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Weierstrass representation for semi-discrete minimal surfaces

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Abstract

We give a Weierstrass type representation for semi-discrete minimal surfaces in Euclidean 3-space. In this paper, we introduce this result, and another related work about discretized maximal surfaces with singularities in Minkowski 3-space.

1 Introduction

Discrete differential geometry is a quickly developing research area, especially in regard to connections with integrable systems techniques. Bobenko and Pinkall [1], [2] defined discrete isothermic surfaces using a cross ratio condition, and the class of discrete isothermic surfaces was broadened via the use of transformation theory, see [3], [4], [5], [6], [9].

Recently Mueller and Wallner [8] defined semi-discrete isothermic surfaces and semi-discrete minimal surfaces via integrable system techniques, where a semi-discrete surface is a map parametrized by one discrete parameter $k \in \mathbb{Z}$ and one smooth parameter $t \in \mathbb{R}$. Rossman and the author [10] gave an explicit formula for constructing semi-discrete minimal surfaces, which can be called a Weierstrass representation.

The first purpose of this paper is to introduce discrete differential geometry of surfaces, especially focusing on the theories of discretized isothermic surfaces and discretized minimal surfaces in Euclidean 3-space, in Sections 2 and 3. The second purpose is to introduce discretized isothermic surfaces and discretized maximal surfaces in Minkowski 3-space. However, for semi-discrete isothermic surfaces and maximal surfaces, there are remaining problems. We give some of these problems here, in Section 4.

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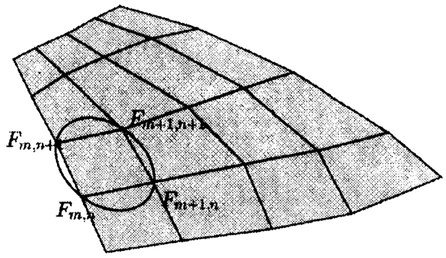

Figure 1: discrete isothermic surface

2 Discrete minimal surfaces

In this section, we identify $\mathbb{R}^3$ with the Lie algebra

$$\mathrm{su}_2 := \left\{ \begin{pmatrix} \mathrm{i}a & -\mathrm{i}b \\ -\mathrm{i}\bar{b} & -\mathrm{i}a \end{pmatrix} \middle| \, a \in \mathbb{R}, \ b \in \mathbb{C} \right\}$$

of the Lie group

$$\mathrm{SU}_2 := \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 + |\beta|^2 = 1 \right\}$$

by identifying $(x_1, x_2, x_3) \in \mathbb{R}^{2,1}$ with the matrix

$$\begin{pmatrix} \mathrm{i}x_3 & -\mathrm{i}(x_1 + \mathrm{i}x_2) \\ -\mathrm{i}(x_1 - \mathrm{i}x_2) & -\mathrm{i}x_3 \end{pmatrix} \in \mathrm{su}_2.$$

The following definition was first given in [1].

Definition 1.

- Let X_1, X_2, X_3, X_4 be points in $\in \mathbb{R}^3$. The unordered pair $\{q, \bar{q}\}$ of eigenvalues of the following matrix

$$Q(X_1, X_2, X_3, X_4) = (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1}$$

  is called the cross ratio of the quadrilateral (X_1, X_2, X_3, X_4).

- Let $F : \mathbb{Z}^2 \to \mathbb{R}^3$ (parametrized by two parameters m, $n \in \mathbb{Z}$) be a discrete surface. F is discrete isothermic if the cross ratios of quadrilaterals of F are $-\frac{\alpha_m}{\beta_n}$, where α_m (depending only on m) and β_n (depending only on n) are scalar positive functions.

- Let F be a discrete isothermic surface. F is minimal if its dual surface F^* is inscribed in $\mathbb{S}^2$.

Remark

1. Let $F : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete surface. If $(F_{m,n}, F_{m+1,n}, F_{m+1,n+1}, F_{m,n+1})$ are concircular, $Q \in \mathbb{R} \setminus \{0\}$. So if F is isothermic, then the quadrilaterals of F are concircular.
2. A discrete isothermic surface g is a *discrete holomorphic function* if g lies in $\mathbb{C}$.

□

The notion of discrete isothermic surfaces can be found in [1], [2]. In this paper, we introduce the Weierstrass representation for discrete minimal surfaces.

Theorem 1. Any discrete minimal surface x in $\mathbb{R}^3$ can be constructed by solving the following difference equations:

$$\begin{aligned} \Delta_1 x &= -\mathrm{Re}\left(\frac{\alpha_m}{g_{m+1,n} - g_{m,n}} \begin{pmatrix} 1 - g_{m+1,n} g_{m,n} \\ \mathrm{i} + \mathrm{i} g_{m+1,n} g_{m,n} \\ g_{m+1,n} + g_{m,n} \end{pmatrix} \right), \\ \Delta_2 x &= \mathrm{Re}\left(\frac{\beta_n}{g_{m,n+1} - g_{m,n}} \begin{pmatrix} 1 - g_{m,n+1} g_{m,n} \\ \mathrm{i} + \mathrm{i} g_{m,n+1} g_{m,n} \\ g_{m,n+1} + g_{m,n} \end{pmatrix} \right) \end{aligned} \tag{1}$$

with α_m and β_n the cross ratio factorizing functions for g, where

$$\Delta_1 x := x_{m+1,n} - x_{m,n}, \ \Delta_2 x := x_{m,n+1} - x_{m,n}.$$

In order to show Theorem 1, we need to show the following:

1. determine the formula,
2. check that x satisfies the compatibility condition, i.e. $\Delta_1\Delta_2 x = \Delta_2\Delta_1 x$.
3. show concircularity of x,
4. check the converse.

In this paper, we show the outline of the proof of item 1 above. We can show item 1 by

- taking a discrete holomorphic function,
- mapping to $\mathbb{S}^2$ by stereographic projection,
- dualizing.

A detailed proof and applications can be found in [1], [2], [10]. The pictures in Figure 2 are such discrete minimal surfaces.

3 Semi-discrete minimal surfaces

Let $x = x(k,t)$ be a map from a domain in $\mathbb{Z} \times \mathbb{R}$ to $\mathbb{R}^3$ ($k \in \mathbb{Z}$, $t \in \mathbb{R}$). We call x a *semi-discrete* surface. Semi-discrete surfaces are the surfaces discretized in only

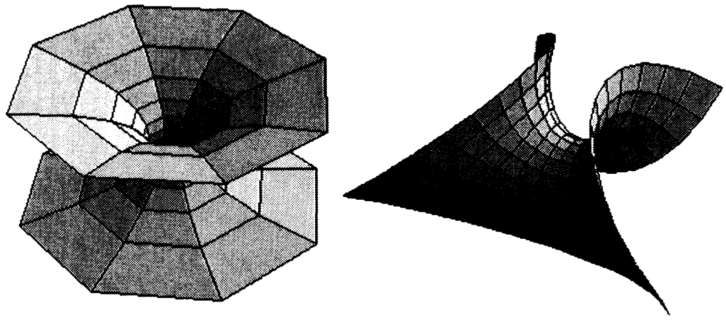

Figure 2: discrete minimal surfaces

one of the two parameter directions ,or are the surfaces obtained by taking limit in only one of the two parameters. (See Figure 3.)

Set

$$\partial x = \frac{\partial x}{\partial t}, \ \Delta x = x_1 - x, \ \partial\Delta x = \partial x_1 - \partial x,$$

where $x_1 = x(k+1, t)$. The following notations can be found in [8], and are all naturally motivated by the case of smooth surfaces:

- x is *conjugate* if ∂x, Δx and $\partial\Delta x$ are linearly dependent.
- x is *circular* if there exists a circle $\mathscr{C}$ passing through x and x_1 that is tangent to ∂x, ∂x_1 there (for all k, t).
- A circular semi-discrete surface x is *isothermic* if there exist positive functions ν, σ, τ such that

$$\|\Delta x\|^2 = \sigma\nu\nu_1, \ \|\partial x\|^2 = \tau\nu^2, \text{ with } \partial\sigma = \Delta\tau = 0. \tag{2}$$

- Suppose x, x^* are conjugate semi-discrete surfaces. Then x and x^* are *dual surfaces* if there exists a function $\nu : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^+$ so that

$$\partial x^* = -\frac{1}{\nu^2}\partial x, \ \Delta x^* = \frac{1}{\nu\nu_1}\Delta x.$$

- A circular semi-discrete isothermic surface x is *minimal* if x^* is inscribed in a sphere.

We introduce the following recipe for constructing semi-discrete minimal surfaces. In the following theorem, $g' = \partial g$.

Theorem 2 (Weierstrass representation). We can construct a semi-discrete minimal surface from a semi-discrete isothermic surface g lying in a plane by solving

$$\partial x = -\frac{\tau}{2}\mathrm{Re}\begin{pmatrix} \frac{1-g^2}{g'} \\ \frac{\mathrm{i}(1+g^2)}{g'} \\ \frac{2g}{g'} \end{pmatrix}, \quad \Delta x = \frac{\sigma}{2}\mathrm{Re}\begin{pmatrix} \frac{1-gg_1}{\Delta g} \\ \frac{\mathrm{i}(1+gg_1)}{\Delta g} \\ \frac{g+g_1}{\Delta g} \end{pmatrix}, \tag{3}$$

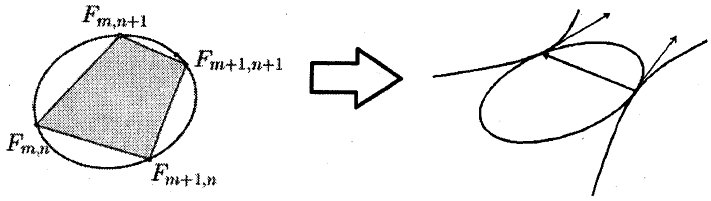

Figure 3: semi-discrete surfaces obtained by taking limit in only one of the two parameters

with τ and σ as in (2) for g. Conversely, any semi-discrete minimal surface is described in this way by some semi-discrete isothermic surface g lying in a plane.

In order to show Theorem 2, we need to show the following:

1. determine the formula,
2. check that x satisfies the compatibility condition, i.e. $\partial\Delta x = \Delta\partial x$.
3. show circularity of x,
4. check the converse.

In this paper, we show the outline of the proof of item 1 above. Like in the case of discrete minimal surfaces, we can show item 1 by

- taking a semi-discrete holomorphic function,
- mapping to $\mathbb{S}^2$ by stereographic projection,
- dualizing.

A detailed proof and applications can be found in [10]. Here we give two examples of semi-discrete minimal surfaces.

Example The semi-discrete Enneper surfaces can be constructed via Theorem 2 with

$$g(k,t) = c(k + \mathrm{i}t) \quad (\text{resp. } g(k,t) = c(t + \mathrm{i}k)),$$

for the right choice of $c \in \mathbb{R} \setminus \{0\}$.

Example The $MW^{in}_{pd,rs}$ (resp. $MW^{in}_{ps,rd}$) catenoid can be constructed via Theorem 2 with $g(k,t) = ce^{\alpha k + \mathrm{i}\beta t}$ (resp. $g(k,t) = ce^{\alpha t + \mathrm{i}\beta k}$), for the right choices of c, α, $\beta \in \mathbb{R} \setminus \{0\}$. We call them MW-catenoids after Mueller and Wallner.

The pictures in Figure 4 are such semi-discrete minimal surfaces.

Remark Recently Mueller constructed new examples of semi-discrete minimal helicoids via new semi-discrete holomorphic functions. These are pictured in [7]. $\square$

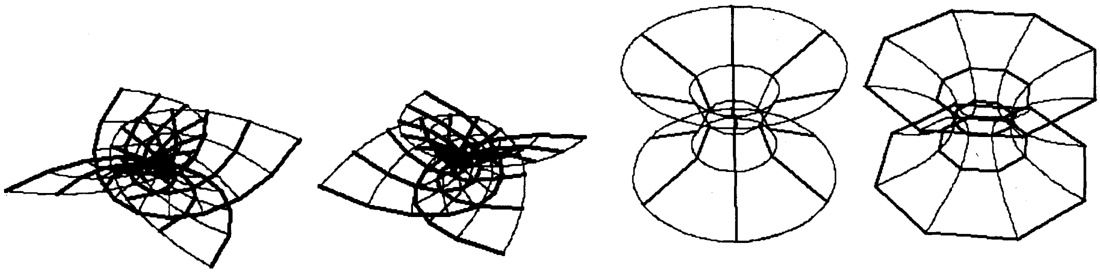

Figure 4: semi-discrete Enneper surfaces and semi-discrete catenoids

4 Related works

Recently we are studying discrete maximal surfaces with some analog of singularities, for both discrete maximal surfaces and semi-discrete maximal surfaces. In this section, we introduce these results and some remaining problems.

4.1 Discrete maximal surfaces

First, we give a brief introduction to discrete maximal surfaces. Let $\mathbb{R}^{2,1} := (\{(x_1, x_2, x_3)|x_j \in \mathbb{R}\}, \langle\cdot,\cdot\rangle)$ be 3-dimensional Minkowski space with the Lorentz metric

$$\langle(x_1, x_2, x_0), (y_1, y_2, y_0)\rangle = x_1y_1 + x_2y_2 - x_0y_0.$$

Now, we identify $\mathbb{R}^{2,1}$ with the Lie algebra

$$\mathrm{su}_{1,1} := \left\{ \begin{pmatrix} \mathrm{i}a & b \\ \bar{b} & -\mathrm{i}a \end{pmatrix} \middle| \, a \in \mathbb{R}, \ b \in \mathbb{C} \right\}$$

of the Lie group

$$\mathrm{SU}_{1,1} := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 - |\beta|^2 = 1 \right\}$$

by identifying $(x_1, x_2, x_0) \in \mathbb{R}^{2,1}$ with the matrix

$$\begin{pmatrix} \mathrm{i}x_0 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -\mathrm{i}x_0 \end{pmatrix} \in \mathrm{su}_{1,1}.$$

we give an analogy of the cross ratio defined by Bobenko and Pinkall [1], [2].

Definition 2. Let X_1, X_2, X_3, X_4 be points in $\in \mathbb{R}^{2,1}$. The unordered pair $\{q, \bar{q}\}$ of eigenvalues of the following matrix

$$Q(X_1, X_2, X_3, X_4) = (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1}$$

is called the cross ratio of the quadrilateral (X_1, X_2, X_3, X_4).

In [11], we show that if the cross-ratio of a quadrilateral (X_1, X_2, X_3, X_4) is real, then (X_1, X_2, X_3, X_4) lies on a quadric in a plane in $\mathbb{R}^{2,1}$. Discrete spacelike isothermic surfaces in Minkowski 3-space are the surfaces for which every quadrilateral face has vertices lying on a "circle" (strictly, on an ellipse), and for which the cross ratios of those faces are $-\frac{\alpha_m}{\beta_n}$, where $\alpha_m > 0$ (resp. $\beta_n > 0$) depends only on discrete parameter m (resp. n). A face is called a singular face if it lies on a hyperbola or parabola. The notion of singular faces is the natural discretization of singularities of spacelike maximal surfaces in the smooth case. This fact can be found in [11].

Let $F : \mathbb{Z}^2 \to \mathbb{R}^{2,1}$ be a discrete isothermic surface. F is discrete maximal if its dual F^* lies in $\mathbb{H}^2_\pm$. Then we have the following theorem, where $\mathbb{H}^2_+$ (resp. $\mathbb{H}^2_-$) denotes

$$\mathbb{H}^2_+ := \{x = (x_1, x_2, x_0) \in \mathbb{R}^{2,1} \mid \|x\|^2 = -1,\ x_0 > 0\}$$
$$(\text{resp. } \mathbb{H}^2_- := \{x = (x_1, x_2, x_0) \in \mathbb{R}^{2,1} \mid \|x\|^2 = -1,\ x_0 < 0\}).$$

Theorem 3.

- We can construct a discrete maximal surface from a discrete holomorphic function g by solving

$$\begin{aligned} \Delta_1 x &= \mathrm{Re}\left(\frac{\alpha_m}{\Delta_1 g}\begin{pmatrix} 1 + g_{m+1,n}g_{m,n} \\ \mathrm{i} - \mathrm{i}g_{m+1,n}g_{m,n} \\ -(g_{m+1,n} + g_{m,n}) \end{pmatrix}\right), \\ \Delta_2 x &= -\mathrm{Re}\left(\frac{\beta_n}{\Delta_2 g}\begin{pmatrix} 1 + g_{m,n+1}g_{m,n} \\ \mathrm{i} - \mathrm{i}g_{m,n+1}g_{m,n} \\ -(g_{m,n+1} + g_{m,n}) \end{pmatrix}\right) \end{aligned} \tag{4}$$

  with α_m and β_n the cross ratio factorizing functions for g, where

$$\Delta_1 x := x_{m+1,n} - x_{m,n},\ \Delta_2 x := x_{m,n+1} - x_{m,n}.$$

  Conversely, any discrete maximal surface is described in this way using some discrete holomorphic function g.

- Let g be a discrete holomorphic function and let F be a discrete maximal surface determined from g. Then the face $\mathcal{F} = (F, F_1, F_{12}, F_2)$ is a singular face if and only if the circle passing through the vertices of the face $\mathcal{G} = (g, g_1, g_{12}, g_2)$ intersects $\mathbb{S}^1$.

The proof of this theorem can be found in [11]. We now give two examples of discrete maximal surfaces.

Example The discrete maximal Enneper surfaces can be constructed via Theorem 3 with

$$g(m, n) = c(m + \mathrm{i}n),$$

where $c \in \mathbb{R} \setminus \{0\}$. The pictures in Figure 5 are such a maximal Enneper surface, and the discrete holomorphic function with the corresponding singular faces marked.

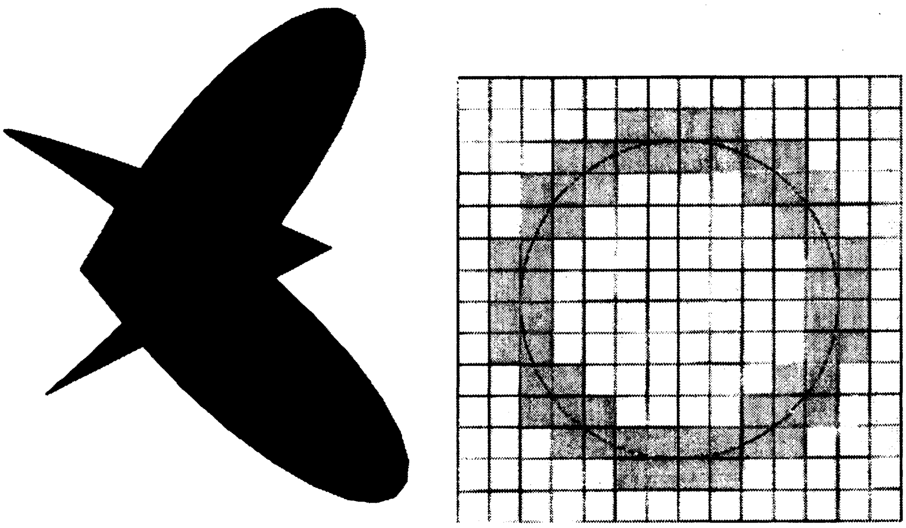

Figure 5: a discrete maximal Enneper surface with $c = \frac{1}{5}$, and its discrete holomorphic function

Example The discrete maximal catenoids can be constructed via Theorem 3 with

$$g(m,n) = \exp(\alpha m + \mathrm{i}\beta n),$$

for the right choices of $\alpha,\ \beta \in \mathbb{R} \setminus \{0\}$. The picture in Figure 6 is such a catenoid, and the discrete holomorphic function with the corresponding singular faces marked.

4.2 Semi-discrete maximal surface

In [12], we give a new classes of semi-discrete surfaces called semi-discrete maximal surfaces. We do not give the definition of these surfaces, which can be found in [12]. We introduce one result we have here.

Like in the discrete case, we have the following theorem.

Theorem 4. We can construct a semi-discrete maximal surface from a semi-discrete holomorphic function g by solving

$$\partial x = -\mathrm{Re}\left(\frac{\tau}{g'}\begin{pmatrix}1+g^2\\ \mathrm{i}-\mathrm{i}g^2\\ -2g\end{pmatrix}\right), \Delta x = \mathrm{Re}\left(\frac{\sigma}{\Delta g}\begin{pmatrix}1+gg_1\\ \mathrm{i}-\mathrm{i}gg_1\\ -(g+g_1)\end{pmatrix}\right) \tag{5}$$

with τ and σ given by g. Conversely, any semi-discrete maximal surface is described in this way using some semi-discrete holomorphic function g.

Example The semi-discrete maximal Enneper surfaces can be constructed via Theorem 5 with

$$g(k,t) = c(t+\mathrm{i}k),\ (\text{resp. } g(k,t) = c(k+\mathrm{i}t)),$$

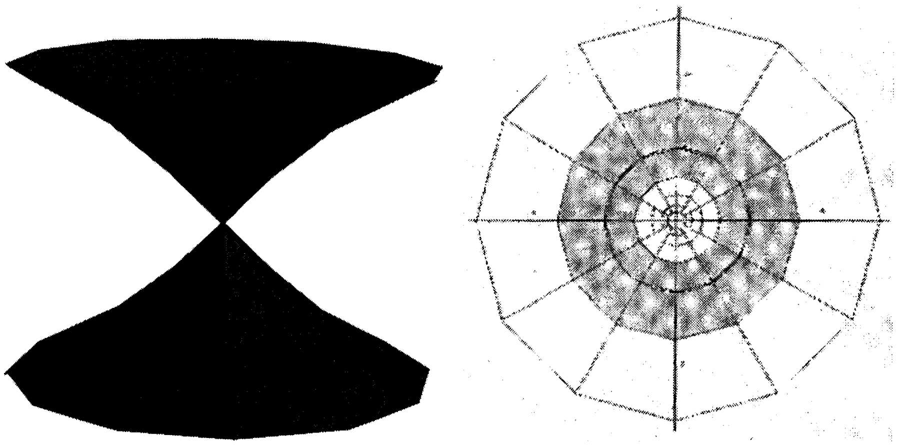

Figure 6: a discrete maximal catenoid and its discrete holomorphic function

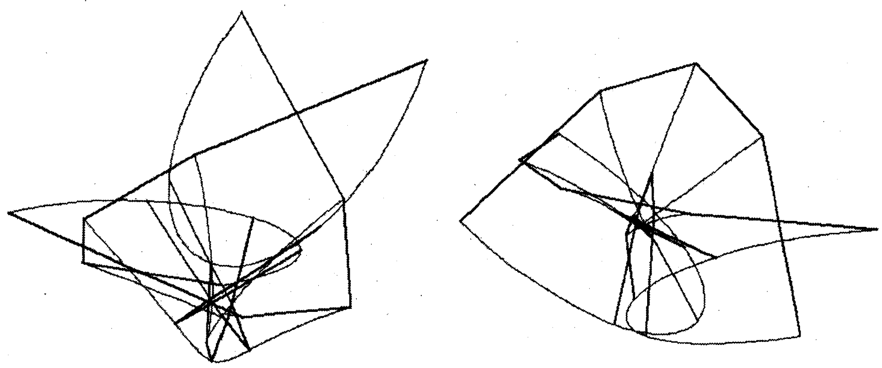

Figure 7: two discrete maximal Enneper surfaces

where $c \in \mathbb{R} \setminus \{0\}$. The pictures in Figure 7 are such maximal Enneper surfaces.

Example The semi-discrete maximal catenoids can be constructed via Theorem 5 with

$$g(k,t) = \exp(\alpha k + i\beta t) \text{ (resp. } g(k,t) = \exp(\alpha t + i\beta k))$$

for the right choices of α, $\beta \in \mathbb{R} \setminus \{0\}$. The pictures in Figure 8 are such catenoids.

As for semi-discrete maximal surfaces, we mention two remaining problems. The second item is being considered in [12], where we are finding the best choice of the definition of singularities for these surfaces.

Problem 1.

- What are the curvatures of semi-discrete surfaces in $\mathbb{R}^{2,1}$?
- What is the definition of singularities for semi-discrete maximal surfaces?

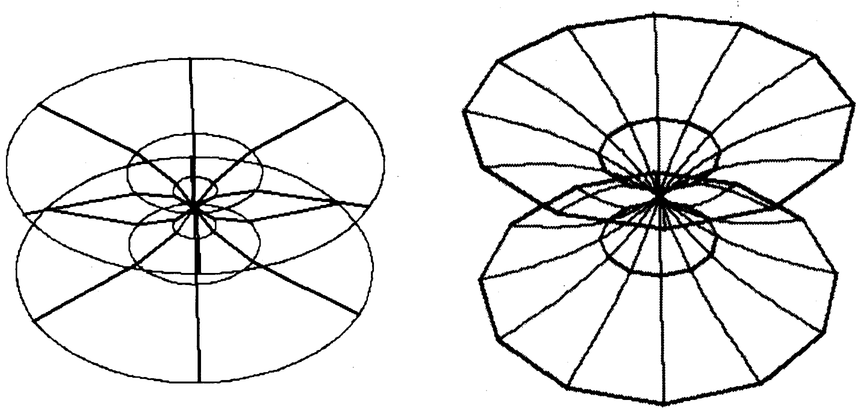

Figure 8: two semi-discrete maximal catenoids

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