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$(1,1)$-BRIDGE SPLITTINGS WITH DISTANCE EXECTLY n

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1. Introduction

Hempel [4] introduced the concept of *distance* of a Heegaard splitting by using curve complex, and showed that there exist arbitrarily high distance Heegaard splittings for closed 3-manifolds by using a construction of Kobayashi [8]. Abrams and Schleimer [1] gave a sharper estimation for the distance of the Heegaard splitting given in [4] by using the result of Masur and Minsky [9], and Evans [3] gave a combinatorial method to construct Heegaard splittings of high distance.

On the other hand, the above concept and results have been extended to *bridge splittings* for links in closed 3-manifolds (for definitions, see subsection 2.3), and have been studied by several authors. For example, Saito [11] showed that for any closed 3-manifold admitting a Heegaard splitting of genus one, there is a knot in the manifold with a $(1,1)$-*bridge splitting* of arbitrary high distance. Recently, Blair, Tomova and Yoshizawa [2] showed that for given integers b, c, g, and n, there exists a manifold M containing a c-component link L so that (M, L) admits a (g, b)-bridge splitting of distance at least n. Moreover, Ichihara and Saito [7] showed that for any given closed orientable 3-manifold M with a Heegaard surface of genus g, and for any positive integers b and n, there exists a knot K in M which admits a (g, b)-bridge splitting of distance greater than n.

In [6], we showed that there exists a Heegaard splitting of a closed orientable 3-manifold with distance exactly n for each positive integer n. To prove this, we gave a method to extend a geodesic in the curve complex of a closed orientable surface to a geodesic with given length, and constructed a concrete example (for details, see [6, Section 4]).

In this paper, we apply the idea of [6, Section 4] to construct a geodesic of any given length in the curve complex of a twice-punctured torus, and show the following.

Theorem 1.1. *For any integer $n>0$, there exists a (1,1)-bridge splitting with distance exactly n.*

2. Definitions and notations

2.1. **Curve complexes.** Let S be an orientable surface with genus g, b boundary components and p punctures. A simple closed curve in S is *essential* if it does not bound a disk or a once-punctured disk in S and is not parallel to a component of ∂S. An arc properly embedded in S is *essential* if it does not co-bound a disk in S together with an arc on ∂S. We say that S is *sporadic* if $g=0, b+p\leq 4$ or $g=1, b+p\leq 1$.

Except in sporadic cases, the *curve complex* $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in S. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. We assume that S is a torus, a torus with one boundary component, or a sphere with 4 boundary components since, otherwise, there are no essential simple closed curves in

S. When S is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of $k+1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by curves in S which mutually intersect exactly once (resp. twice). The *arc-and-curve complex* $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{AC}(S)$ if they can be realized by disjoint arcs or simple closed curves in S.

We can define the *distance* between two vertices in the curve complex $\mathcal{C}(S)$ to be the minimal number of 1-simplexes of a simplicial path in $\mathcal{C}(S)$ joining the two vertices. We denote by $d_{\mathcal{C}(S)}(x,y)$, or $d_S(x,y)$ in brief, the distance in $\mathcal{C}(S)$ between the vertices x and y. For subsets X and Y of the vertices of $\mathcal{C}(S)$, we define $\mathrm{diam}_S(X,Y) = \mathrm{diam}_S(X \cup Y)$. Similarly, we can define the distance $d_{\mathcal{AC}(S)}(x,y)$ and $\mathrm{diam}_{\mathcal{AC}(S)}(X,Y)$. We denote by $[a_0, a_1, \ldots, a_n]$ the path in $\mathcal{C}(S)$ with vertices $a_0, a_1, \ldots, a_n$ such that $a_i \cap a_{i+1} = \emptyset$ $(i = 0, 1, \ldots, n-1)$. We call a path $[a_0, a_1, \ldots, a_n]$ a *geodesic* if $n = d_s(a_0, a_n)$.

2.2. **Subsurface projections.** Let $\mathcal{P}(Y)$ denote the power set of a set Y. Suppose that X is an essential subsurface of S that contains an essential simple closed curve. We call the composition $\pi_0 \circ \pi_A$ of maps $\pi_A : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X))$ and $\pi_0 : \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X))$ a *subsurface projection* if they satisfy the following (see Figure 1): for a vertex α, take a representative α so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_A(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union for all $i = 1, \ldots, n$ of the set of all isotopy classes of the components of $\partial N(\alpha_i \cup \partial X)$ which are essential in X, where $N(\alpha_i \cup \partial X)$ is a regular neighborhood of $\alpha_i \cup \partial X$ in X.

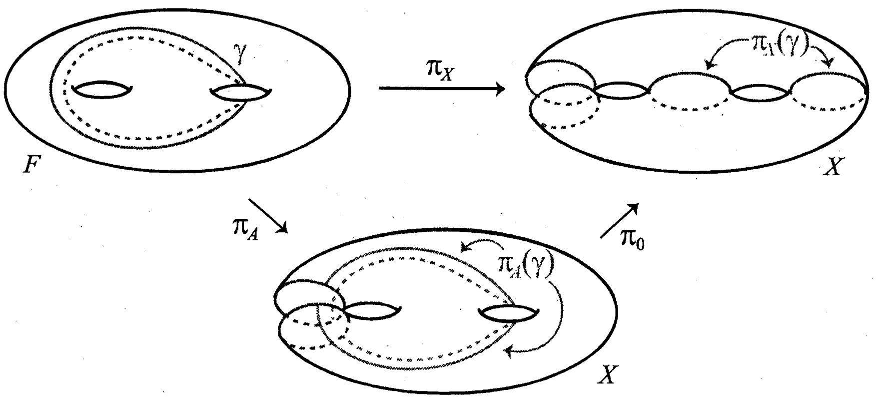

FIGURE 1

2.3. (g,b)**-bridge splittings.** Let H be a genus-$g(\geq 0)$ handlebody. We say that a set of n arcs $\{t_1, \ldots, t_n\}$ properly embedded in H is a *set of trivial n arcs* if $t_1 \cup \cdots \cup t_n$ is parallel to ∂H. Let H be a handlebody and $\tau = \{t_1, \ldots, t_n\}$ a set of trivial n arcs in H. Then τ can be isotoped in H so that the projection from $\partial H \times [0,1)$ to $[0,1)$ has exactly one critical point in each t_i.

It is well known that every closed orientable 3-manifold M has a genus-g Heegaard splitting for some $g(\geq 0)$, i.e., $M = V_1 \cup_P V_2$, where V_1 and V_2 are genus-g handlebodies such that $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = P$. Let L be a link in M. We say that $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is a (g, b)-*bridge splitting* (or (g,b)-*splitting* for short) for the pair (M, L) if P separates (M, L) into two components (V_1, τ_1) and (V_2, τ_2) where $\tau_1 = L \cap V_1$ (resp. $\tau_2 = L \cap V_2$) is a set of trivial b arcs in A (resp. B). Then we say that P is a (g, b)-*bridge surface* (or a *bridge surface* for short). It is known that each (M, L) has a (g, b)-bridge splitting for some g and b. (For a detailed discussion, see [5, Lemma 2.1]).

For $i = 1$ or 2, $\mathcal{D}(V_i)$ denotes the subset of $\mathcal{C}^0(\partial V_i - \tau_i)$ consisting of the vertices with representatives bounding disks in $V_i - \tau_i$. Then the *(Hempel) distance* of $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is defined by $d_{P'}(\mathcal{D}(V_1), \mathcal{D}(V_2))$, where $P' = \partial V_1 - \tau_1 = \partial V_2 - \tau_2$.

3. Extending geodesics

Let F be a twice-punctured torus. The following two propositions can be shown by using arguments in the proof of [6, Propositions 4.1 and 4.4].

Proposition 3.1 (cf. [6, Proposition 4.1]). *For an integer $n(\geq 4)$, let $[\alpha_0, \alpha_1, \dots, \alpha_n]$ be a path in $\mathcal{C}(F)$ satisfying the following.*

(H1) *$[\alpha_0, \dots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $\mathcal{C}(F)$,*

(H2) *$\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \geq 4n$, where X_{n-2} is the component of $\mathrm{Cl}(F \setminus N(\alpha_{n-2}))$ that contains an essential simple closed curve.*

Then $[\alpha_0, \alpha_1, \dots, \alpha_n]$ is a geodesic in $\mathcal{C}(F)$.

Proposition 3.2 (cf. [6, Proposition 4.4]). *For an integer $n(\geq 3)$, let $[\alpha_0, \alpha_1, \dots, \alpha_n]$ be a path in $\mathcal{C}(F)$ satisfying the following.*

(H1) *$[\alpha_0, \dots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $\mathcal{C}(F)$,*

(H2') *$\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_n)) > 2n$, where X_{n-2} is the component of $\mathrm{Cl}(F \setminus N(\alpha_{n-2}))$ that contains an essential simple closed curve.*

Then $[\alpha_0, \alpha_1, \dots, \alpha_n]$ is a geodesic in $\mathcal{C}(F)$.

By using Propositions 3.1 and 3.2, we construct a geodesic $[\alpha_0, \alpha_1, \dots, \alpha_n]$ in $\mathcal{C}(F)$, i.e., $d_F(\alpha_0, \alpha_n) = n$, for a positive integer n.

3.1. A construction of a concrete example: the case when n is even.

We first assume that n is even. Let α_0, α_2 be essential non-separating simple closed curves on F which intersect transversely in one point, and let α_1 be an essential simple closed curve on S which is disjoint from $\alpha_0 \cup \alpha_2$. Let $X_2 = \mathrm{Cl}(F \setminus N(\alpha_2))$. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $\mathcal{C}(F)$. Choose a homeomorphism $f_2 : F \to F$ such that $f_2(N(\alpha_2)) = N(\alpha_2)$ and that $\mathrm{diam}_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(f_2(\alpha_0))) \geq 4n$. This is possible by [9, Proposition 4.6]. Let $\alpha_3 = f_2(\alpha_1)$ and $\alpha_4 = f_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $\mathcal{C}(F)$ and α_2 intersects α_4 transversely in one point.

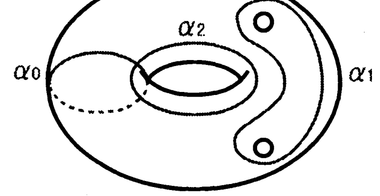

FIGURE 2

We repeat this process to construct a path $[a_0, a_1, \ldots, a_n]$ inductively as follows. Suppose that we have constructed a path $[a_0, a_1, \ldots, a_i]$ with $|\alpha_{i-2} \cap \alpha_i| = 1$ for each even $i(< n)$. Then let $X_i = \mathrm{Cl}(F \setminus N(\alpha_i))$. Choose a homeomorphism $f_i : F \to F$ such that $f_i(N(\alpha_i)) = N(\alpha_i)$ and that

$$\text{(1)} \qquad \mathrm{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(f_i(\alpha_{i-2}))) \geq 4n.$$

Then we let $\alpha_{i+1} = f_i(\alpha_{i-1})$ and $\alpha_{i+2} = f_i(\alpha_{i-2})$. Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic of length two in $\mathcal{C}(F)$, and we have obtained a path $[a_0, a_1, \ldots, a_{i+2}]$ with $|\alpha_i \cap \alpha_{i+2}| = 1$.

Claim 3.3. *For each $k \in \{2, 4, \ldots, n\}$, the path $[\alpha_0, \alpha_1, \ldots, \alpha_k]$ constructed above is a geodesic in $\mathcal{C}(F)$.*

Proof. We prove the claim by mathematical induction on k. It is clear that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $\mathcal{C}(F)$. Hence, Claim 3.1 holds for $k = 2$. Assume that $[\alpha_0, \alpha_1, \ldots, \alpha_k]$ is a geodesic in $\mathcal{C}(F)$ for some $k \in \{2, 4, \ldots, n-2\}$. We note that $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(F)$. Furthermore, by the inequality (1), we have $\mathrm{diam}_{X_k}(\pi_{X_k}(\alpha_{k-2}), \pi_{X_k}(\alpha_{k+2})) \geq 4n > 4k$. Hence, by Proposition 3.1, the path $[\alpha_0, \alpha_1, \ldots, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(F)$, which shows that Claim 3.3 holds for $k + 2$. This completes the proof of Claim 3.3. □

3.2. **A construction of a concrete example: the case when n is odd.** Suppose that n is odd. Let α_2, α_3 be essential non-separating simple closed curves on F which are mutually disjoint. Let x be an essential simple closed curve which intersects α_2 and α_3 transversely in one point, respectively. Choose an essential simple closed curve y_1 on F that is disjoint from α_2 and x. Let $X_2 = \mathrm{Cl}(F \setminus N(\alpha_2))$. By [10, Proposition 4.6], there exists homeomorphism $f_2 : F \to F$ such that $f_2(N(\alpha_2)) = N(\alpha_2)$ and that

$$\text{(2)} \qquad \mathrm{diam}_{X_2}(\pi_{X_2}(\alpha_3), \pi_{X_2}(f_2(x))) > 2n.$$

Let $\alpha_0 = f_2(x)$ and $\alpha_1 = f_2(y_1)$. Note that $\alpha_0 \cap \alpha_1 = \emptyset$, $\alpha_1 \cap \alpha_2 = \emptyset$ and α_0 intersects α_2 transversely in one point, which implies that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $\mathcal{C}(F)$. On the other hand, choose an essential simple closed curve y_2 on F that is disjoint from α_3 and x. Let $X_3 = \mathrm{Cl}(F \setminus N(\alpha_3))$. By [10, Proposition 4.6], there exists a homeomorphism $f_3 : F \to F$ such that $f_3(N(\alpha_3)) = N(\alpha_3)$ and that

$$\text{(3)} \qquad \mathrm{diam}_{X_3}(\pi_{X_3}(\alpha_0), \pi_{X_3}(f_3(x))) > 2n.$$

Let $\alpha_4 = f_3(y_2)$ and $\alpha_5 = f_3(x)$. Note that $\alpha_3 \cap \alpha_4 = \emptyset$, $\alpha_4 \cap \alpha_5 = \emptyset$ and α_3 intersects α_5 transversely in one point, which implies that $[\alpha_3, \alpha_4, \alpha_5]$ is a geodesic in $\mathcal{C}(F)$.

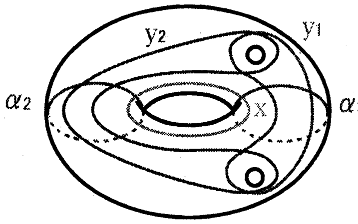

FIGURE 3

Claim 3.4. *The path $[\alpha_0, \alpha_1, \ldots, \alpha_5]$ constructed above is a geodesic in $\mathcal{C}(F)$.*

Proof. By Proposition 3.2 together with the inequality (2), the path $[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ is a geodesic. By Proposition 3.2 again together with the inequality (3), we see that the path $[\alpha_0, \alpha_1, \ldots, \alpha_5]$ is also a geodesic in $\mathcal{C}(F)$. □

We extend the above geodesic $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$ as follows. Suppose that we have constructed a path $[\alpha_0, \alpha_1, \ldots, \alpha_i]$ for an odd integer i with $5 \leq i < n$. Let $X_i = \mathrm{Cl}(F \setminus N(\alpha_i))$. Then there exists a homeomorphism $f_i : F \to F$ such that $f_i(N(\alpha_i)) = N(\alpha_i)$ and that $\mathrm{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(f_i(\alpha_{i-2}))) \geq 4n$. Let $\alpha_{i+1} = f_i(\alpha_{i-1})$ and $\alpha_{i+2} = f_i(\alpha_{i-2})$. Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic in $\mathcal{C}(F)$ and that α_i intersects α_{i+2} transversely in one point. By Proposition 3.1, the path $[\alpha_1, \alpha_2, \ldots, \alpha_{i+2}]$ is a geodesic in $\mathcal{C}(F)$. We repeat this process until we obtain a geodesic $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ of length n. Note that α_{n-2} intersects α_n transversely in one point.

4. Proof of Theorem 1.1

Basically we mimic the proof of [6, Theorem 1.1]. For $i = 1, 2$, let V_i be a solid torus and t_i a trivial arc properly embedded in V_i. The following assertion is proved by Saito [11, Proposition 3.8].

Assertion 4.1. *Let D_i be an essential disk in $V_i - t_i$ as in Figure 4. Then any non-separating essential disk in $V_i - t_i$ is isotopic to D_i and any separating essential disk in $V_i - t_i$ can be isotoped to be disjoint from D_i.*

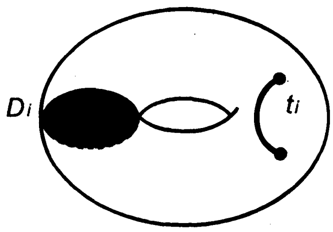

Figure 4

Let $P = \partial V_1$. Then starting with a geodesic $[\alpha_0(= \partial D_1), \alpha_1, \alpha_2]$ in $\mathcal{C}(P - t_1)$ with $|\alpha_0 \cap \alpha_2| = 1$, we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]$ with $|\alpha_n \cap \alpha_{n+2}| = 1$ as in Subsections 3.1 and 3.2. We glue ∂V_1 and ∂V_2 by a homeomorphism $h : \partial V_1 \to \partial V_2$ such that $h(\partial t_1) = \partial t_2$ and $h(\alpha_{n+2}) = \partial D_2$. Then the argument in the proof of [6, Theorem 1.1], together with Assertion 4.1, enables us to show that the distance of the (1,1)-bridge splitting $(V_1, t_1) \cup_P (V_2, t_2)$ is exactly n.

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