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Yokota type invariants derived from Costantino-Murakami's Invariants

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This is a survey of [4]. In this note, we define invariants for colored oriented spatial graphs from Costantino-Murakami's invariants ([2]) following the method defining Yokota's invariants ([7]). We call these invariants Yokota type invariants. Then we propose a volume conjecture between the Yokota type invariants and volumes of hyperbolic polyhedra.

1 Knots and spatial graphs

In this section, we quickly review knots, spatial graphs, the Reidemeister moves for them, the volume conjecture and Yokota's invariants.

Definition 1.1. A *knot* is an embedding of a circle into the three-sphere. A *spatial graph* (a *knotted graph*) is an embedding of a graph (V, E) into the three-sphere. Where V is a set of vertices and E is a set of edges. A *plane graph* is a spatial graph which can be embedded to the two-sphere.

We treat them through diagrams derived by regular projections to the two-sphere.

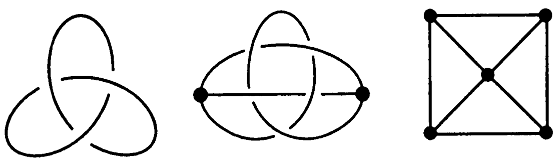
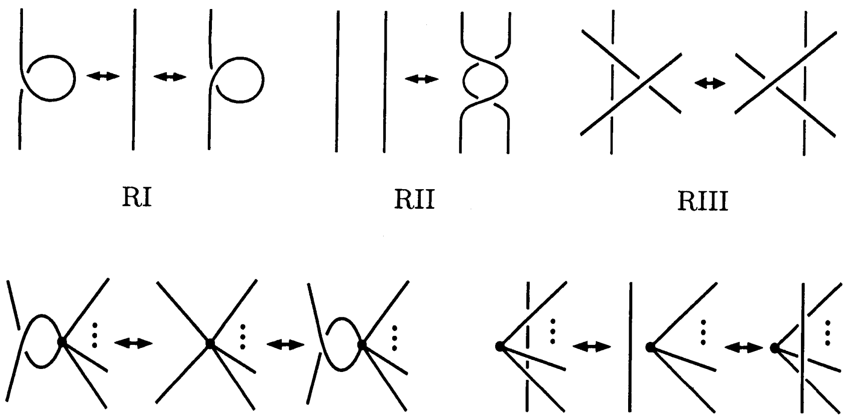


Figure 1: Diagrams of a knot, a spatial graph and a plane graph.

There are 5 local moves called *Reidemeister moves* for knot and spatial graph diagrams (Figure 2). Here RIV and RV moves appear only for spatial graph diagrams.

Theorem 1.2. *Two diagrams represent the same knot or spatial graph if and only if the two diagrams are transformed to each other by a sequence of the Reidemeister moves.*



RI RII RIII

RIV RV

Figure 2: Reidemeister moves.

From Theorem 1.2, values or properties derived from diagrams of knots (resp. spatial graphs) that do not change under the Reidemeister moves are *invariants* of knots (resp. spatial graphs).

The *N-th colored Jones polynomial* $J_N(\,\cdot\,;q)$ is an invariant for knots defined through an N-dimensional representation of quantum group $\mathcal{U}_q(sl_2)$. When $N=2$, it corresponds to the *Jones polynomial.*

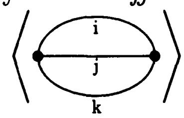
Conjecture 1.3 (Volume conjecture [3] [5]). *Let K be a hyperbolic knot in S^3 (i.e. the complement of K has a complete hyperbolic structure). Then the value of the colored Jones polynomial of K at N-th root of unity* $\exp(2\pi\sqrt{-1}/N)$ *in the next form converges to the hyperbolic volume of complement of K.*

$$2\pi \lim_{N\to\infty} \frac{\log|J_N(K;\exp(2\pi\sqrt{-1}/N))|}{N} = \mathrm{Vol}(S^3\setminus K),$$

where Vol$(\cdot\,)$ is the hyperbolic volume.

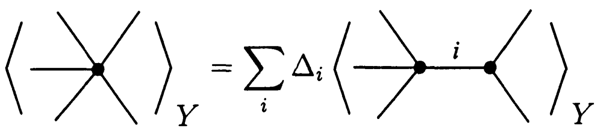
The volume conjecture is generalized for any knots by using the simplicial volume (Gromov norm) instead of the hyperbolic volume ([5]).

In [7], Y. Yokota defined invariants for colored spatial graphs. A *color* is a non-negative integer added to the graph edges. The colors correspond to the dimension of the representation of $\mathcal{U}_q(sl_2)$. A triple (i,j,k) of colors is called *admissible* (for this representation) if they satisfy $|i-j|\le k\le i+j$ and $i+j+k\in 2\mathbb{Z}$. Yokota's invariants are first defined for trivalent graphs then generalized for any graphs.

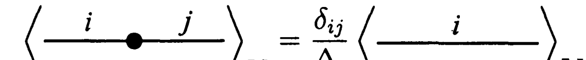
Definition 1.4 (Yokota's invariants). Let Γ be a trivalent spatial graph. We add colors to edges of Γ such that at each vertex the colors of three edges are admissible. The diagram D of admissibly colored Γ can be estimated by the *Kauffman bracket* $\langle\cdot\rangle$ (see [7] for details). We put $\Delta_a =$  and $\theta(i,j,k) =$  for admissible triple (i,j,k). Yokota's invariants $\langle\cdot\rangle_Y$ for colored trivalent graph Γ are defined as

$$\langle\Gamma\rangle_Y = \langle D\rangle\langle\overline{D}\rangle \Big/ \prod_{\substack{\text{Triples of} \\ \text{colors at vertices}}} \theta(i,j,k),$$

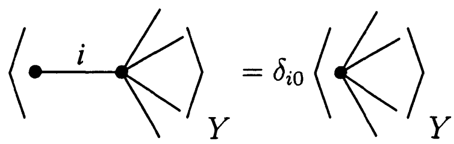
where $\overline{\cdot}$ means the mirror image of a diagram. Yokota's invariants are generalized for spatial graphs which have 1, 2, n-valent $(4 \leq n)$ vertices with the next relations at vertices.



for an n-valent vertex $(4 \leq n)$ where color i moves all admissible colors for the right-hand side diagram. This relation is independent of the ways extending the edge.



for a 2-valent vertex, and



for an 1-valent vertex.

2 Costantino-Murakami's invariants

In this section we review the invariants for colored oriented framed trivalent spatial graphs defined by F. Costantino and J. Murakami in [2]. Here trivalent graphs may have circle components. These invariants have properties of the volume conjecture between tetrahedron graphs and hyperbolic volumes of tetrahedra.

2.1 Definition of Costantino-Murakami's invariants

Costantino-Murakami's invariants are defined through the non-integral representations of the quantum group $\mathcal{U}_q(sl_2)$ where q is at a root of unity. $\mathcal{U}_q(sl_2)$ is the Hopf algebra defined as follows.

Generators: E, F, K, K^{-1}.

Relations: $[E,F]=\dfrac{K^2-K^{-2}}{q-q^{-1}}$, $KE=qEK$, $KF=q^{-1}FK$, $KK^{-1}=K^{-1}K=1$.

Structure of the Hopf algebra:

$$\Delta(E)=E\otimes K+K^{-1}\otimes E,\ \Delta(F)=F\otimes K+K^{-1}\otimes F,\ \Delta(K^{\pm 1})=K^{\pm 1}\otimes K^{\pm 1},$$

$$S(E)=-qE,\ S(F)=-q^{-1}F,\ S(K)=K^{-1},$$

$$\epsilon(E)=\epsilon(F)=0,\ \epsilon(K)=1,$$

where Δ is the coproduct, S is the antipode and ϵ is the counit.

Let $n\in\mathbb{N}$ and ξ_n be a $2n$-th primitive root of unity $\exp(\pi\sqrt{-1}/n)$. We prepare notations:

$$\{a\}=\xi_n^a-\xi_n^{-a}\ (a\in\mathbb{C}),\quad [a]=\frac{\{a\}}{\{1\}},\quad \{k\}!=\prod_{j=1}^{k}\{j\}\ (k\in\mathbb{N}),$$

$$\begin{bmatrix} a \\ b \end{bmatrix}=\prod_{j=0}^{a-b-1}\frac{\{a-j\}}{\{a-b-j\}}\ (a,b\in\mathbb{C}\text{ s.t. } a-b\in\{0,1,\dots,n-1\}).$$

For each non-half-integer complex number $a\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}$, there is a simple n-dimensional representation of $\mathcal{U}_{\xi_n}(sl_2)$ on a representation space V^a which is an n-dimensional vector space whose basis is $\{e_0^a,\dots,e_{n-1}^a\}$. The actions of this representation are given by

$$E(e_j^a)=[j]e_{j-1}^a,\ F(e_j^a)=[2a-j]e_{j+1}^a,\ K(e_j^a)=\xi_n^{a-j}e_j^a\quad (e_{-1}^a=e_n^a=0).$$

For two complex numbers $a,b\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}$, the two representations related to them are isomorphic if and only if $a-b\in 2n\mathbb{Z}$. The dual representation on $(V^a)^*$ is isomorphic to the representation on V^{n-1-a}.

Let Γ be an oriented framed trivalent graph. We add non-half-integer complex numbers to each oriented edge of Γ (Figure 3 left). The numbers are called *colors*. In the calculations of Costantino-Murakami's invariants (see below), an a colored downward edge corresponds to the representation space V^a and an a colored upward edge corresponds to the dual space $(V^a)^*$. From the isomorphism between the representations on $(V^a)^*$ and V^{n-1-a}, we can identify an a colored edge and $n-1-a$ colored opposite direction edge (Figure 3 right). We put $\overline{a}=n-1-a$. We define admissible condition for the colors of these representations.

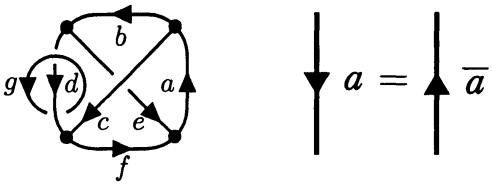
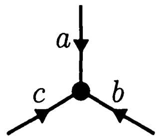


Figure 3: A colored oriented trivalent graph (left) and identification of colored edges (right).

Definition 2.1 (Admissible condition). If three colors a, b, c of edges at a vertex satisfy the next condition, the triple (a, b, c) of the colors is called *admissible.*

$$a + b + c \in \{n-1, n, \ldots, 2n-2\},$$



here the orientations of the three edges are all toward the vertex.

If three colors of a vertex are admissible, we can give a representation canonically at the vertex. From now on, unless otherwise noted, variable colors in summations $\sum$ move all admissible colors.

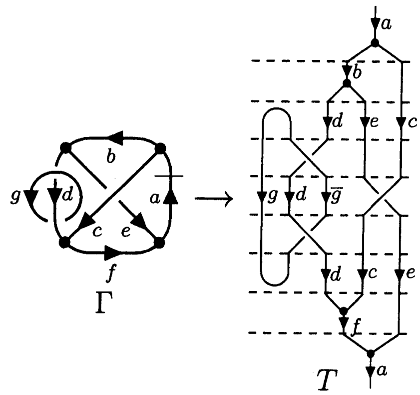
To calculate Costantino-Murakami's invariants for admissibly colored oriented framed trivalent spatial graph Γ, we cut an edge of Γ and make an $(1,1)$-tangle diagram T so that the boundary edges of T are oriented downward. Then we cut T to slices so that in each slice there is just one singular point, which is a maximal, minimal, crossing or vertex point (Figure 4 left). A boundary of an a colored strand in a slice is related to the representation space V^a if the strand is oriented downward and to $(V^a)^*$ or V^{n-1-a} if the strand is oriented upward. From the representation of $\mathcal{U}_{\xi_n}(sl_2)$, each slice is regarded as a map from the tensor product of representation spaces corresponding to the bottom boundary of strands in the slice to the one corresponding to the upper boundary (Figure 4 right). Here the maps are defined as follows.

$${}_a^bR(e_i^a \otimes e_j^b) = \sum_m \{m\}! \xi_n^{2(a-i)(b-j)-m(a-b-i+j)-\frac{m(m+1)}{2}} \begin{bmatrix} i \\ i-m \end{bmatrix} \begin{bmatrix} 2b-j \\ 2b-j-m \end{bmatrix} e_{j+m}^b \otimes e_{i-m}^a,$$

where $m \in [0, \min(i, n-j-1)] \cap \mathbb{N}$.

$$\cap_{a,b}(e_i^a \otimes e_j^b) = \delta_{b,n-1-a}\delta_{i,n-1-j}\xi_n^{-(a-i)(n-1)}, \quad \cup_{a,b} = \delta_{b,n-1-a}\sum_{i=0}^{n-1} \xi_n^{-(a-i)(n-1)} e_i^a \otimes e_{n-1-i}^b.$$

$$Y_c^{a,b}(e_k^c) = \sum_{\substack{i,j: \\ i+j-k \\ =a+b-c}} C_{i,j,k}^{a,b,c} e_i^a \otimes e_j^b, \quad Y_{a,b}^c(e_i^a \otimes e_j^b) = \sum_{\substack{k: \\ i+j-k \\ =a+b-c}} C_{n-1-j,n-1-i,n-1-k}^{n-1-b,n-1-a,n-1-c} e_k^c,$$



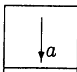
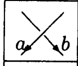
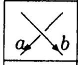
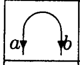
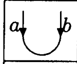
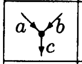
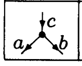
	$id_a : V^a \to V^a$
	${}^b_a R : V^a \otimes V^b \to V^b \otimes V^a$
	${}^b_a(R^{-1}) : V^a \otimes V^b \to V^b \otimes V^a$
	$\cap_{a,b} : V^a \otimes V^b \to \mathbb{C}$
	$\cup_{a,b} : \mathbb{C} \to V^a \otimes V^b$
	$Y^{a,b}_c : V^c \to V^a \otimes V^b$
	$Y^c_{a,b} : V^a \otimes V^b \to V^c$

Figure 4: Γ and T (left), the maps related to the singular points (right).

where

$$C^{a,b,c}_{i,j,k} = \sqrt{-1}^{\,c-a-b}(-1)^{j-k}\xi_n^{\frac{j(2b-j+1)-i(2a-i+1)}{2}} \begin{bmatrix} 2c \\ 2c-k \end{bmatrix}^{-1} \begin{bmatrix} 2c \\ a+b+c-(n-1) \end{bmatrix}$$
$$\sum_{z+w=k} (-1)^z \xi_n^{\frac{(2z-k)(2c-k+1)}{2}} \begin{bmatrix} a+b-c \\ i-z \end{bmatrix} \begin{bmatrix} 2a-i+z \\ 2a-i \end{bmatrix} \begin{bmatrix} 2b-j+w \\ 2b-j \end{bmatrix}.$$

Then we have a map $op(T)$ from the representation space related to the color, say a, of the cutting edge of Γ to itself by composing the maps related to the slices of T; $op(T) : V^a \to V^a$. By Schur's lemma, $op(T)$ is equal to a scalar $\lambda(T)(\in \mathbb{C})$ multiplied identity $\lambda(T)id_a$. Costantino-Murakami's invariants $\langle \cdot \rangle_{CM}$ of Γ are defined by

$$\langle \Gamma \rangle_{CM} = \lambda(T) \begin{bmatrix} 2a+n \\ 2a+1 \end{bmatrix}^{-1}.$$

Theorem 2.2 ([2]). *For colored oriented framed trivalent spatial graph Γ, the value $\langle \Gamma \rangle_{CM}$ does not depend on the choice of the cutting edge and (1,1)-tangle diagram T. Therefore $\langle \cdot \rangle_{CM}$ is an invariants for colored oriented framed trivalent spatial graph.*

Remark 2.3. 1. For a half-integer $a \in \frac{1}{2}\mathbb{Z}$,

$$\begin{bmatrix} 2a+n \\ 2a+1 \end{bmatrix} = 0.$$

Hence for half-integer colors, Costantino-Murakami's invariants may become infinity.

2. If graphs are restricted to links, Costantino-Murakami's invariants correspond to the Akutsu-Deguchi-Ohtsuki (colored Alexander) invariants. For the Akutsu-Deguchi-Ohtsuki invariants, the properties of the volume conjecture between links and cone manifolds whose singular sets are the links are observed in [1] and [6].

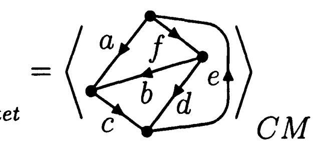
2.2 Relations of Costantino-Murakami's invariants

In this subsection, we review relations of Costantino-Murakami's invariants. Using the relations, Costantino-Murakami's invariants are calculated axiomatically.

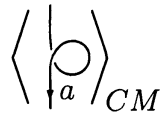
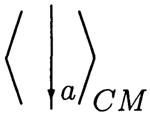
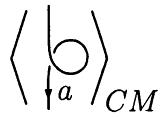
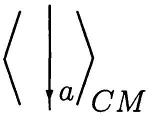
The $6j$*-symbols* $\{\cdot\}$ are the values determined by 6 colors. The $6j$-symbols are defined as coefficients of the relation (3) below. For $a,b,c,d,e,f \in \mathbb{C}\setminus\frac{1}{2}\mathbb{Z}$ and $a+b-c, a+f-e, b+d-f, d+c-e \in \mathbb{Z}$, the $6j$-symbols are calculated by the next formula.

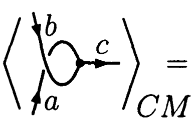
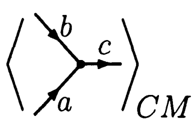
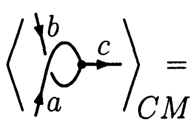
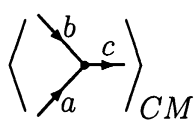
$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{n-1+B_{afe}} \begin{bmatrix} 2f+n \\ 2f+1 \end{bmatrix}^{-1} \frac{\{B_{dce}\}!\{B_{abc}\}!}{\{B_{bdf}\}!\{B_{afe}\}!} \begin{bmatrix} 2c \\ A_{abc}+1-n \end{bmatrix} \begin{bmatrix} 2c \\ B_{ced} \end{bmatrix}^{-1}$$
$$\times \sum_{z=s}^{S} (-1)^z \begin{bmatrix} A_{afe}+1 \\ 2e+z+1 \end{bmatrix} \begin{bmatrix} B_{aef}+z \\ B_{aef} \end{bmatrix} \begin{bmatrix} B_{bfd}+B_{dce}-z \\ B_{bfd} \end{bmatrix} \begin{bmatrix} B_{dec}+z \\ B_{dfb} \end{bmatrix},$$

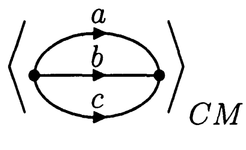
where $s = \max(0, -B_{bdf}+B_{dce})$, $S = \min(B_{dce}, B_{afe})$, $A_{xyz} = x+y+z$, $B_{xyz} = x+y-z$. Costantino-Murakami's invariants of tetrahedron graphs are described by using the $6j$-symbols and we denote them by $\{\cdot\}_{tet}$.

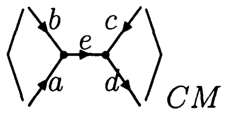
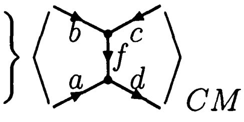
$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_{tet} =$$  $$= \begin{bmatrix} 2f+n \\ 2f+1 \end{bmatrix} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}.$$

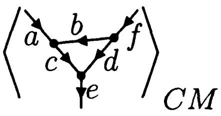
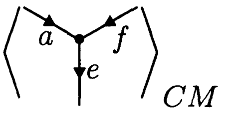
It was proved that $\{\cdot\}_{tet}$ is well-defined for half-integer colors. The next local relations hold for Costantino-Murakami's invariants.

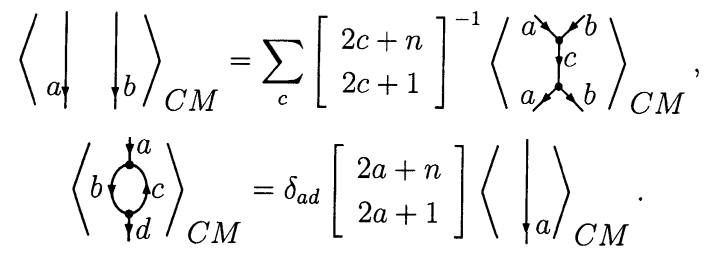
 $= \xi_n^{-2a\bar{a}}$ ,  $= \xi_n^{2a\bar{a}}$ , (1)

 $= \xi_n^{a\bar{a}+b\bar{b}-c\bar{c}}$ ,  $= \xi_n^{-a\bar{a}-b\bar{b}+c\bar{c}}$ , (2)

 $= 1,$

 $= \sum_f \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}$ , (3)

 $= \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_{tet}$ , (4)



2.3 Properties of the volume conjecture

Costantino-Murakami's invariants have properties of the volume conjecture between tetrahedron graphs and hyperbolic volumes of *ideal* and *truncated* tetrahedra (Figure 5 left). Shapes of tetrahedra in the hyperbolic space are determined by their 6 dihedral angles of edges. The ideal tetrahedra are the hyperbolic tetrahedra whose vertices are all at infinity points of hyperbolic space. The two dihedral angles of opposite edges of ideal tetrahedra are equal. Therefore the shapes of ideal tetrahedra are determined by 3 dihedral angles α, β, γ. It is known that they satisfy $\alpha + \beta + \gamma = \pi$. In the Klein model of hyperbolic space, we can consider the tetrahedron whose vertices are "outside" the hyperbolic space (Figure 5 right). For each vertex of this tetrahedron, there is just one geodesic surface which intersects perpendicularly to each of three adjacent faces of the vertex. Cutting the tetrahedron by the surfaces at every vertex, we have a finite polyhedron in the hyperbolic space. This polyhedron is called truncated tetrahedron. Three dihedral angles α, β, γ of edges adjacent to an "outside" vertex satisfy $\alpha+\beta+\gamma < \pi$.

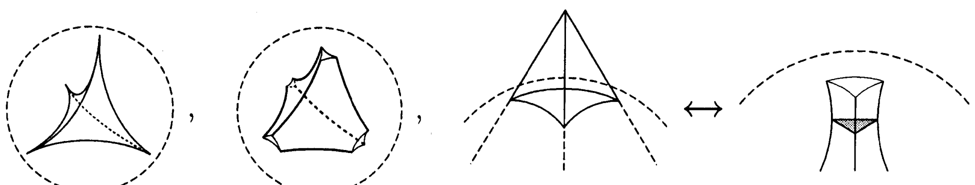


Figure 5: Images of ideal and truncated tetrahedra(left), a vertex outside the hyperbolic space and a cutting surface (gray) (right).

Theorem 2.4 ([2]). *Let S be a hyperbolic tetrahedron, $\theta_a, \cdots, \theta_f$ be dihedral angles of S and $a_n, \cdots, f_n$ be sequences of integer colors such that $\lim_{n\to\infty} \frac{2\pi a_n}{n} = \pi - \theta_a, \cdots,$ $\lim_{n\to\infty} \frac{2\pi f_n}{n} = \pi - \theta_f$. Costantino-Murakami's invariants of tetrahedron graphs $\{\cdot\}_{tet}$ with corresponding colors to the dihedral angles of S hold the next equations. If S is ideal* *(i.e. dihedral angles of opposite edges are equal),*

$$\begin{aligned}\mathrm{Vol}(S) &= \lim_{n\to\infty}\frac{\pi}{n}\log\left((-1)^{n-1}\left\{\begin{matrix} a_n & b_n & c_n \\ a_n & b_n & c_n \end{matrix}\right\}_{tet}\right) \\ &= \lim_{n\to\infty}\frac{\pi}{n}\log\left((-1)^{n-1}\left\{\begin{matrix} \overline{a_n} & \overline{b_n} & \overline{c_n} \\ \overline{a_n} & \overline{b_n} & \overline{c_n} \end{matrix}\right\}_{tet}\right).\end{aligned}$$

If S is a truncated tetrahedron,

$$\mathrm{Vol}(S) = \lim_{n\to\infty}\frac{\pi}{2n}\log\left(\left\{\begin{matrix} a_n & b_n & c_n \\ d_n & e_n & f_n \end{matrix}\right\}_{tet}\left\{\begin{matrix} \overline{a_n} & \overline{b_n} & \overline{c_n} \\ \overline{d_n} & \overline{e_n} & \overline{f_n} \end{matrix}\right\}_{tet}\right). \tag{5}$$

3 Yokota type invariants and numerical calculations

In this section, we define Yokota type invariants for colored oriented spatial graphs with more than or equal to 3-valent vertices from Costantino-Murakami's invariants. We also propose a volume conjecture for the Yokota type invariants between plane graphs and hyperbolic convex tetrahedra. By numerical calculations, we observe regularities of the Yokota type invariants for integer colors and asymptotic behaviors of them.

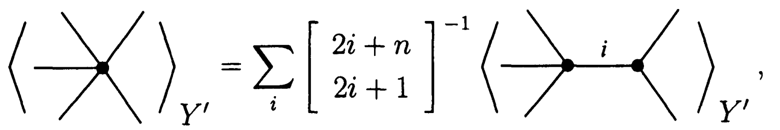
3.1 Definition of Yokota type invariants

Using the similar way to define Yokota's invariants, Costantino-Murakami's invariants are generalized to invariants for non-framed colored oriented spatial graphs with more than or equal to 3-valent vertices. Like the Yokota's invariants, these invariants are first defined for trivalent graphs then generalized for graphs with more than 3-valrent vertices.

Definition 3.1 (Yokota type invariants). Let Γ be admissibly colored oriented trivalent graph and D be its diagram. *Yokota type invariants* $\langle\cdot\rangle_{Y'}$ are defined from Costantino-Murakami's invariants by the next relation.

$$\langle\Gamma\rangle_{Y'} = \langle D\rangle_{CM}\,\langle\overline{D}^{r}\rangle_{CM},$$

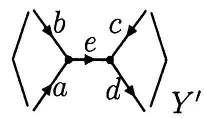
where $\overline{\cdot}$ means the mirror image, $\cdot^{r}$ means reversing orientations of all edges. Using the next relation, we define the Yokota type invariants for graphs with more than 3-valent vertices.

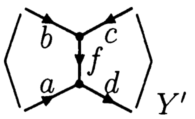


where the surrounding edges have the same colors and orientations on the both sides and the orientation of the i colored edge is arbitrary.

Theorem 3.2. *The values of the Yokota type invariants are independent of the choice of the diagrams to calculate and of the ways to extend edges at more than 3-valent vertices.*

Proof. This is a sketch of proof. The invariance of the Yokota type invariants for RII, RIII and RV moves come from that of Costantino-Murakami's invariants. The invariance for RI and RIV moves come from direct calculations using the relations (1) and (2) respectively. The next equation holds for the Yokota type invariants.

$$\sum_e \begin{bmatrix} 2e+n \\ 2e+1 \end{bmatrix}^{-1}$$


$$= \sum_f \begin{bmatrix} 2f+n \\ 2f+1 \end{bmatrix}^{-1}$$


By extending edges at a more than 3-valent vertex recursively, it changes to a trivalent tree. The shape of the tree depends on the ways to extend the edges. The values of the result graphs are, however, the same because the trees are transformed to each other by a sequence of the moves in the equation. □

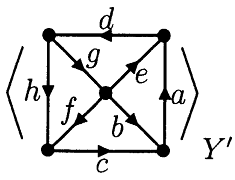
In Theorem 2.4, the value inside log(·) of Equation (5) is the value of the Yokota type invariants for tetrahedron graphs. Using the Yokota type invariants, we conjecture the extension of Theorem 2.4.

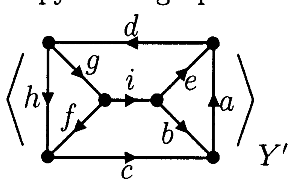
Conjecture 3.3. *Let* Γ *be a plane graph and* S_Γ *be a hyperbolic convex polyhedron which is bounded by* Γ*. If sequences of integer colors of* Γ *are taken as in Theorem 2.4 for corresponding dihedral angles of* S_Γ*, then*

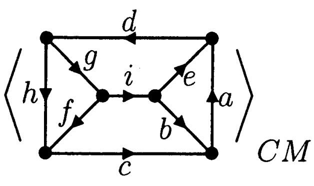
$$\mathrm{Vol}(S_\Gamma) = \lim_{n\to\infty} \frac{\pi}{2n} \log\left(\langle\Gamma\rangle_{Y'}\right).$$

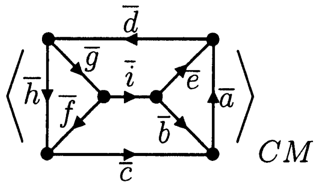
3.2 Numerical calculations

We show the numerical calculations of the Yokota type invariants for square pyramid graphs and observe regularities of the Yokota type invariants for integer colors and asymptotic behaviors. The value of the square pyramid graph is calculated as follows.



$$= \sum_i \begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1}$$


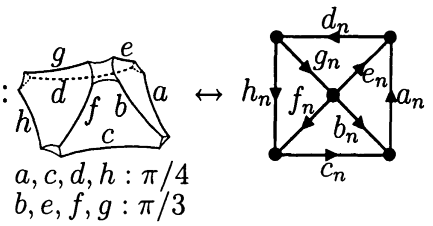
$$= \sum_i \begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1}$$




$$= \sum_i \begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1} \begin{Bmatrix} a & e & d \\ i & c & b \end{Bmatrix}_{tet} \begin{Bmatrix} d & g & h \\ f & c & i \end{Bmatrix}_{tet} \begin{Bmatrix} \bar{a} & \bar{e} & \bar{d} \\ \bar{i} & \bar{c} & \bar{b} \end{Bmatrix}_{tet} \begin{Bmatrix} \bar{d} & \bar{g} & \bar{h} \\ \bar{f} & \bar{c} & \bar{i} \end{Bmatrix}_{tet},$$

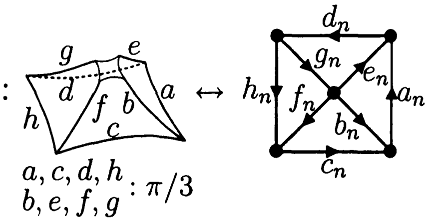
where the third equation uses the relation (4). We consider colored square pyramid graphs $\Gamma_{1,n}$ and $\Gamma_{2,n}$ corresponding to the following square pyramids.

$\Gamma_{1,n}$:



$$\begin{cases} a_n = 3n/8\,(+\varepsilon) & b_n = n/3\ \ (+2\varepsilon) \\ c_n = 3n/8\,(+3\varepsilon) & d_n = 3n/8\,(+4\varepsilon) \\ e_n = n/3\ \ (+3\varepsilon) & f_n = n/3\ \ (-6\varepsilon) \\ g_n = n/3\ \ (+5\varepsilon) & h_n = 3n/8\,(+9\varepsilon) \end{cases},$$

$\Gamma_{2,n}$:



$$\begin{cases} a_n = n/3\,(+\varepsilon) & b_n = n/3\,(+2\varepsilon) \\ c_n = n/3\,(+3\varepsilon) & d_n = n/3\,(+4\varepsilon) \\ e_n = n/3\,(+3\varepsilon) & f_n = n/3\,(-6\varepsilon) \\ g_n = n/3\,(+5\varepsilon) & h_n = n/3\,(+9\varepsilon) \end{cases},$$

where all vertices of the square pyramid of $\Gamma_{1,n}$ are truncated and the 4 bottom vertices of the square pyramid of $\Gamma_{2,n}$ are ideal vertices. The sequences of colors become integers by taking appropriate integer n. For the integral colors, the formula of the square pyramid graph looks diverging to infinity because of the coefficient $\begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1}$ and even the regularity at the integer colors of the square pyramid formula is not proved yet. When we do numerical calculations, we slightly differ the integer colors using small real number ε preserving admissible conditions.

Before the numerical calculations, we show algebraic computation of the formula of the square pyramids. We calculated the formula as a rational function of q by not substituting ξ_n to q (i.e. defining $\{a\} = q^a - q^{-a}$) and reduced the numerator and the denominator by common factors then substituted $q = \xi_n$. The results are as follows.

$\Gamma_{1,n} : n = 24, \{a,b,c,d,e,f,g,h\} = \{9,8,9,9,8,8,8,9\}$,

$$\frac{2702553921462776104873773262573943868288}{4144454025633775}.$$

$\Gamma_{2,n} : n = 12, \{a,b,c,d,e,f,g,h\} = \{4,4,4,4,4,4,4,4\}$,

$$\frac{9478552239158866648400}{206606306907}.$$

$\Gamma_{2,n} : n = 24, \{a, b, c, d, e, f, g, h\} = \{8, 8, 8, 8, 8, 8, 8, 8\}$,

$$\frac{18417276716781939060567652343662582870272\,00}{1974379602081567900828\,7}.$$

The values are finite and the regularities for these n's are shown. However, the above computation for large n takes too long time. We did numerical calculations at $\varepsilon = 0.0000001$ and observed asymptotic behavior of the formula for $\Gamma_{1,n}$ and $\Gamma_{2,n}$. The results are in Table 1, where we calculated to 9th decimal places. We took the absolute values of the Yokota type invariants to kill the multivalency of log because in the result of the calculations the value of Yokota type invariant for each n was a negative real number. We need more discussions here. The results seem to tend to the volume of each square pyramid. These near-integer colors calculations also show that the formula may have regularities at integer colors. The results are not so strong supporting evidences for Conjecture 3.3. We propose the next problem.

n	$\pi/2n * \log(\lvert\langle \Gamma_{1,n} \rangle_{Y'}\rvert)$	n	$\pi/2n * \log(\lvert\langle \Gamma_{2,n} \rangle_{Y'}\rvert)$
24	3.440464669	24	2.597872961
48	3.653713460	48	2.603015626
72	3.741391100	72	2.594719877
120	3.824413802	120	2.581962148
240	3.900859202	240	2.566523650
600	3.959111190	600	2.552634909
900	3.986845579	900	2.548604997
1200	3.983212953	1200	2.546357950
Vol.	4.01536	Vol.	2.53735

Table 1: Numerical calculations at $\varepsilon = 0.0000001$

Problem 3.4. *Prove Conjecture 3.3 for some polyhedra which have more than 3-valent vertices.*

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