuous family of representations of G. For $s > 0$, let $\rho_s : G \to SL_2(\mathbb{R})$ be the representation defined by the correspondence

$$\rho_s(x) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad \rho_s(y) = \begin{pmatrix} \frac{t-s-1}{\sqrt{t}-\frac{1}{\sqrt{t}}} & \frac{s}{(\sqrt{t}-\frac{1}{\sqrt{t}})^2} - 1 \\ -s & \frac{s+1-\frac{1}{t}}{\sqrt{t}-\frac{1}{\sqrt{t}}} \end{pmatrix}. \tag{2.3}$$

Since ρ_s is conjugate with ρ, if s and t satisfy Riley's equation $\phi_K(s,t) = 0$ then ρ_s gives a non-abelian representation of G as well as ρ (see [8, 14]).

Proposition 2.4 *For the longitude $\mathcal{L}$ of G, the matrix $\rho_s(\mathcal{L})$ is diagonal, and the $(1,1)$-entry of $\rho_s(\mathcal{L})$ is a positive real number.*

The first conclusion is easy, but the second is important. To show it, the character variety theory was used in [4, Lemma 7], but we can establish it through a direct calculation.

Let B_s be the $(1,1)$-entry of the matrix $\rho_s(\mathcal{L})$.

Proposition 2.5

$$B_s = \frac{-f_m + tf_{m-1}}{-f_{m-1} + tf_m}.$$

This conclusion is interesting, because the parameter n disappears.

Let $r = p/q$ be a rational number, and let $K(r)$ denote the resulting manifold by r-surgery on K. In other words, $K(r)$ is obtained by attaching a solid torus V to the knot exterior $E(K)$ along their boundaries so that the loop $x^p\mathcal{L}^q$ bounds a meridian disk of V, where x and $\mathcal{L}$ are a meridian and longitude of K.

Our representation $\rho_s : G \to SL_2(\mathbb{R})$ induces a homomorphism $\pi_1(K(r)) \to SL_2(\mathbb{R})$ if and only if $\rho_s(x)^p\rho_s(\mathcal{L})^q = I$. Since both of $\rho_s(x)$ and $\rho_s(\mathcal{L})$ are diagonal (see (2.3) and Proposition 2.4), this is equivalent to the single equation

$$A_s^p B_s^q = 1, \tag{2.4}$$

where A_s and B_s are the $(1,1)$-entries of $\rho_s(x)$ and $\rho_s(\mathcal{L})$, respectively. We remark that $A_s = \sqrt{t}$ (> 1) is a positive real number, so is B_s by Proposition 2.4. Hence the equation (2.4) is furthermore equivalent to the equation

$$-\frac{\log B_s}{\log A_s} = \frac{p}{q}. \tag{2.5}$$

Let $g : (0, \infty) \to \mathbb{R}$ be a function defined by

$$g(s) = -\frac{\log B_s}{\log A_s}.$$

By calculating limits, we obtain the following.

Proposition 2.6 *The image of g contains an open interval* $(0, 4m)$.

The next is the key in [4], which is originally claimed in [14], for the figure-eight knot. Our proof most follows that of [4].

The universal covering group $\widetilde{SL_2(\mathbb{R})}$ can be described as

$$\widetilde{SL_2(\mathbb{R})} = \{(\gamma, \omega) \mid |\gamma| < 1, -\infty < \omega < \infty\}.$$

See [1, 14]. Let $\chi : \widetilde{SL_2(\mathbb{R})} \to SL_2(\mathbb{R})$ be the covering projection. Then $\ker\chi = \{(0, 2j\pi) \mid j \in \mathbb{Z}\}$.

Lemma 2.7 *Let $\tilde{\rho} : G \to \widetilde{SL_2(\mathbb{R})}$ be a lift of ρ_s. Then replacing $\tilde{\rho}$ by a representation $\tilde{\rho}' = h \cdot \tilde{\rho}$ for some $h : G \to \widetilde{SL_2(\mathbb{R})}$, we can suppose that $\tilde{\rho}(\pi_1(\partial E(K)))$ is contained in the subgroup $(-1, 1) \times \{0\}$ of $\widetilde{SL_2(\mathbb{R})}$.*

Proof of Theorem 1.1 Suppose $n \neq -1$. Let $r = p/q \in (0, 4m)$. By Proposition 2.6, we can find s so that $g(s) = r$. Choose a lift $\tilde{\rho}_s$ of ρ_s so that $\tilde{\rho}_s(\pi_1(\partial E(K))) \subset (-1, 1) \times \{0\}$ (Lemma 2.7). Then $\rho_s(x^p\mathcal{L}^q) = I$, so $\chi(\tilde{\rho}_s(x^p\mathcal{L}^q)) = I$. This means that $\tilde{\rho}_s(x^p\mathcal{L}^q)$ lies in $\ker\chi = \{(0, 2j\pi) \mid j \in \mathbb{Z}\}$. Hence $\tilde{\rho}_s(x^p\mathcal{L}^q) = (0, 0)$. Then $\tilde{\rho}_s$ can induce a homomorphism $\pi_1(K(r)) \to \widetilde{SL_2(\mathbb{R})}$ with non-abelian image. Recall that $\widetilde{SL_2(\mathbb{R})}$ is left-orderable ([2]) and any (non-trivial) subgroup of a left-orderable group is left-orderable. Since $K(r)$ is irreducible [12], $\pi_1(K(r))$ is left-orderable by [5, Theorem 1.1]. For $r = 0$, $K(0)$ is irreducible ([10]) and has positive betti number. Hence $\pi_1(K(0))$ is left-orderable by [5, Corollary 3.4]. Thus we have shown that any slope in $[0, 4m)$ is left-orderable for $K = K(m, n)$.

Suppose $n > 0$. If we apply the above argument for $K(n, m)$, then any slope in $[0, 4n)$ is shown to be left-orderable. Since $K(n, m)$ is equivalent to the mirror image of $K(m, n)$, any slope in $(-4n, 0]$ is left-orderable for $K(m, n)$. Thus we can conclude that $(-4n, 4m)$ consists of left-orderable slopes for $K = K(m, n)$ with $n > 0$.

Suppose $m > 1$ and $n < -1$. Since $K(m, n)$ is equivalent to $K(-n, -m)$, the argument in the first paragraph shows that any slope in $[0, -4n)$ is left-orderable. In this case, we obtain $[0, \max\{4m, -4n\})$ consisting of left-orderable slopes.

Finally, consider the remaining cases. They are $K(1, n)$ with $n < -1$ and $K(m, -1)$ with $m > 1$. Since $K(m, -1)$ is isotopic to $K(1, -m)$, two cases coincide. We obtain $[0, 4)$ consisting of left-orderable slopes by the argument in the first paragraph. Furthermore, since these knots are twist knots, the slope 4 is also left-orderable by [19]. □

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