Left-orderable fundamental groups and Dehn surgery on two-bridge knots

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1 Introduction

In Heegaard Floer homology theory, $L$-spaces introduced in [17] have an important role. A rational homology 3-sphere $Y$ is called an $L$-space if $\widehat{HF}(Y)$ is a free abelian group whose rank is equal to the order of $H_1(Y)$. Lens spaces are typical $L$-spaces, and several other families of $L$-spaces are known so far. However, it is still an open problem to give a characterization of $L$-spaces without involving Heegaard Floer homology.

In [4], Boyer, Gordon and Watson conjecture that an irreducible rational homology 3-sphere is an $L$-space if and only if its fundamental group is not left-orderable. This would be an algebraic characterization of $L$-spaces. Here, a non-trivial group $G$ is said to be left-orderable if it admits a strict total ordering "<" which is invariant under left-multiplication. That is, if $g < h$ then $fg < fh$ for any $f, g, h \in G$. As a convention, the trivial group is defined to be not left-orderable. It is easy to see that $G$ is left-orderable if and only if $G$ is right-orderable, which is defined similarly. The history of research on orderable groups is long, and many groups which appear in topology are left-orderable. For example, free groups, free abelian groups, knot or link groups, braid groups are left-orderable. Also, the fundamental groups of surfaces but the projective plane are left-orderable. Since left-orderable groups are torsion-free, the fundamental groups of lens spaces, elliptic manifolds are not left-orderable. It is natural to ask which 3-manifolds have left-orderable fundamental groups. As a classical fact, the free products of left-orderable groups are left-orderable. Hence we may restrict ourselves to prime 3-manifolds. Boyer, Rolfsen and Wiest [5] prove that if a compact connected orientable 3-manifold has non-zero first betti number, then its fundamental group is left-orderable. Thus irreducible rational homology 3-spheres remain to be done.

Dehn surgery might be the easiest way to create rational homology 3-spheres. For a given knot $K$ in the 3-sphere $S^3$, $r$-surgery yields a rational homology sphere whenever $r \neq 0$. By considering the cabling conjecture, the resulting rational homology sphere
would be irreducible if $K$ is not cabled. On the other hand, there are some strong
constraints for knots which admit Dehn surgery yielding $L$-spaces. For example, such
knots are fibered ([16]), and their Alexander polynomials have a specified form ([17]).
Thus the above conjecture by Boyer, Gordon and Watson suggests that any non-trivial
Dehn surgery on $K$ yields a $3$-manifold with left-orderable fundamental group, unless $K$
passes such criteria.

Any knot group is left-orderable. The fundamental group of the resulting manifold by
Dehn surgery on a knot is a quotient of the knot group. Although any subgroup of a left-
orderable group is left-orderable, a quotient may not be left-orderable. For torus knots,
the resulting manifold by Dehn surgery is either a Seifert fibered manifold or the con-
affirmatively for Seifert fibered manifolds, the left-orderability of the fundamental groups
of the resulting manifolds by Dehn surgery is completely understandable for torus knots.

The simplest hyperbolic knot is the figure-eight knot. By [17], it does not admit Dehn
surgery yielding an $L$-space. Hence we may expect that any non-trivial Dehn surgery
yields a $3$-manifold whose fundamental group is left-orderable. Toward this direction,
Boyer, Gordon and Watson [4] showed if the surgery slope $r$ lies in the interval $(-4,4)$,
then $r$-surgery yields a manifold with left-orderable fundamental group. Later, Clay,
Watson and Lidman [6] confirmed the same conclusion for $r = \pm 4$. (We remark that as
noted in [4], this is also true for any integral surgery by [9].) These two arguments are
quite different. The former builds a non-trivial representation of the fundamental group of
the resulting manifold by $r$-surgery into $\text{SL}_2(\mathbb{R})$, which is known to be left-orderable ([2]).
But the latter makes use of the torus decomposition of the resulting (graph) manifold into
two Seifert fibered pieces and some gluing technique of left-orderings ([3]). The argument
of [6] was generalized to all hyperbolic twist knots in [19]. We showed that 4-surgery on
a hyperbolic twist knot yields a manifold with left-orderable fundamental group. (Here, the
hook of a twist knot is assumed to be left-handed.) Furthermore, we extended the
argument for any exceptional Dehn surgery on hyperbolic two-bridge knots in [7].

In this note, we report a generalization of the argument of [4] from the figure-eight knot
to hyperbolic genus one two-bridge knots. Details are found in [11]. Let $K = K(m,n)$ be
a hyperbolic genus one two-bridge knot $S(4mn + 1,2m)$ as shown in Figure 1. Here, the
twists in the vertical box is left-handed (resp. right-handed) if $m > 0$ (resp. $m < 0$), but
those in the horizontal box is right-handed (resp. left-handed) if $n > 0$ (resp. $n < 0$). By
symmetry, $K(m,n)$ is equivalent to $K(-n, -m)$. Also, $K(-m, -n)$ is the mirror image
of $K(m,n)$. Hence we may assume that $m > 0$. Thus $K(1,1)$ is the figure-eight knot,
and $K(1,-1)$ is the right-handed trefoil.

For a knot $K$, a slope $r$ is said to be left-orderable if the resulting manifold $K(r)$ by
$r$-surgery has a left-orderable fundamental group.

**Theorem 1.1 ([11])** Let $K(m,n)$ be a hyperbolic genus one two-bridge knot $S(4mn+1,2m)$ in the 3-sphere $S^3$. Let $I$ be the interval defined by

$$I = \begin{cases} (-4n,4m) & \text{if } n > 0, \\ [0,\max\{4m,-4n\}) & \text{if } m > 1 \text{ and } n < -1, \\ [0,4] & \text{otherwise.} \end{cases}$$

Then any slope in $I$ is left-orderable. That is, the fundamental group of the resulting manifold by $r$-surgery on $K(m,n)$ is left-orderable if $r \in I$.

Among $K(m,n)$, $K(1,n)$ and $K(m,\pm 1)$ are twist knots. Moreover, $K(m,-1)$ is equivalent to $K(1,-m)$, and $K(m,1)$ is the mirror image of $K(1,m)$.

**Corollary 1.2** Let $K(1,n)$ be the $n$-twist knot with $n \neq -1$. If $n > 0$, then any slope in the interval $(-4n,4]$ is left-orderable. If $n < -1$, then any slope in $[0,4]$ is left-orderable.

Our argument works for the figure-eight knot, and it is much simpler than one in [4], which involves character varieties. The fact that a knot has genus one is crucial in our argument as well as that of [4]. In general, the longitude of a knot group is a product of commutators. If a knot has genus one, then the longitude is a single commutator. For a representation of a knot group into the universal covering group $\widetilde{SL_2}(\mathbb{R})$, we need to control the image of the longitude, by using Wood’s inequality [21]. See Lemma 2.7.

Anh Tran [20] obtained independently a similar result to Theorem 1.1.
2 Outline

Let $K = K(m, n)$ and let $G = \pi_1(S^3 - K)$ be its knot group. We always assume that $m > 0$ and $n \neq 0$, unless specified otherwise.

**Proposition 2.1** The knot group $G$ admits a presentation

$$G = \langle x, y \mid w^n x = yw^n \rangle,$$

where $x$ and $y$ are meridians and $w = (xy^{-1})^m(x^{-1}y)^m$. Furthermore, the longitude $\mathcal{L}$ is given as $\mathcal{L} = w_* w^n$, where $w_* = (yx^{-1})^m(y^{-1}x)^m$ is obtained from $w$ by reversing the order of letters.

This is slightly different from that in [13, Proposition 1], but both are isomorphic. It is derived from a surgery diagram of $K$ as illustrated in Figure 2, where $1/m$-surgery and $-1/n$-surgery are performed along the second and third components, respectively.

Let $s$ and $t$ be real numbers such that $s > 0$ and $t > 1$. Let $\rho : G \to SL_2(\mathbb{R})$ be a representation of $G$ defined by

$$\rho(x) = \begin{pmatrix} \sqrt{t} & 1/\sqrt{t} \\ 0 & 1/\sqrt{t} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \sqrt{t} & 0 \\ -s\sqrt{t} & 1/\sqrt{t} \end{pmatrix}.$$

By [18], $\rho$ gives a non-abelian representation if $s$ and $t$ are a pair of solutions of the Riley polynomial. Let $W = \rho(w)$ and $z_{i,j}$ be the $(i,j)$-entry of $W^n$. Then the Riley polynomial of $K$ is given by $\phi_K(s, t) = z_{1,1} + (1 - t)z_{1,2}$. (See also [8].) Since $s$ and $t$ are limited to be positive real numbers in our setting, it is not obvious that there exist solutions for Riley’s equation $\phi_K(s, t) = 0$. However, this will be verified in Proposition 2.3 under some condition.

To describe the Riley polynomial of $K$ explicitly, we need two sequences of polynomials with a single variable $s$. 

![Surgery Diagram of K(m, n)](image-url)
For non-negative integer \( m \), let \( f_m \in \mathbb{Z}[s] \) be defined by the recursion
\[
f_{m+2} - (s + 2)f_{m+1} + f_m = 0 \tag{2.1}
\]
with initial conditions \( f_0 = 1 \) and \( f_1 = s + 1 \). Also, let \( g_m \in \mathbb{Z}[s] \) be defined by the same recursion
\[
g_{m+2} - (s + 2)g_{m+1} + g_m = 0 \tag{2.2}
\]
with slightly different initial conditions \( g_0 = 1 \) and \( g_1 = s + 2 \). We remark that \( g_m \) is equivalent to the Chebyshev polynomial of the second kind.

The closed formulæ for \( f_m \) and \( g_m \) are
\[
f_m = \sum_{i=0}^{m} \binom{m + im - i}{m - i} s^i, \quad g_m = \sum_{i=0}^{m} \binom{m + l+i - im}{m - i} s^i.
\]

In particular, all coefficients of \( f_m \) and \( g_m \) are positive integers, and the degree of \( f_m \) and \( g_m \) is \( m \). Also, \( f_m \) and \( g_m \) are monic.

Let \( \lambda_{\pm} \in \mathbb{C} \) be the eigenvalues of \( W = \rho(w) \). For any integer \( k \), set \( \tau_k = (\lambda_+^k - \lambda_-^k)/(\lambda_+ - \lambda_-) \).

**Proposition 2.2** The Riley polynomial of \( K \) is
\[
\phi_K(s, t) = (\tau_{n+1} - \tau_n) + (s + 2 - t - 1/t)f_m g_m \tau_n.
\]

For convenience, we introduce a variable \( T = t + 1/t \). Then the Riley polynomial of \( K \) is \( \phi_K(s, T) = (\tau_{n+1} - \tau_n) + (s + 2 - T)f_m g_m \tau_n \).

For example, if \( n = 1 \) then
\[
\phi_K(s, T) = (\tau_2 - \tau_1) + (s + 2 - T)f_m g_m \tau_1
\]
\[
= (\text{tr}W - 1) + (s + 2 - T)f_m g_m
\]
\[
= s(s + 2 - T)g_m^2 + 1 + (s + 2 - T)f_m g_m
\]
\[
= (s + 2 - T)g_m(s g_m + f_m) + 1
\]
\[
= (s + 2 - T)g_m f_m + 1.
\]

Thus Riley’s equation \( \phi_K(s, T) = 0 \) has the unique solution \( T = s + 2 + 1/(f_m g_m) \) for any \( s > 0 \). Then \( T > s + 2 > 2 \), because \( f_m > 0 \) and \( g_m > 0 \). Hence we have a real solution \( t = (T + \sqrt{T^2 - 4})/2 > 1 \). In fact, we have \( s + 2 < T < s + 2 + 4/(sg_m^2) \).

**Proposition 2.3** Suppose \( n \neq \pm 1 \). For any \( s > 0 \), Riley’s equation \( \phi_K(s, T) = 0 \) has a solution \( T \) satisfying \( s + 2 + c/(sg_m^2) < T < s + 2 + d/(sg_m^2) \), where \( c \) and \( d \) are constants in \((0, 4)\) depending only on \( n \). In particular, \( \phi_K(s, t) = 0 \) has a solution \( t > 1 \) for any \( s > 0 \).
Now, we introduce a continuous family of representations of $G$. For $s > 0$, let $\rho_s : G \to \text{SL}_2(\mathbb{R})$ be the representation defined by the correspondence

$$\rho_s(x) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad \rho_s(y) = \begin{pmatrix} \frac{t-s-1}{\sqrt{t}} & -s \\ \frac{s+1-1}{\sqrt{t}} & \sqrt{t} \end{pmatrix}.$$  \hspace{1cm} (2.3)

Since $\rho_s$ is conjugate with $\rho$, if $s$ and $t$ satisfy Riley's equation $\phi_K(s, t) = 0$ then $\rho_s$ gives a non-abelian representation of $G$ as well as $\rho$ (see [8, 14]).

**Proposition 2.4** For the longitude $\mathcal{L}$ of $G$, the matrix $\rho_s(\mathcal{L})$ is diagonal, and the $(1,1)$-entry of $\rho_s(\mathcal{L})$ is a positive real number.

The first conclusion is easy, but the second is important. To show it, the character variety theory was used in [4, Lemma 7], but we can establish it through a direct calculation. Let $B_s$ be the $(1,1)$-entry of the matrix $\rho_s(\mathcal{L})$.

**Proposition 2.5**

$$B_s = \frac{-f_m + tf_{m-1}}{-f_{m-1} + tf_m}.$$  \hspace{1cm} (2.4)

This conclusion is interesting, because the parameter $n$ disappears.

Let $r = p/q$ be a rational number, and let $K(r)$ denote the resulting manifold by $r$-surgery on $K$. In other words, $K(r)$ is obtained by attaching a solid torus $V$ to the knot exterior $E(K)$ along their boundaries so that the loop $x^p\mathcal{L}^q$ bounds a meridian disk of $V$, where $x$ and $\mathcal{L}$ are a meridian and longitude of $K$.

Our representation $\rho_s : G \to \text{SL}_2(\mathbb{R})$ induces a homomorphism $\pi_1(K(r)) \to \text{SL}_2(\mathbb{R})$ if and only if $\rho_s(x)^p \rho_s(\mathcal{L})^q = I$. Since both of $\rho_s(x)$ and $\rho_s(\mathcal{L})$ are diagonal (see (2.3) and Proposition 2.4), this is equivalent to the single equation

$$A_s^p B_s^q = 1,$$  \hspace{1cm} (2.4)

where $A_s$ and $B_s$ are the $(1,1)$-entries of $\rho_s(x)$ and $\rho_s(\mathcal{L})$, respectively. We remark that $A_s = \sqrt{t} (>1)$ is a positive real number, so is $B_s$ by Proposition 2.4. Hence the equation (2.4) is furthermore equivalent to the equation

$$\frac{-\log B_s}{\log A_s} = \frac{p}{q}.$$  \hspace{1cm} (2.5)

Let $g : (0, \infty) \to \mathbb{R}$ be a function defined by

$$g(s) = \frac{-\log B_s}{\log A_s}.$$  

By calculating limits, we obtain the following.
Proposition 2.6 The image of $g$ contains an open interval $(0, 4m)$.

The next is the key in [4], which is originally claimed in [14], for the figure-eight knot. Our proof most follows that of [4].

The universal covering group $\hat{SL}_2(\mathbb{R})$ can be described as

$$\hat{SL}_2(\mathbb{R}) = \{(\gamma, \omega) \mid |\gamma| < 1, -\infty < \omega < \infty\}.$$ 

See [1, 14]. Let $\chi : \hat{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ be the covering projection. Then $\ker \chi = \{(0, 2j\pi) \mid j \in \mathbb{Z}\}$.

Lemma 2.7 Let $\tilde{\rho} : G \to \hat{SL}_2(\mathbb{R})$ be a lift of $\rho_s$. Then replacing $\tilde{\rho}$ by a representation $\tilde{\rho}' = h \cdot \tilde{\rho}$ for some $h : G \to SL_2(\mathbb{R})$, we can suppose that $\tilde{\rho}(\pi_1(\partial E(K)))$ is contained in the subgroup $(-1, 1) \times \{0\}$ of $SL_2(\mathbb{R})$.

Proof of Theorem 1.1 Suppose $n \neq -1$. Let $r = p/q \in (0, 4m)$. By Proposition 2.6, we can find $s$ so that $g(s) = r$. Choose a lift $\tilde{\rho}_s$ of $\rho_s$ so that $\tilde{\rho}_s(\pi_1(\partial E(K))) \subset (-1, 1) \times \{0\}$ (Lemma 2.7). Then $\rho_s(x^pL^q) = I$, so $\chi(\tilde{\rho}_s(x^pL^q)) = I$. This means that $\tilde{\rho}_s(x^pL^q)$ lies in $\ker \chi = \{(0, 2j\pi) \mid j \in \mathbb{Z}\}$. Hence $\tilde{\rho}_s(x^pL^q) = (0, 0)$. Then $\tilde{\rho}_s$ can induce a homomorphism $\pi_1(K(r)) \to \hat{SL}_2(\mathbb{R})$ with non-abelian image. Recall that $\hat{SL}_2(\mathbb{R})$ is left-orderable ([2]) and any (non-trivial) subgroup of a left-orderable group is left-orderable. Since $K(r)$ is reducible [12], $\pi_1(K(r))$ is left-orderable by [5, Theorem 1.1]. For $r = 0$, $K(0)$ is irreducible ([10]) and has positive betti number. Hence $\pi_1(K(0))$ is left-orderable by [5, Corollary 3.4]. Thus we have shown that any slope in $[0, 4m)$ is left-orderable for $K = K(m, n)$.

Suppose $n > 0$. If we apply the above argument for $K(n, m)$, then any slope in $[0, 4n)$ is shown to be left-orderable. Since $K(n, m)$ is equivalent to the mirror image of $K(m, n)$, any slope in $(-4n, 0]$ is left-orderable for $K(m, n)$. Thus we can conclude that $(-4n, 4m)$ consists of left-orderable slopes for $K = K(m, n)$ with $n > 0$.

Suppose $m > 1$ and $n < -1$. Since $K(m, n)$ is equivalent to $K(-n, -m)$, the argument in the first paragraph shows that any slope in $[0, -4n)$ is left-orderable. In this case, we obtain $[0, \max\{4m, -4n\}]$ consisting of left-orderable slopes.

Finally, consider the remaining cases. They are $K(n, 1)$ with $n < -1$ and $K(m, -1)$ with $m > 1$. Since $K(m, -1)$ is isotopic to $K(1, -m)$, two cases coincide. We obtain $[0, 4)$ consisting of left-orderable slopes by the argument in the first paragraph. Furthermore, since these knots are twist knots, the slope 4 is also left-orderable by [19].
References


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