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On $PGL_3(\mathbb{C})$-torsions

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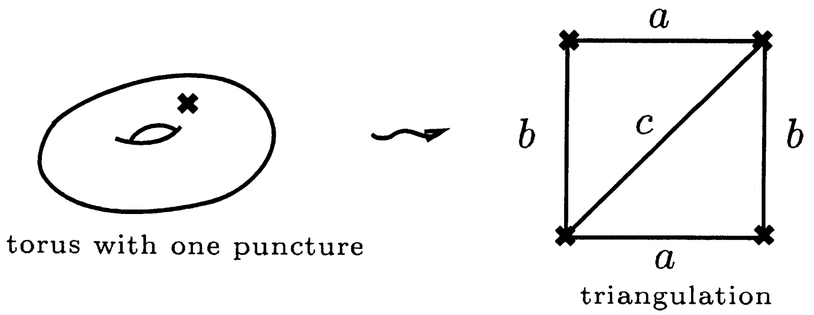
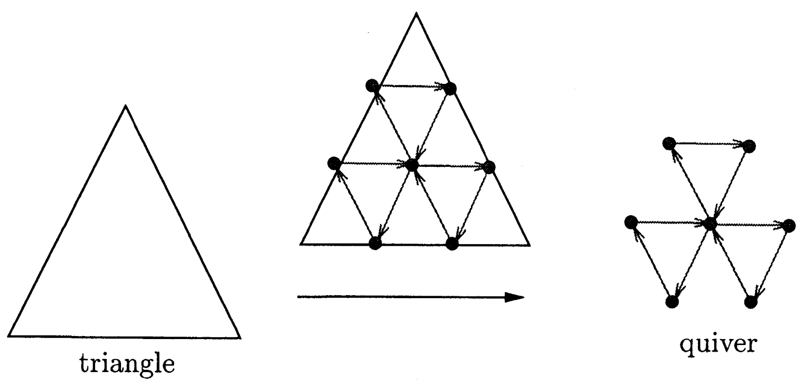
In this note, we explain how to obtain a $PGL_3(\mathbb{C})$-torsion for a mapping torus of a surface with punctures, using a concrete description of the action of a mapping class on Fock-Goncharov parameters.

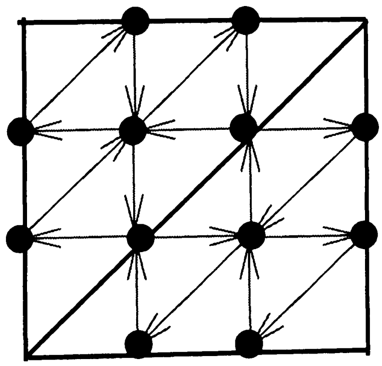
Step 1. First, we fix an ideal triangulation of a surface S with punctures. In the case of a torus with one puncture, we choose the following triangulation:



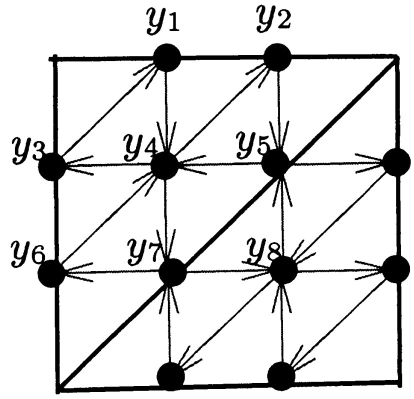
Step 2. For each triangle in the ideal triangulation, we put the following quiver:



In the case of the triangulation of a torus with one puncture in Step 1, we have the following quiver:



Step 3. For each vertex v in the quiver, we put a variable y_v. These variable $\{y_v\}$ are called Fock-Goncharov parameters [FG]. In the case of the quiver for a torus with one puncture in Step 2, we have variables $y_1, y_2, \ldots, y_8$:



Step 4. For a Dehn twist φ on a surface S, we write down the action $\varphi^{\#}$ on Fock-Goncharov parameters. This step can be done concretely according to the appendex in [TY] (see also [NTY]). In the case of the Dehn twists L, R on a torus with one puncture, we have the following

action:

$$L^{\#}(y_1) = \frac{y_1 y_3^2 (1+y_6) y_8}{(1+y_3)(1+y_3+y_3y_8+y_3y_6y_8)}$$
$$L^{\#}(y_2) = \frac{y_2(1+y_3)y_4y_6^2}{(1+y_6)(1+y_6+y_4y_6+y_3y_4y_6)}$$
$$L^{\#}(y_3) = \frac{(1+y_3)y_5(1+y_6+y_4y_6+y_3y_4y_6)}{1+y_6}$$
$$L^{\#}(y_4) = \frac{(1+y_6+y_4y_6+y_3y_4y_6)y_8}{1+y_3+y_3y_8+y_3y_6y_8}$$
$$L^{\#}(y_5) = \frac{1+y_6}{(1+y_3)y_4y_6}$$
$$L^{\#}(y_6) = \frac{(1+y_6)y_7(1+y_3+y_3y_8+y_3y_6y_8)}{1+y_3}$$
$$L^{\#}(y_7) = \frac{1+y_3}{y_3(1+y_6)y_8}$$
$$L^{\#}(y_8) = \frac{y_4(1+y_3+y_3y_8+y_3y_6y_8)}{1+y_6+y_4y_6+y_3y_4y_6}$$

$$R^{\#}(y_1) = \frac{y_1^2(1+y_2)y_4y_7}{(1+y_1)(1+y_1+y_1y_4+y_1y_2y_4)}$$
$$R^{\#}(y_2) = \frac{(1+y_1)y_2^2y_5y_8}{(1+y_2)(1+y_2+y_2y_8+y_1y_2y_8)}$$
$$R^{\#}(y_3) = \frac{(1+y_1)y_3(1+y_2+y_2y_8+y_1y_2y_8)}{1+y_2}$$
$$R^{\#}(y_4) = \frac{y_4(1+y_2+y_2y_8+y_1y_2y_8)}{1+y_1+y_1y_4+y_1y_2y_4}$$
$$R^{\#}(y_5) = \frac{1+y_2}{(1+y_1)y_2y_8}$$
$$R^{\#}(y_6) = \frac{(1+y_2)(1+y_1+y_1y_4+y_1y_2y_4)y_6}{1+y_1}$$
$$R^{\#}(y_7) = \frac{1+y_1}{y_1(1+y_2)y_4}$$
$$R^{\#}(y_8) = \frac{(1+y_1+y_1y_4+y_1y_2y_4)y_8}{1+y_2+y_2y_8+y_1y_2y_8}$$

Step 5. For a mapping class φ of a surface S, we consider the equation of fixed points:

$$y_i = \varphi^{\#}(y_i)$$

This equation should be the equation of a "geometric part" of the moduli space of $PGL(3;\mathbb{C})$-representations of the fundamental group of the mapping torus M_φ with φ. In the case of the mapping class LR of a torus with one puncture, we have the following equation:

$$
\begin{aligned}
y_1 =&(y_1^2y_3^3(1+2y_6+y_4y_6+y_3y_4y_6+y_6^2+y_4y_6^2+y_2y_4y_6^2+y_3y_4y_6^2+y_2y_3y_4y_6^2)y_8^2)/\\
&((1+2y_3+y_3^2+y_3y_8+y_3^2y_8+y_1y_3^2y_8+y_3y_6y_8+y_3^2y_6y_8+y_1y_3^2y_6y_8)\\
&(1+2y_3+y_3^2+2y_3y_8+2y_3^2y_8+y_1y_3^2y_8+2y_3y_6y_8+2y_3^2y_6y_8+y_1y_3^2y_6y_8+\\
&y_3^2y_8^2+y_1y_3^2y_8^2+2y_3^2y_6y_8^2+2y_1y_3^2y_6y_8^2+y_1y_3^2y_4y_6y_8^2+\\
&y_3^2y_6^2y_8^2+y_1y_3^2y_6^2y_8^2+y_1y_3^2y_4y_6^2y_8^2+y_1y_2y_3^2y_4y_6^2y_8^2))\\
y_2 =&(y_2^2y_4^2y_6^3(1+2y_3+y_3^2+y_3y_8+y_3^2y_8+y_1y_3^2y_8+y_3y_6y_8+y_3^2y_6y_8+y_1y_3^2y_6y_8))/\\
&((1+2y_6+y_4y_6+y_3y_4y_6+y_6^2+y_4y_6^2+y_2y_4y_6^2+y_3y_4y_6^2+y_2y_3y_4y_6^2)\\
&(1+2y_6+2y_4y_6+2y_3y_4y_6+y_6^2+2y_4y_6^2+y_2y_4y_6^2+2y_3y_4y_6^2+\\
&y_2y_3y_4y_6^2+y_4^2y_6^2+y_2y_4^2y_6^2+2y_3y_4^2y_6^2+2y_2y_3y_4^2y_6^2+y_3^2y_4^2y_6^2+\\
&y_2y_3^2y_4^2y_6^2+y_2y_3y_4^2y_6^2y_8+y_2y_3^2y_4^2y_6^2y_8+y_1y_2y_3^2y_4^2y_6^2y_8))\\
y_3 =&(y_5(1+2y_3+y_3^2+y_3y_8+y_3^2y_8+y_1y_3^2y_8+y_3y_6y_8+y_3^2y_6y_8+y_1y_3^2y_6y_8)\\
&(1+2y_6+2y_4y_6+2y_3y_4y_6+y_6^2+2y_4y_6^2+y_2y_4y_6^2+2y_3y_4y_6^2+\\
&y_2y_3y_4y_6^2+y_4^2y_6^2+y_2y_4^2y_6^2+2y_3y_4^2y_6^2+2y_2y_3y_4^2y_6^2+y_3^2y_4^2y_6^2+\\
&y_2y_3^2y_4^2y_6^2+y_2y_3y_4^2y_6^2y_8+y_2y_3^2y_4^2y_6^2y_8+y_1y_2y_3^2y_4^2y_6^2y_8))/\\
&((1+2y_6+y_4y_6+y_3y_4y_6+y_6^2+y_4y_6^2+y_2y_4y_6^2+y_3y_4y_6^2+y_2y_3y_4y_6^2)\\
&(1+y_3+y_3y_8+y_3y_6y_8))\\
y_4 =&(y_8(1+y_3+y_3y_8+y_3y_6y_8)(1+2y_6+2y_4y_6+2y_3y_4y_6+y_6^2+2y_4y_6^2+y_2y_4y_6^2+\\
&2y_3y_4y_6^2+y_2y_3y_4y_6^2+y_4^2y_6^2+y_2y_4^2y_6^2+2y_3y_4^2y_6^2+2y_2y_3y_4^2y_6^2+\\
&y_3^2y_4^2y_6^2+y_2y_3^2y_4^2y_6^2+y_2y_3y_4^2y_6^2y_8+y_2y_3^2y_4^2y_6^2y_8+y_1y_2y_3^2y_4^2y_6^2y_8))/\\
&((1+y_6+y_4y_6+y_3y_4y_6)(1+2y_3+y_3^2+2y_3y_8+2y_3^2y_8+y_1y_3^2y_8+2y_3y_6y_8+\\
&2y_3^2y_6y_8+y_1y_3^2y_6y_8+y_3^2y_8^2+y_1y_3^2y_8^2+2y_3^2y_6y_8^2+2y_1y_3^2y_6y_8^2+\\
&y_1y_3^2y_4y_6y_8^2+y_3^2y_6^2y_8^2+y_1y_3^2y_6^2y_8^2+y_1y_3^2y_4y_6^2y_8^2+y_1y_2y_3^2y_4y_6^2y_8^2))\\
y_5 =&((1+y_6+y_4y_6+y_3y_4y_6)(1+2y_6+y_4y_6+y_3y_4y_6+y_6^2+y_4y_6^2+\\
&y_2y_4y_6^2+y_3y_4y_6^2+y_2y_3y_4y_6^2))/\\
&(y_2y_4^2y_6^2(1+2y_3+y_3^2+y_3y_8+y_3^2y_8+y_1y_3^2y_8+y_3y_6y_8+y_3^2y_6y_8+y_1y_3^2y_6y_8))\\
y_6 =&((1+2y_6+y_4y_6+y_3y_4y_6+y_6^2+y_4y_6^2+y_2y_4y_6^2+y_3y_4y_6^2+y_2y_3y_4y_6^2)y_7\\
&(1+2y_3+y_3^2+2y_3y_8+2y_3^2y_8+y_1y_3^2y_8+2y_3y_6y_8+2y_3^2y_6y_8+y_1y_3^2y_6y_8+\\
&y_3^2y_8^2+y_1y_3^2y_8^2+2y_3^2y_6y_8^2+2y_1y_3^2y_6y_8^2+y_1y_3^2y_4y_6y_8^2+y_3^2y_6^2y_8^2+\\
&y_1y_3^2y_6^2y_8^2+y_1y_3^2y_4y_6^2y_8^2+y_1y_2y_3^2y_4y_6^2y_8^2))/((1+y_6+y_4y_6+y_3y_4y_6)\\
&(1+2y_3+y_3^2+y_3y_8+y_3^2y_8+y_1y_3^2y_8+y_3y_6y_8+y_3^2y_6y_8+y_1y_3^2y_6y_8))
\end{aligned}
$$

$$
\begin{aligned}
y_7 =&((1+y_3+y_3y_8+y_3y_6y_8)(1+2y_3+y_3^2+y_3y_8+y_3^2y_8+y_1y_3^2y_8+\\
&y_3y_6y_8+y_3^2y_6y_8+y_1y_3^2y_6y_8))/\\
&(y_1y_3^2(1+2y_6+y_4y_6+y_3y_4y_6+y_6^2+y_4y_6^2+y_2y_4y_6^2+y_3y_4y_6^2+y_2y_3y_4y_6^2)y_8^2)\\
y_8 =&(y_4(1+y_6+y_4y_6+y_3y_4y_6)(1+2y_3+y_3^2+2y_3y_8+2y_3^2y_8+y_1y_3^2y_8+2y_3y_6y_8+\\
&2y_3^2y_6y_8+y_1y_3^2y_6y_8+y_3^2y_8^2+y_1y_3^2y_8^2+2y_3^2y_6y_8^2+2y_1y_3^2y_6y_8^2+\\
&y_1y_3^2y_4y_6y_8^2+y_3^2y_6^2y_8^2+y_1y_3^2y_6^2y_8^2+y_1y_3^2y_4y_6^2y_8^2+y_1y_2y_3^2y_4y_6^2y_8^2))/\\
&((1+y_3+y_3y_8+y_3y_6y_8)(1+2y_6+2y_4y_6+2y_3y_4y_6+y_6^2+2y_4y_6^2+y_2y_4y_6^2+\\
&2y_3y_4y_6^2+y_2y_3y_4y_6^2+y_4^2y_6^2+y_2y_4^2y_6^2+2y_3y_4^2y_6^2+2y_2y_3y_4^2y_6^2+\\
&y_3^2y_4^2y_6^2+y_2y_3^2y_4^2y_6^2+y_2y_3y_4^2y_6^2y_8+y_2y_3^2y_4^2y_6^2y_8+y_1y_2y_3^2y_4^2y_6^2y_8))
\end{aligned}
$$

Remark that the mapping torus in this example is the complement of the figure-eight knot.

Step 6. For the equation of fixed points, we choose a solution. We can show that, for a pesudo-Anosov mapping class φ, we have always a solution of the equation of fixed points [KT].

In the case of LR, we have a solution:

$$
\begin{aligned}
y_1 =&1\\
y_2 =&1\\
y_3 =&\frac{1}{2}\left(-1+\sqrt{-3}\right)\\
y_4 =&1\\
y_5 =&\frac{1}{2}\left(-1-\sqrt{-3}\right)\\
y_6 =&\frac{1}{2}\left(-1+\sqrt{-3}\right)\\
y_7 =&\frac{1}{2}\left(-1-\sqrt{-3}\right)\\
y_8 =&1
\end{aligned}
$$

Step 7. For a mapping class φ of a surface S, we evaluate at a solution $y = y^{\mathrm{sol}}$ the characteristic polynomial $\Delta(t)$ of the Jacobi matrix of $\varphi^{\#}$:

$$
\Delta(t) = \det\left(t - \left(\frac{\partial \varphi^{\#}(y_i)}{\partial y_j}\right)\right)\Bigg|_{y=y^{\mathrm{sol}}}
$$

In the case of the mapping class LR and the solution $y = y^{\mathrm{sol}}$ in step 6, we have the following polynomial.

$$
\Delta(t) = (-1+t)^2(1-5t+t^2)(1-9t+44t^2-9t^3+t^4)
$$

Step 8. Last, we consider the following limit:

$$\tau := \lim_{t\to 1} \frac{\Delta(t)}{(t-1)^{2p}},$$

where p is the number of punctures on S. One of our results is that this number is identified with $PGL_3(\mathbb{C})$-torsion with the adjoint representation of the representation which corresponds the solution, except sign ambiguity [KT]. A $PGL_3(\mathbb{C})$-torsion is a generalization of a Porti's torsion for a $PSL_2(\mathbb{C})$-representation [P]. In the case of LR, we have:

$$\begin{aligned}\tau &= \lim_{t\to 1} \frac{\Delta(t)}{(t-1)^2} \\ &= -3\times 28 \\ &= -84\end{aligned}$$

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