数理解析研究所講究録
第 1868 巻 2013 年 83-100

Evolutes and involutes of fronts in the Euclidean plane

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Abstract

This is a survey on evolutes and involutes of curves in the Euclidean plane. The evolutes and the involutes for regular curves are the classical object. Even if a curve is regular, the evolute and the involute of the curve may have singularities. By using a moving frame of the front and the curvature of the Legendre immersion, we define an evolute and an involute of the front (the Legendre immersion in the unit tangent bundle) in the Euclidean plane and discuss properties of them. We also consider about relationship between evolutes and involutes of fronts. We can observe that the evolutes and the involutes of fronts are corresponding to the differential and integral in classical calculus.

1 Introduction

The notions of evolutes and involutes (also known as evolvents) have studied by C. Huygens in his work [13] and they have studied in classical analysis, differential geometry and singularity theory of planar curves (cf. [3, 8, 10, 11, 19, 20]). The evolute of a regular curve in the Euclidean plane is given by not only the locus of all its centres of the curvature (the caustics of the regular curve), but also the envelope of normal lines of the regular curve, namely, the locus of singular loci of parallel curves (the wave front of the regular curve). On the other hand, the involute of a regular curve is the trajectory described by the end of stretched string unwinding from a point of the curve. Alternatively, another way to construct the involute of a curve is to replace the taut string by a line segment that is tangent to the curve on one end, while the other end traces out the involute. The length of the line segment is changed by an amount equal to the arc length traversed by the tangent point as it moves along the curve.

In §2, we give a brief review on the theory of regular curves, define the classical evolutes and involutes. It is well-known that the relationship between evolutes and involutes of regular plane curves. In §3, we consider Legendre curves and Legendre immersions in the unit tangent bundle and give the curvature of the Legendre curve (cf. [5]). We give the existence and the uniqueness Theorems for Legendre curves like as regular curves. By using the curvature of the Legendre immersion, we define evolutes and involutes of fronts in §4 and §5 respectively. We see that the evolute of the front is not only a (wave) front but also a caustic in §4. Moreover, the involute of the front is not only a (wave) front but also a caustic in §5. The study of singularities of (wave) fronts and caustics is the starting point of the theory of Legendrian and Lagrangian singularities developed by several mathematicians and physicists [1, 2, 4, 9, 12, 15, 16, 17, 18, 21, 22, 23, 24] etc. Furthermore, we can observe that the evolutes and the involutes of fronts are corresponding to the differential and integral in classical calculus in §6.

This is the announcement of results obtained in [5, 6, 7]. Refer [5, 6, 7] for detailed proofs, further properties and examples.

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

Acknowledgement. We would like to thank Professors Takashi Nishimura and Kentaro Saji for holding of the workshop. The author was supported by a Grant-in-Aid for Young Scientists (B) No. 23740041.

2 Regular plane curves

Let I be an interval or $\mathbb{R}$. Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular plane curve, that is, $\dot{\gamma}(t) \neq 0$ for any $t \in I$. If s is the arc-length parameter of γ, we denote $\boldsymbol{t}(s)$ by the unit tangent vector $\boldsymbol{t}(s) = \gamma'(s) = (d\gamma/ds)(s)$ and $\boldsymbol{n}(s)$ by the unit normal vector $\boldsymbol{n}(s) = J(\boldsymbol{t}(s))$ of $\gamma(s)$, where J is the anticlockwise rotation by $\pi/2$. Then we have the Frenet formula as follows:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \end{pmatrix},$$

where

$$\kappa(s) = \boldsymbol{t}'(s) \cdot \boldsymbol{n}(s) = \det(\gamma'(s), \gamma''(s))$$

is the curvature of γ and $\cdot$ is the inner product on $\mathbb{R}^2$.

Even if t is not the arc-length parameter, we have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$, the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$ and the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where $\dot{\gamma}(t) = (d\gamma/dt)(t)$, $|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$ and the curvature is given by

$$\kappa(t) = \frac{\dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)}{|\dot{\gamma}(t)|} = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}.$$

Note that the curvature $\kappa(t)$ is independent on the choice of a parametrisation.

Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2$ be regular curves. We say that γ and $\widetilde{\gamma}$ are *congruent* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t))$ for all $t \in I$, where the *congruence* C is a composition of a rotation and a translation on $\mathbb{R}^2$.

As well-known results, the existence and the uniqueness for regular plane curves are as follows (cf. [8, 10]):

Theorem 2.1 (The Existence Theorem) *Let $\kappa : I \to \mathbb{R}$ be a smooth function. There exists a regular parametrised curve $\gamma : I \to \mathbb{R}^2$ whose associated curvature function is κ.*

Theorem 2.2 (The Uniqueness Theorem) *Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2$ be regular curves whose speeds $s = |\dot{\gamma}(t)|$ and $\widetilde{s} = |\dot{\widetilde{\gamma}}(t)|$, and also curvatures κ and $\widetilde{\kappa}$ each coincide. Then γ and $\widetilde{\gamma}$ are congruent.*

In fact, the regular curve whose associated curvature function is κ, is given by the form

$$\gamma(t) = \left(\int \cos \left(\int \kappa(t) dt \right) dt, \int \sin \left(\int \kappa(t) dt \right) dt \right).$$

In this paper, we consider evolutes and involutes of plane curves. The *evolute $Ev(\gamma) : I \to \mathbb{R}^2$ of a regular plane curve $\gamma : I \to \mathbb{R}^2$* is given by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)} \boldsymbol{n}(t), \tag{1}$$

away from the point $\kappa(t) = 0$, i.e., without inflection points (cf. [3, 8, 10]).

On the other hand, the *involute $Inv(\gamma, t_0) : I \to \mathbb{R}^2$ of a regular plane curve $\gamma : I \to \mathbb{R}^2$ at $t_0 \in I$* is given by

$$Inv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} |\dot{\gamma}(s)| ds \right) \boldsymbol{t}(t). \tag{2}$$

Example 2.3 (1) Let $\gamma : [0, 2\pi) \to \mathbb{R}^2$ be an ellipse $\gamma(t) = (a \cos t, b \sin t)$ with $a \neq b$. Then the evolute of the ellipse is

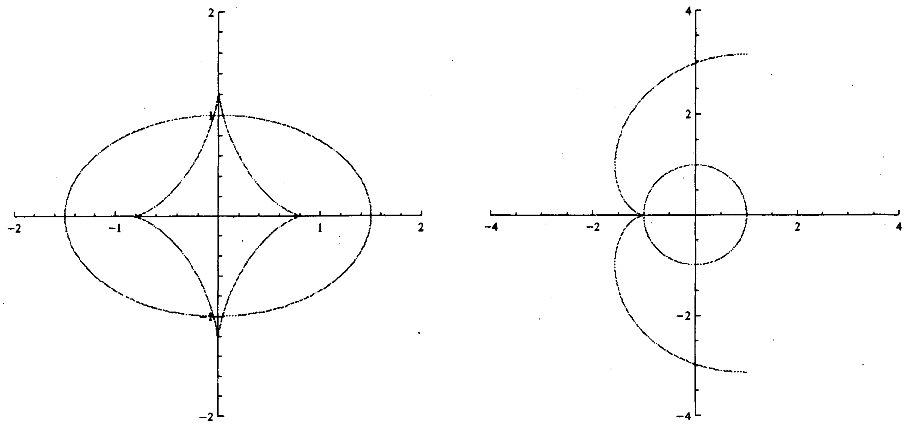
$$Ev(\gamma)(t) = \left(\frac{a^2 - b^2}{a} \cos^3 t, -\frac{a^2 - b^2}{b} \sin^3 t \right).$$

The evolute of the ellipse with $a = 3/2, b = 1$ is pictured as Figure 1 left.

(2) Let $\gamma : [0, 2\pi) \to \mathbb{R}^2$ be a circle $\gamma(t) = (r\cos t, r\sin t)$. Then the involute of the circle at t_0 is

$$Inv(\gamma, t_0)(t) = (r\cos t - r(t - t_0)\sin t, r\sin t + r(t - t_0)\cos t).$$

The involute of the circle with $r = 1$ at $t_0 = \pi$ is pictured as Figure 1 right.



(1) the evolute of an ellipse (2) the involute of a circle at π
Figure 1.

The following properties are also well-known in the classical differential geometry of curves:

Proposition 2.4 *Let $\gamma : I \to \mathbb{R}^2$ be a regular curve and $t_0 \in I$.*

(1) *If t is a regular point of $Inv(\gamma, t_0)$, then $Ev(Inv(\gamma, t_0))(t) = \gamma(t)$.*

(2) *Suppose that t_0 is a regular point of $Ev(\gamma)$. If t is a regular point of $Ev(\gamma)$, then $Inv(Ev(\gamma), t_0)(t) = \gamma(t) - (1/\kappa(t_0))\boldsymbol{n}(t)$.*

Note that even if γ is a regular curve, $Ev(\gamma)$ may have singularities and also t_0 is a singular point of $Inv(\gamma, t_0)$, see Figure 1. For a singular point of $Ev(\gamma)$ (respectively, $Inv(\gamma, t_0)$), the involute $Inv(Ev(\gamma), t_0)(t)$ (respectively, the evolute $Ev(Inv(\gamma, t_0))(t)$) can not define by the definition of the evolute and the involute. In general, if γ is not a regular curve, then we can not define the evolute and the involute of the curve.

In this paper, we define the evolutes and the involutes with singular points, see §4 and §5. In order to describe these definitions, we introduce the notion of fronts in the next section.

3 Legendre curves and Legendre immersions

We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact 1-form on the unit tangent bundle

$T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [1, 2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) *a Legendre immersion*. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* (respectively, *a front* or *a wave front*) if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. Then we have the Frenet formula of the frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of the frontal* $\gamma(t)$ in $\mathbb{R}^2$ and the Frenet formula of the frontal (or, the Legendre curve) which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, if $\dot{\gamma}(t) = \alpha(t)\nu(t) + \beta(t)\boldsymbol{\mu}(t)$ for some smooth functions $\alpha(t), \beta(t)$, then $\alpha(t) = 0$ follows from the condition $\dot{\gamma}(t) \cdot \nu(t) = 0$. Hence, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t).$$

The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals). We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t).

Definition 3.1 Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t)) = A(\gamma(t)) + \boldsymbol{b}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$, where C is given by the rotation A and the translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

We have the existence and the uniqueness for Legendre curves in the unit tangent bundle like as regular plane curves, see in [5].

Theorem 3.2 (The Existence Theorem) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).*

Theorem 3.3 (The Uniqueness Theorem) *Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$ coincide. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are congruent as Legendre curves.*

In fact, the Legendre curve whose associated curvature of the Legendre curve is (ℓ, β), is given by the form

$$\begin{aligned}\gamma(t) &= \left(-\int \beta(t)\sin\left(\int \ell(t)dt\right)dt,\ \int \beta(t)\cos\left(\int \ell(t)dt\right)dt\right),\\ \nu(t) &= \left(\cos\int \ell(t)dt,\ \sin\int \ell(t)dt\right).\end{aligned}$$

Remark 3.4 By definition of the Legendre curve, if (γ, ν) is a Legendre curve, then $(\gamma, -\nu)$ is also. In this case, $\ell(t)$ does not change, but $\beta(t)$ changes to $-\beta(t)$.

Let I and $\overline{I}$ be intervals. A smooth function $s : \overline{I} \to I$ is a (*positive*) *change of parameter* when s is surjective and has a positive derivative at every point. It follows that s is a diffeomorphism map by calculus.

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ and $(\overline{\gamma}, \overline{\nu}) : \overline{I} \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of the Legendre curves are (ℓ, β) and $(\overline{\ell}, \overline{\beta})$ respectively. Suppose (γ, ν) and $(\overline{\gamma}, \overline{\nu})$ are parametrically equivalent via the change of parameter $s : \overline{I} \to I$. Thus $(\overline{\gamma}(t), \overline{\nu}(t)) = (\gamma(s(t)), \nu(s(t)))$ for all $t \in \overline{I}$. By differentiation, we have

$$\overline{\ell}(t) = \ell(s(t))\dot{s}(t),\ \overline{\beta}(t) = \beta(s(t))\dot{s}(t).$$

Therefore, the curvature of the Legendre curve is depended on a parametrisation. We give examples of Legendre curves.

Example 3.5 One of the typical example of a front (and hence a frontal) is a regular plane curve. Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. In this case, we may take $\nu : I \to S^1$ by $\nu(t) = \boldsymbol{n}(t)$. Then it is easy to check that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion (a Legendre curve).

By a direct calculation, we give a relationship between the curvature of the Legendre curve $(\ell(t), \beta(t))$ and the curvature $\kappa(t)$ if γ is a regular curve.

Proposition 3.6 ([6, Lemma 3.1]) *Under the above notions, if γ is a regular curve, then $\ell(t) = |\beta(t)|\kappa(t)$.*

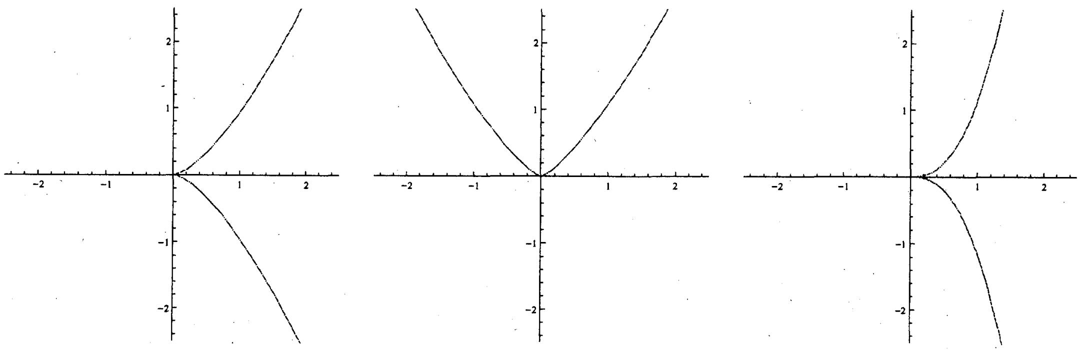
Example 3.7 Let n, m and k be natural numbers with $m = n + k$. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be

$$\gamma(t) = \left(\frac{1}{n}t^n, \frac{1}{m}t^m\right),\ \nu(t) = \frac{1}{\sqrt{t^{2k}+1}}\left(-t^k, 1\right).$$

It is easy to see that (γ, ν) is a Legendre curve, and a Legendre immersion when $k = 1$. We call γ is of *type* (n, m). For example, the frontal of type

(2,3) has the 3/2 cusp (A_2 singularity) at $t = 0$, of type (3,4) has the 4/3 cusp (E_6 singularity) at $t = 0$ and of type (2,5) has the 5/2 cusp (A_4 singularity) at $t = 0$, see Figure 2 (cf. [2, 3, 14]). By definition, we have $\boldsymbol{\mu}(t) = (1/\sqrt{t^{2k}+1})(-1, -t^k)$ and

$$\ell(t) = \frac{kt^{k-1}}{t^{2k}+1}, \ \beta(t) = -t^{n-1}\sqrt{t^{2k}+1}.$$



the 3/2 cusp the 4/3 cusp the 5/2 cusp

Figure 2.

More generally, we see that analytic curves $\gamma : I \to \mathbb{R}^2$ are frontals.

Now, we consider Legendre immersions in the unit tangent bundle. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. Then the curvature of the Legendre immersion $(\ell(t), \beta(t)) \neq (0,0)$ for all $t \in I$. In this case, we define *the normalized curvature for the Legendre immersion* by

$$(\overline{\ell}(t), \overline{\beta}(t)) = \left(\frac{\ell(t)}{\sqrt{\ell(t)^2 + \beta(t)^2}}, \frac{\beta(t)}{\sqrt{\ell(t)^2 + \beta(t)^2}} \right).$$

Then the normalized curvature $(\overline{\ell}(t), \overline{\beta}(t))$ is independent on the choice of a parametrisation. Moreover, since $\overline{\ell}(t)^2 + \overline{\beta}(t)^2 = 1$, there exists a smooth function $\theta(t)$ such that

$$\overline{\ell}(t) = \cos\theta(t), \ \overline{\beta}(t) = \sin\theta(t).$$

It is helpful to introduce the notion of the arc-length parameter of Legendre immersions. In general, we can not consider the arc-length parameter of the front γ, since γ may have singularities. However, (γ, ν) is an immersion, we introduce the arc-length parameter for the Legendre immersion (γ, ν). The *speed* $s(t)$ of the Legendre immersion at the parameter t is defined to be the length of the tangent vector at t, namely,

$$s(t) = |(\dot{\gamma}(t), \dot{\nu}(t))| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t) + \dot{\nu}(t) \cdot \dot{\nu}(t)}.$$

Given scalars $a, b \in I$, we define the arc-length from $t = a$ to $t = b$ to be the integral of the speed,

$$L(\gamma, \nu) = \int_a^b s(t)\, dt.$$

By the same method for the are-length parameter of a regular plane curve, one can prove the following:

Proposition 3.8 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t), \nu(t))$ be a Legendre immersion, and let $t_0 \in I$. Then (γ, ν) is parametrically equivalent to the unit speed curve*

$$(\overline{\gamma}, \overline{\nu}) : \overline{I} \to \mathbb{R}^2 \times S^1; s \mapsto (\overline{\gamma}(s), \overline{\nu}(s)) = (\gamma \circ u(s), \nu \circ u(s)),$$

under a positive change of parameter $u : \overline{I} \to I$ with $u(0) = t_0$ and with $u'(s) > 0$.

We call the above parameter s in Proposition 3.8 *the arc-length parameter for the Legendre immersion* (γ, ν). Let s be the are-length parameter for (γ, ν). By definition, we have $\gamma'(s) \cdot \gamma'(s) + \nu'(s) \cdot \nu'(s) = 1$, where $'$ is the derivation with respect to s. It follows that $\ell(s)^2 + \beta(s)^2 = 1$. Then there exists a smooth function $\theta(s)$ such that

$$\ell(s) = \cos\theta(s),\ \beta(s) = \sin\theta(s).$$

In the last of this section, we consider the other special parametrisation for Legendre immersions without inflection points. We define inflection points. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature of the Legendre curve (ℓ, β).

Definition 3.9 We say that a point $t_0 \in I$ is an *inflection point* of the frontal γ (or, the Legendre curve (γ, ν)) if $\ell(t_0) = 0$.

Remark that the definition of the inflection point of the frontal is a generalisation of the definition of the inflection point of a regular curve, namely, $\kappa(t) = 0$ by Proposition 3.6.

If $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve without inflection points, then (γ, ν) is a Legendre immersion.

Under the assumption $\ell(t) \neq 0$ for all $t \in I$, we can choose the special parameter t so that $|\dot{\nu}(t)| = 1$, similarly to the arc-length parameter of regular curves, namely, if $\beta(t) \neq 0$ for all $t \in I$, we can choose the arc-length parameter s so that $|\gamma'(s)| = 1$. Since $|\dot{\nu}(t)| = 1$, $\nu(t)$ (and also $\boldsymbol{\mu}(t)$) is the unit speed. By the same method for the are-length parameter of regular plane curves, one can also prove the following:

Proposition 3.10 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points, and let $t_0 \in I$. Then ν is parametrically equivalent to the unit speed curve*

$$\overline{\nu} : \overline{I} \to S^1; s \mapsto \overline{\nu}(s) = \nu \circ u(s),$$

under a positive change of parameter $u : \overline{I} \to I$ with $u(0) = t_0$ and with $u'(s) > 0$.

We call the above parameter s in Proposition 3.10 *the arc-length parameter for ν* (or, *the harmonic parameter for the Legendre immersion*). If t is the are-length parameter for ν, then we have $|\ell(t)| = 1$ for all $t \in I$. Note that we have $\ell(t) = 1$ for all $t \in I$, if necessary, a change of parameter $t \mapsto -t$.

Hereafter we consider the Legendre immersion $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ without inflection points.

4 Evolutes of fronts

In [6], we have defined an evolute of the front in the Euclidean plane by using parallel curves of the front. Here, we recall an alternative definition of the evolutes of fronts as follows, see Theorem 3.3 in [6].

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β). Assume that (γ, ν) dose not have inflection points, namely, $\ell(t) \neq 0$ for all $t \in I$.

Definition 4.1 We define the *evolute* $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ of γ,

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t). \tag{3}$$

Remark that the definition of the evolute of the front (3) is a generalisation of the definition of the evolute of a regular curve (1).

Proposition 4.2 *Under the above notations, the evolute $\mathcal{E}v(\gamma)$ is also a front. More precisely, $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature*

$$\left(\ell(t), \frac{d}{dt}\frac{\beta(t)}{\ell(t)}\right).$$

By Proposition 4.2, t is a singular point of the evolute $\mathcal{E}v(\gamma)$ if and only if $(d/dt)(\beta/\ell)(t) = 0$.

Definition 4.3 We say that t_0 is *a vertex of the front γ* (or, *the Legendre immersion (γ, ν)*) if $(d/dt)(\beta/\ell)(t_0) = 0$, namely, $(d/dt)(\mathcal{E}v(\gamma))(t_0) = 0$.

Remark that if t_0 is a regular point of γ, the definition of the vertex coincides with usual vertex for regular curves. Therefore, this is a generalisation of the notion of the vertex of a regular plane curve. We have some results for the Four vertex Theorem of the front, see in [6, 7].

We now consider the evolute of the front as a (wave) front of a Legendre immersion, and as a caustic of a Lagrange immersion by using the following families of functions.

We define two families of functions

$$F_\mu : I \times \mathbb{R}^2 \to \mathbb{R}, \quad (t,x,y) \mapsto (\gamma(t) - (x,y)) \cdot \boldsymbol{\mu}(t)$$

and

$$F_\nu : I \times \mathbb{R}^2 \to \mathbb{R}, \quad (t,x,y) \mapsto (\gamma(t) - (x,y)) \cdot \nu(t).$$

Then we have the following results:

Proposition 4.4 (1) $F_\mu(t,x,y) = 0$ *if and only if there exists a real number* λ *such that* $(x,y) = \gamma(t) - \lambda\nu(t)$.

(2) $F_\mu(t,x,y) = (\partial F_\mu/\partial t)(t,x,y) = 0$ *if and only if* $(x,y) = \gamma(t) - (\beta(t)/\ell(t))\nu(t)$.

One can show that F_μ is a *Morse family*, in the sense of Legendrian (cf. [1, 16, 17, 18, 21, 23]), namely, $(F_\mu, \partial F_\mu/\partial t) : I \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ is a submersion at $(t,x,y) \in \Sigma(F_\mu)$, where

$$\Sigma(F_\mu) = \left\{ (t,x,y) \;\middle|\; F_\mu(t,x,y) = \frac{\partial F_\mu}{\partial t}(t,x,y) = 0 \right\}.$$

It follows that the evolute of the front $\mathcal{E}v(\gamma)$ is a (wave) front of a Legendre immersion.

Moreover, since $(\partial F_\nu/\partial t)(t,x,y) = \ell(t)F_\mu(t,x,y)$, we have the following:

Proposition 4.5 (1) $(\partial F_\nu/\partial t)(t,x,y) = 0$ *if and only if there exists a real number* λ *such that* $(x,y) = \gamma(t) - \lambda\nu(t)$.

(2) $(\partial F_\nu/\partial t)(t,x,y) = (\partial^2 F_\nu/\partial t^2)(t,x,y) = 0$ *if and only if* $(x,y) = \gamma(t) - (\beta(t)/\ell(t))\nu(t)$.

One can also show that F_ν is a *Morse family*, in the sense of Lagrangian (cf. [1, 16, 17, 18, 21, 23]), namely, $\partial F_\nu/\partial t : I \times \mathbb{R}^2 \to \mathbb{R}$ is a submersion at $(t,x,y) \in C(F_\nu)$, where

$$C(F_\nu) = \left\{ (t,x,y) \;\middle|\; \frac{\partial F_\mu}{\partial t}(t,x,y) = 0 \right\}.$$

It also follows that the evolute of the front $\mathcal{E}v(\gamma)$ is a caustic of a Lagrange immersion.

By Proposition 4.2, if $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion without inflection points, then $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ is also a Legendre immersion without inflection points. Therefore, we can repeat the evolute of the front.

Theorem 4.6 *The evolute of the evolute of the front is given by*

$$\mathcal{E}v(\mathcal{E}v(\gamma))(t) = \mathcal{E}v(\gamma)(t) - \frac{\dot{\beta}(t)\ell(t) - \beta(t)\dot{\ell}(t)}{\ell(t)^3}\boldsymbol{\mu}(t).$$

The following result give the relationship between the singular point of γ and the properties of the evolutes.

Proposition 4.7 (1) *Suppose that t_0 is a singular point of γ. Then γ is diffeomorphic to the* 3/2 *cusp at t_0 if and only if t_0 is a regular point of $\mathcal{E}v(\gamma)$.*

(2) *Suppose that t_0 is a singular point of both γ and $\mathcal{E}v(\gamma)$. Then γ is diffeomorphic to the* 4/3 *cusp at t_0 if and only if t_0 is a regular point of $\mathcal{E}v(\mathcal{E}v(\gamma))$.*

We give the form of the n-th evolute of the front, where n is a natural number. We denote $\mathcal{E}v^0(\gamma)(t) = \gamma(t)$ and $\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v(\gamma)(t)$ for convenience. We define $\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v(\mathcal{E}v^{n-1}(\gamma))(t)$ and

$$\beta_0(t) = \frac{\beta(t)}{\ell(t)}, \quad \beta_n(t) = \frac{\dot{\beta}_{n-1}(t)}{\ell(t)},$$

inductively.

Theorem 4.8 *The n-th evolute of the front is given by*

$$\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v^{n-1}(\gamma)(t) - \beta_{n-1}(t)J^{n-1}(\nu(t)),$$

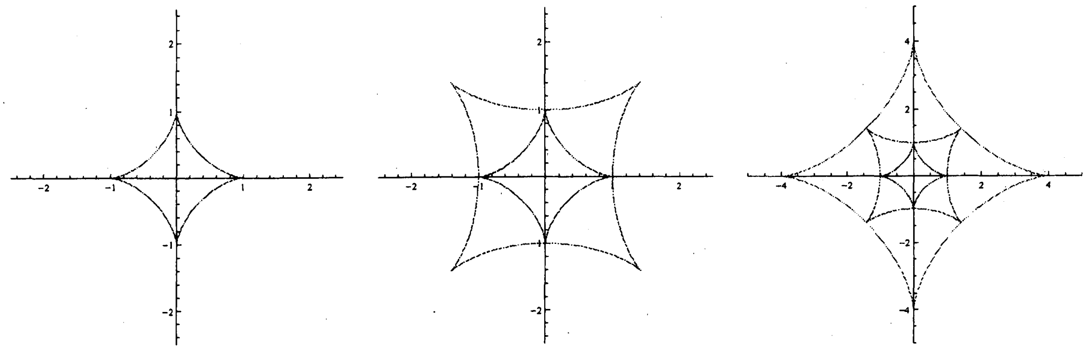
where J^{n-1} is $(n-1)$-times of J.

Example 4.9 Let $\gamma : [0, 2\pi) \to \mathbb{R}^2$ be the asteroid $\gamma(t) = (\cos^3 t, \sin^3 t)$, Figure 3 left. We can choose the unit normal $\nu(t) = (-\sin t, -\cos t)$ and $\boldsymbol{\mu}(t) = (\cos t, -\sin t)$. Then (γ, ν) is a Legendre immersion and the curvature of the Legendre immersion is given by

$$\ell(t) = -1, \quad \beta(t) = -3\cos t \sin t.$$

The evolute and the second evolute of the asteroid are as follows, see Figure 3 centre and right:

$$\begin{aligned}\mathcal{E}v(\gamma)(t) &= \left(\cos^3 t + 3\cos t\sin^2 t, \sin^3 t + 3\cos^2 t\sin t\right),\\ \mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \left(4\cos^3 t, 4\sin^3 t\right) = 4\gamma(t).\end{aligned}$$



the asteroid the evolute the second evolute

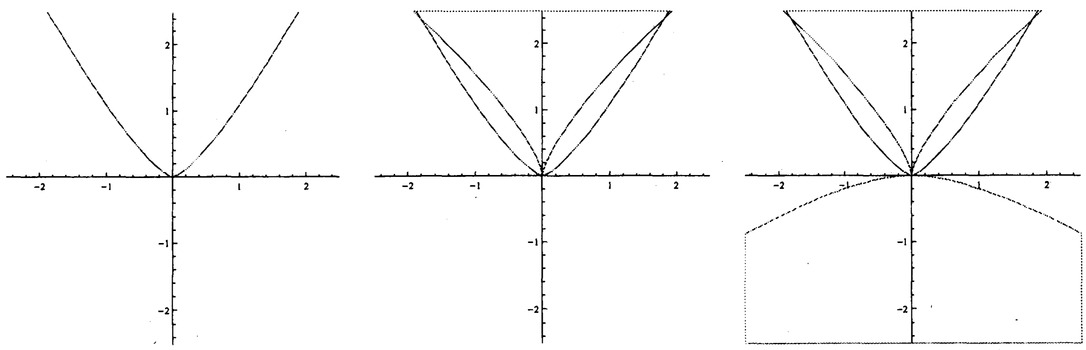
Figure 4. The asteroid and evolutes.

Example 4.10 Let $\gamma(t) = ((1/3)t^3, (1/4)t^4)$ be of type $(3,4)$ in Example 3.7, Figure 4 left. Then $\nu(t) = (1/\sqrt{t^2+1})(-t,1)$, $\boldsymbol{\mu}(t) = (1/\sqrt{t^2+1})(-1,-t)$, and the curvature of the Legendre immersion is given by

$$\ell(t) = \frac{1}{t^2+1}, \quad \beta(t) = -t^2\sqrt{t^2+1}.$$

The evolute and the second evolute of the 4/3 cusp are as follows, see Figure 4 centre and right:

$$\begin{aligned}\mathcal{E}v(\gamma)(t) &= \left(-\frac{2}{3}t^3 - t^5, t^2 + \frac{5}{4}t^4\right),\\ \mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \left(-2t - \frac{23}{3}t^3 - 6t^5, -t^2 - \frac{23}{4}t^4 - 5t^6\right).\end{aligned}$$



the 4/3 cusp the evolute the second evolute

Figure 4. The 4/3 cusp and evolutes.

5 Involutes of fronts

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β). Assume that (γ, ν) dose not have inflection points, namely, $\ell(t) \neq 0$ for all $t \in I$.

Definition 5.1 We define the *involute* $\mathcal{I}nv(\gamma, t_0) : I \to \mathbb{R}^2$ of γ at t_0,

$$\mathcal{I}nv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} \beta(s)ds\right) \boldsymbol{\mu}(t). \tag{4}$$

Remark that the definition of the involute of the front (4) is a generalisation of the definition of the involute of a regular curve (2).

Proposition 5.2 *Under the above notations, the involute $\mathcal{I}nv(\gamma, t_0)$ is also a front for each $t_0 \in I$. More precisely, $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature*

$$\left(\ell(t), \left(\int_{t_0}^{t} \beta(s)ds\right) \ell(t)\right).$$

By Proposition 5.2, t is a singular point of the involute $\mathcal{I}nv(\gamma, t_0)$ if and only if $\displaystyle\int_{t_0}^{t} \beta(s)ds = 0$. Especially, t_0 is a singular point of the involute $\mathcal{I}nv(\gamma, t_0)$.

We consider the involute of the front as a (wave) front of a Legendre immersion, and as a caustic of a Lagrange immersion by using the following families of functions. We also define two families of functions.

$$\widetilde{F}_\mu : I \times \mathbb{R}^2 \to \mathbb{R}, \quad (t, x, y) \mapsto (\gamma(t) - (x, y)) \cdot \boldsymbol{\mu}(t) - \int_{t_0}^{t} \beta(s)ds$$

and

$$\widetilde{F}_\nu : I \times \mathbb{R}^2 \to \mathbb{R}, \quad (t, x, y) \mapsto (\gamma(t) - (x, y)) \cdot \nu(t) - \int_{t_0}^{t} \left(\ell(u) \int_{t_0}^{u} \beta(s)ds\right) du.$$

Then we have the following results:

Proposition 5.3 (1) $\widetilde{F}_\mu(t, x, y) = 0$ *if and only if there exists a real number λ such that* $(x, y) = \gamma(t) - \lambda\nu(t) - (\int_{t_0}^{t} \beta(s)ds)\boldsymbol{\mu}(t)$.

(2) $\widetilde{F}_\mu(t, x, y) = (\partial \widetilde{F}_\mu / \partial t)(t, x, y) = 0$ *if and only if*

$$(x, y) = \gamma(t) - \left(\int_{t_0}^{t} \beta(s)ds\right) \boldsymbol{\mu}(t).$$

One can show that $\widetilde{F}_\mu$ is a *Morse family*, in the sense of Legendrian and the involute of the front $\mathcal{I}nv(\gamma, t_0)$ is a (wave) front of a Legendre immersion.

Moreover, since $(\partial\widetilde{F}_\nu/\partial t)(t,x,y) = \ell(t)\widetilde{F}_\mu(t,x,y)$, we have the following:

Proposition 5.4 (1) $(\partial\widetilde{F}_\nu/\partial t)(t,x,y) = 0$ *if and only if there exists a real number* λ *such that* $(x,y) = \gamma(t) - \lambda\nu(t) - (\int_{t_0}^t \beta(s)ds)\boldsymbol{\mu}(t)$.

(2) $(\partial\widetilde{F}_\nu/\partial t)(t,x,y) = (\partial^2\widetilde{F}_\nu/\partial t^2)(t,x,y) = 0$ *if and only if*

$$(x,y) = \gamma(t) - \left(\int_{t_0}^t \beta(s)ds\right)\boldsymbol{\mu}(t).$$

One can also show that $\widetilde{F}_\nu$ is a *Morse family*, in the sense of Lagrangian and the involute of the front $\mathcal{I}nv(\gamma, t_0)$ is a caustic of a Lagrange immersion.

We analyse singular points of the involute of the front.

Proposition 5.5 (1) *Suppose that* t *is a singular point of* $\mathcal{I}nv(\gamma, t_0)$. *Then* $\mathcal{I}nv(\gamma, t_0)$ *is diffeomorphic to the* $3/2$ *cusp at* t *if and only if* $\beta(t) \neq 0$.

(2) *Suppose that* t *is a singular point of* $\mathcal{I}nv(\gamma, t_0)$. *Then* $\mathcal{I}nv(\gamma, t_0)$ *is diffeomorphic to the* $4/3$ *cusp at* t *if and only if* $\beta(t) = 0$ *and* $\dot\beta(t) \neq 0$.

As a corollary of Proposition 5.5, we have the following.

Corollary 5.6 (1) $\mathcal{I}nv(\gamma, t_0)$ *is diffeomorphic to the* $3/2$ *cusp at* t_0 *if and only if* t_0 *is a regular point of* γ.

(2) $\mathcal{I}nv(\gamma, t_0)$ *is diffeomorphic to the* $4/3$ *cusp at* t_0 *if and only if* γ *is diffeomorphic to the* $3/2$ *cusp at* t_0.

By Proposition 5.2, if $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion without inflection points, then $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ is also a Legendre immersion without inflection points. Therefore we can also repeat the involute of the front. We give a form of the n-th involute of the front, where n is a natural number. We denote $\mathcal{I}nv^0(\gamma, t_0)(t) = \gamma(t)$ and $\mathcal{I}nv^1(\gamma, t_0)(t) = \mathcal{I}nv(\gamma, t_0)(t)$ for convenience. We define $\mathcal{I}nv^n(\gamma, t_0)(t) = \mathcal{I}nv(\mathcal{I}nv^{n-1}(\gamma, t_0), t_0)(t)$ and

$$\beta_{-1}(t) = \left(\int_{t_0}^t \beta(s)ds\right)\ell(t), \quad \beta_{-n}(t) = \left(\int_{t_0}^t \beta_{-n+1}(s)ds\right)\ell(t)$$

inductively.

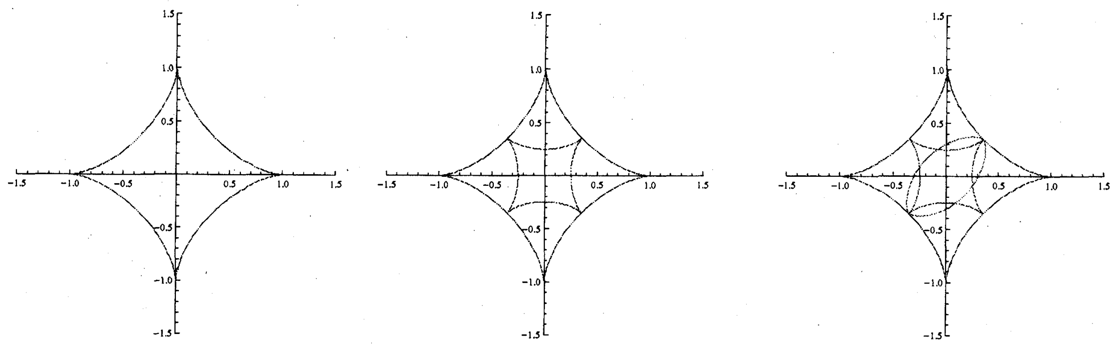
Theorem 5.7 *The n-th involute of the front γ at t_0 is given by*

$$\mathcal{I}nv^n(\gamma, t_0)(t) = \mathcal{I}nv^{n-1}(\gamma, t_0)(t) + \frac{\beta_{-n}(t)}{\ell(t)} J^{-n}(\nu(t)),$$

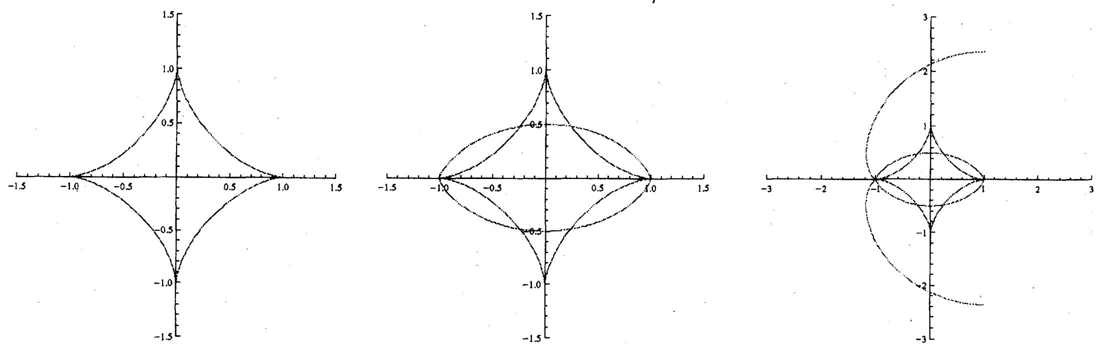
where J^{-n} is n-times of J^{-1}.

Example 5.8 Let $\gamma : [0, 2\pi) \to \mathbb{R}^2$ be the asteroid $\gamma(t) = (\cos^3 t, \sin^3 t)$ in Example 4.9 and $t_0 \in [0, 2\pi)$. Then the involute and the second involute of the asteroid at t_0 are as follows, see Figure 5 at $t_0 = \pi/4$ and at $t_0 = \pi$.

$$\begin{aligned}
\mathcal{I}nv(\gamma, t_0)(t) &= \Big(\frac{1}{4}\cos^3 t + \frac{3}{4}\cos t \sin^2 t + \frac{3}{4}\cos 2t_0 \cos t, \\
&\qquad \frac{1}{4}\sin^3 t + \frac{3}{4}\cos^2 t \sin t - \frac{3}{4}\cos 2t_0 \sin t \Big) \\
\mathcal{I}nv(\mathcal{I}nv(\gamma, t_0), t_0)(t) &= \Big(\frac{1}{4}\cos^3 t + \frac{3}{4}\cos 2t_0 \cos t + \frac{3}{4}(\cos 2t_0) t \sin t \\
&\qquad + \frac{3}{8}\sin 2t_0 \sin t - \frac{3}{4}(\cos 2t_0) t_0 \sin t, \\
&\qquad \frac{1}{4}\sin^3 t - \frac{3}{4}\cos 2t_0 \sin t + \frac{3}{4}(\cos 2t_0) t \cos t \\
&\qquad + \frac{3}{8}\sin 2t_0 \cos t - \frac{3}{4}(\cos 2t_0) t_0 \cos t \Big).
\end{aligned}$$



the asteroid | the involute at $\pi/4$ | the second involute at $\pi/4$



the asteroid | the involute at π | the second involute at π

Figure 5. The asteroid and involutes.

6 Relationship between evolutes and involutes of fronts

In this section, we discuss on relationship between the evolutes and the involutes of fronts. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β). Assume that (γ, ν) dose not have inflection points, namely, $\ell(t) \neq 0$ for all $t \in I$. We give a justification of Proposition 2.4 with singular points.

Proposition 6.1 *Let $t_0 \in I$.*
(1) $\mathcal{E}v(\mathcal{I}nv(\gamma, t_0))(t) = \gamma(t)$.
(2) $\mathcal{I}nv(\mathcal{E}v(\gamma), t_0)(t) = \gamma(t) - (\beta(t_0)/\ell(t_0))\nu(t)$.

For a given Legendre immersion $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, we consider the existence condition of a Legendre immersion $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ such that $\mathcal{E}v(\widetilde{\gamma})(t) = \gamma(t)$ or $\mathcal{I}nv(\widetilde{\gamma}, t_0)(t) = \gamma(t)$ for some t_0. By using Proposition 6.1, we have the following result.

Proposition 6.2 (1) *If $\widetilde{\gamma}(t) = \mathcal{I}nv(\gamma, t_0)(t) + \lambda\boldsymbol{\mu}(t), \widetilde{\nu}(t) = J^{-1}(\nu(t))$ for any $t_0 \in I$ and $\lambda \in \mathbb{R}$, then $\mathcal{E}v(\widetilde{\gamma})(t) = \gamma(t)$.*
(2) *If $\widetilde{\gamma}(t) = \mathcal{E}v(\gamma)(t), \widetilde{\nu}(t) = J(\nu(t))$ and t_0 is a singular point of γ, then $\mathcal{I}nv(\widetilde{\gamma}, t_0)(t) = \gamma(t)$.*

By Theorems 4.8 and 5.7, we have the following sequence of the evolutes and the involutes of the front.

$$\begin{aligned}\cdots &\overset{\mathcal{I}nv}{\leftarrow} (\mathcal{I}nv^2(\gamma, t_0)(t), J^{-2}(\nu)(t)) \overset{\mathcal{I}nv}{\leftarrow} (\mathcal{I}nv(\gamma, t_0)(t), J^{-1}(\nu)(t)) \overset{\mathcal{I}nv}{\leftarrow} \\ &(\gamma(t), \nu(t)) \overset{\mathcal{E}v}{\to} (\mathcal{E}v(\gamma)(t), J(\nu)(t)) \overset{\mathcal{E}v}{\to} (\mathcal{E}v^2(\gamma)(t), J^2(\nu)(t)) \overset{\mathcal{E}v}{\to} \cdots \qquad (5)\end{aligned}$$

It follows that the corresponding sequence of the curvatures of the Legendre immersions (5) is given by

$$\begin{aligned}\cdots \leftarrow (\ell(t), \beta_{-2}(t)) \leftarrow (\ell(t), \beta_{-1}(t)) \leftarrow \\ (\ell(t), \beta(t)) \to (\ell(t), \beta_1(t)) \to (\ell(t), \beta_2(t)) \to \cdots \qquad (6)\end{aligned}$$

Moreover, we may suppose that t is the arc-length parameter for ν, see §3. It follows that $\ell(t) = 1$ for all $t \in I$, if necessary, a change of parameter $t \mapsto -t$. Then the relationship between second components of the curvatures of the Legendre immersions (6) is pictured as follows:

$$\cdots \leftarrow \int_{t_0}^{t} \left(\int_{t_0}^{t} \beta(t) dt \right) dt \leftarrow \int_{t_0}^{t} \beta(t) dt \leftarrow \beta(t) \to \frac{d}{dt}\beta(t) \to \frac{d^2}{dt^2}\beta(t) \to \cdots$$

This is corresponding to the relationship between the differential and integral in classical calculus.

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol.* I. Birkhäuser (1986).

[2] V. I. Arnol'd, *Singularities of Caustics and Wave Fronts.* Mathematics and Its Applications **62** Kluwer Academic Publishers (1990).

[3] J. W. Bruce and P. J. Giblin, *Curves and singularities. A geometrical introduction to singularity theory. Second edition.* Cambridge University Press, Cambridge (1992).

[4] J. Ehlers and E.T. Newman, *The theory of caustics and wave front singularities with physical applications*, J. Math. Physics **41** (2000), 3344–3378.

[5] T. Fukunaga and M. Takahashi, *Existence and uniqueness for Legendre curves.* to appear in J. Geometry (2013).

[6] T. Fukunaga and M. Takahashi, *Evolutes of fronts in the Euclidean plane.* Preprint, Hokkaido University Preprint Series, No.1026 (2012).

[7] T. Fukunaga and M. Takahashi, *Involutes of fronts in the Euclidean plane.* Preprint, (2013).

[8] C. G. Gibson, *Elementary geometry of differentiable curves. An undergraduate introduction.* Cambridge University Press, Cambridge (2001).

[9] V. V. Goryunov and V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Real and complex singularities, Trends Math., Birkhauser, Basel, (2007), 169–185.

[10] A. Gray, E. Abbena and S. Salamon, *Modern differential geometry of curves and surfaces with Mathematica. Third edition.* Studies in Advanced Mathematics. Chapman and Hall/CRC, Boca Raton, FL (2006)

[11] E. Hairer and G. Wanner, *Analysis by its history*, Springer-Verlag, New York, (1996).

[12] W. Hasse, M. Kriele and V. Perlick, *Caustics of wavefronts in general relativity*, Class. Quantum Grav. **13** (1996), 1161–1182.

[13] C. Huygens, *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae*, (1673).

[14] G. Ishikawa, *Zariski's moduli problem for plane branches and the classification of Legendre curve singularities.* Real and complex singularities, World Sci. Publ. Hackensack, NJ, (2007), 56–84.

[15] S. Izumiya, *Differential Geometry from the viewpoint of Lagrangian or Legendrian singularity theory.* in Singularity Theory (ed., D. Chéniot et al), World Scientific (2007), 241–275.

[16] S. Izumiya and M. Takahashi, *Spacelike parallels and evolutes in Minkowski pseudo-spheres.* Journal of Geometry and Physics. **57** (2007), 1569–1600.

[17] S. Izumiya and M. Takahashi, *Caustics and wave front propagations: Applications to differential geometry.* Banach Center Publications. Geometry and topology of caustics. **82** (2008), 125–142.

[18] S. Izumiya and M. Takahashi, *On caustics of submanifolds and canal hypersurfaces in Euclidean space.* Topology Appl. **159** (2012), 501–508.

[19] G. de l'Hospital, *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes*, (1696).

[20] I. Porteous, *Geometric differentiation. For the intelligence of curves and surfaces. Second edition.* Cambridge University Press, Cambridge (2001)

[21] M. Takahashi, *Caustics and wave front propagations.* Singularity theory of smooth maps and related geometry, Suurikenn Kokyuroku **1707** (2010), 135–148.

[22] G. Wassermann, *Stability of Caustics.* Math. Ann. **216** (1975), 43–50.

[23] V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings.* J. Soviet Math. **27** (1983), 2713–2735.

[24] V. M. Zakalyukin, *Envelope of Families of Wave Fronts and Control Theory.* Proc. Steklov Inst. Math. **209** (1995), 114–123.

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