PERIODS OF AUTOMORPHIC FORMS: 
THE CASE OF $(GL_{n+1} \times GL_n, GL_n)$

市野 篤史 (ICHINO, ATSUSHI)

This note is a report on a joint work with Shunsuke Yamana [2]. Details will appear elsewhere.

Let $G$ be a connected reductive algebraic group over a number field $F$ and $G'$ a closed subgroup of $G$ over $F$. Let $\mathcal{A}(G)$ and $\mathcal{A}(G')$ denote the spaces of automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. We will consider the period integral

$$P^{G'}(\varphi \otimes \varphi') := \int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g) dg$$

for $\varphi \in \mathcal{A}(G)$ and $\varphi' \in \mathcal{A}(G')$. Let $\pi \subset \mathcal{A}(G)$ and $\pi' \subset \mathcal{A}(G')$ be irreducible subrepresentations. If $P^{G'}(\varphi \otimes \varphi')$ converges for all $\varphi \in \pi$ and $\varphi' \in \pi'$, then

$$P^{G'}|_{\pi \otimes \pi'} \in \text{Hom}_{\triangle G'(A)}(\pi \otimes \pi'_{\mathbb{C}}).$$

We say that $\pi \otimes \pi'$ is $\Delta G'$-distinguished (with respect to $P^{G'}$) if $P^{G'}|_{\pi \otimes \pi'} \neq 0$.

In this note, we consider the case $G = GL_{n+1}$ and $G' = GL_n$, which was studied by Jacquet, Piatetski-Shapiro and Shalika.

Theorem 1 (Jacquet-Piatetski-Shapiro-Shalika). If $\varphi \in \mathcal{A}^{cusp}(G)$ and $\varphi' \in \mathcal{A}^{cusp}(G')$, then

$$P^{G'}(\varphi \otimes \varphi'_s) = I(s, \varphi, \varphi') := \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})} W^{\psi}(g, \varphi)W^{\overline{\psi}}(g, \varphi')|\det g|^s dg.$$

Here, $\mathcal{A}^{cusp}(G)$ and $\mathcal{A}^{cusp}(G')$ denote the spaces of cusp forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively, $\varphi'_s = \varphi' \cdot |\det|^s$ for $s \in \mathbb{C}$, $N \subset G$ and $N' \subset G'$ are upper triangular unipotent subgroups, $W^{\psi}(g, \varphi)$ is a Whittaker function (with respect to a nontrivial character $\psi$ of $F(\mathbb{A})$) defined by

$$W^{\psi}(g, \varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ug)\overline{\psi(u_{1,2} + u_{2,3} + \cdots + u_{n,n+1})} du$$

and $W^{\overline{\psi}}(g, \varphi')$ is defined similarly. The left-hand side converges for all $s$ and the right-hand side converges for $\Re s \gg 0$. Moreover, if $\varphi =
$\otimes_{v}\varphi_{v} \in \pi \subset \mathscr{A}^{cusp}(G)$ and $\varphi' = \otimes_{v}\varphi_{v}' \in \pi' \subset \mathscr{A}^{cusp}(G')$, then

$$I(s, \varphi, \varphi') = L(s + \frac{1}{2}, \pi \times \pi') \prod_{v} \frac{I(s, W_{\varphi_{v}}^{\psi_{v}}, W_{\varphi_{v}'}^{\overline{\psi}_{v}})}{L(s + \frac{1}{2}, \pi_{v} \times \pi'_{v})}.$$  

In particular, $\pi \otimes \pi'$ is $\Delta G'$-distinguished if and only if

$$L\left(\frac{1}{2}, \pi \times \pi'\right) \neq 0.$$

The last assertion is a special case of the Gan-Gross-Prasad conjecture [1]. We also remark that $I(s, \varphi, \varphi')$ makes sense for any automorphic forms $\varphi$ and $\varphi'$. Our main result is an extension of the above theorem.

**Theorem 2** (I-Yamana). Let $\varphi \in \mathscr{A}(G)$ and $\varphi' \in \mathscr{A}(G')$. Then

$$P_{\text{reg}}^{G'}(\varphi \otimes \varphi_{s}') = I(s, \varphi, \varphi')$$

as meromorphic functions of $s$. Here, $P_{\text{reg}}^{G'}$ is the regularized period integral defined below.

As immediate consequences, we obtain the following corollaries.

**Corollary 3.**

1. $P_{\text{reg}}^{G'}$ is $\Delta G'_{\mathbb{A}}$-invariant.
2. $P_{\text{reg}}^{G'}(\varphi \otimes \varphi_{s}') = 0$ unless $\varphi$ and $\varphi'$ are generic.

**Corollary 4.** Assume that $\pi$ and $\pi'$ are induced from irreducible cuspidal automorphic representations of Levi subgroups of $G$ and $G'$ respectively. Then $\pi \otimes \pi'$ is $\Delta G'$-distinguished (with respect to $P_{\text{reg}}^{G'}$) if and only if

$$L\left(\frac{1}{2}, \pi \times \pi'\right) \neq 0.$$

**Corollary 5.** Let $\varphi \in \pi \subset \mathscr{A}^{\text{disc}}(G)$ and $\varphi' \in \pi' \subset \mathscr{A}^{\text{disc}}(G')$. Here, $\mathscr{A}^{\text{disc}}(G)$ and $\mathscr{A}^{\text{disc}}(G')$ denote the spaces of square integrable automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. Assume that $\pi$ is not 1-dimensional. Then $P_{\text{reg}}^{G'}(\varphi \otimes \varphi')$ converges and

$$P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = 0$$

unless $\pi$ and $\pi'$ are cuspidal.

The original motivation was to study the Gan-Gross-Prasad conjecture in the non-tempered case. We also expect an application to the spectral expansion of the relative trace formula of Jacquet-Rallis [4]. In what follows, we will explain the definition of $P_{\text{reg}}^{G'}$ and the proof of Theorem 2.
Following Jacquet, Lapid and Rogawski [3], we define $P_{\text{reg}}^{G'}$. The construction is based on truncation. Recall that Arthur's truncation is given by

$$
\Lambda^{T}\varphi(g) = \sum_{P}(-1)^{\dim \mathfrak{a}_{P}^{G}} \sum_{\gamma \in P \backslash G} \varphi_{P}(\gamma g) \hat{\tau}_{P}(H_{P}(\gamma g) - T),
$$

which is rapidly decreasing. Here, $P = MU$ is a standard parabolic subgroup of $G$, $\varphi_{P}$ is the constant term of $\varphi$ along $P$, $\mathfrak{a}_{P} = \text{Hom}(X^{*}(M), \mathbb{R})$, $\mathfrak{a}_{P}^{*} = X^{*}(M) \otimes \mathbb{R}$, $\mathfrak{a}_{P} = \mathfrak{a}_{P}^{G} \oplus \mathfrak{a}_{G}$ is the canonical decomposition, $H_{P} : G(\mathbb{A}) \to \mathfrak{a}_{P}$ is a function such that $e^{\langle \chi, H_{P}(m) \rangle} = |\chi(m)|_{\mathbb{A}}$ for $\chi \in X^{*}(M)$, $m \in M(\mathbb{A})$ and extended by the Iwasawa decomposition, $T \in \mathfrak{a}_{0}^{G} = \mathfrak{a}_{B}^{G}$ is sufficiently positive with the standard Borel subgroup $B$, and $\hat{\tau}_{P}$ is the characteristic function of the obtuse cone in $\mathfrak{a}_{P}$ spanned by coroots.

The integral $P_{\text{reg}}^{G'}(\Lambda^{T}\varphi \otimes \varphi')$ converges but is hard to compute. Thus we adopt more suitable "mixed truncation" given by

$$
\Lambda_{m}^{T}\varphi(g) = \sum_{P}(-1)^{\dim \mathfrak{a}_{P}^{G}} \sum_{\gamma \in P \backslash PWG'} \varphi_{P}(\gamma g) \hat{\tau}_{P}(H_{P}(\gamma g) - T),
$$

where $W$ is the Weyl group of $G$.

**Lemma 6.**

1. $\Lambda_{m}^{T}\varphi$ is rapidly decreasing on $G'(F) \backslash G'(\mathbb{A})$.
2. $P_{\text{reg}}^{G'}(\Lambda_{m}^{T}\varphi \otimes \varphi') = \sum_{\lambda} p_{\lambda}(T)e^{\langle \lambda, T \rangle}$, where the right-hand side is a finite sum with $\lambda \in (\mathfrak{a}_{0,\mathbb{C}}^{G})^{*}$ and $p_{\lambda} \in \mathbb{C}[\mathfrak{a}_{0}]$.

We define

$$
P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = p_{0}(T)
$$

if the exponents of $\varphi$ and $\varphi'$ avoid some finitely many hyperplanes. It turns out that $p_{0}(T)$ is constant, i.e., independent of $T$. If $\varphi \in \mathcal{A}_{\text{cusp}}(G)$, then $\Lambda_{m}^{T}\varphi = \varphi$, so that $P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = P^{G'}(\varphi \otimes \varphi')$. This identity holds more generally if the exponents of $\varphi$ and $\varphi'$ satisfy some finitely many negativity conditions. We can define $P_{\text{reg}}^{G'}(\varphi \otimes \varphi_{s}')$ for generic $s$ and obtain a meromorphic function of $s$.

Following Lapid and Rogawski [5], we prove Theorem 2. We may assume that $\varphi$ is a cuspidal Eisenstein series. We want to unfold $P^{G'}(\varphi \otimes \varphi')$ by using the Fourier expansion

$$
\varphi(g) = \sum_{i=0}^{n} \sum_{\gamma \in P_{i}' \backslash G'} W_{Q_{i}}^{\psi}(\gamma g, \varphi_{Q_{i}}).
$$
Here,

\[ Q_i = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}^i \right\}_{n+1-i} \subset G, \]

\[ P_i' = \left\{ \begin{pmatrix} * & * \\ 0 & \nabla \end{pmatrix} \right\}_{n-i} \subset G', \]

and \( W_{Q_i}^\psi \) is the Whittaker function for the \( GL_{n+1-i} \) part. If \( \varphi \in \mathcal{A}_{cusp}(G) \), then only the term \( i = 0 \) survives. Since \( P_0' = N' \) and \( W_{Q_0}^\psi = W^\psi \), we can unfold \( P^G(\varphi \otimes \phi') \) to get \( I(s, \varphi, \varphi') \). In general, we cannot unfold. Instead, we compute the convergent integral \( P^G(\theta_\phi \otimes \varphi') \) in two ways. Here, \( \phi_\lambda = f(\lambda) \cdot \varphi \) for \( \lambda \in (\mathfrak{a}_{P,\mathbb{C}}^G)^* \) with \( f \in \mathcal{P}\mathcal{W}((\mathfrak{a}_{P,\mathbb{C}}^G)^*) \) and \( \varphi \in \mathcal{A}_{cusp}(G) \), \( \theta_\phi \) is a pseudo Eisenstein series given by

\[
\theta_\phi(g) = \int_{\Re \lambda = \kappa} f(\lambda) E(g, \varphi, \lambda) \, d\lambda
\]

with sufficiently positive \( \kappa \in (\mathfrak{a}_{P,\mathbb{C}}^G)^* \) and an Eisenstein series

\[
E(g, \varphi, \lambda) = \sum_{\gamma \in P \setminus G} \varphi(\gamma g) e^{\langle \lambda, H_P(\gamma g) \rangle}.
\]

We can show that

\[
P^G(\theta_\phi \otimes \varphi'_s) = \int_{\Re \lambda = \kappa} f(\lambda) P_{\text{reg}}^G(E(\varphi, \lambda) \otimes \varphi'_s) \, d\lambda
\]

under some mild condition of \( f \). We can unfold \( P^G(\theta_\phi \otimes \varphi'_s) \) to get

\[
\int_{\Re \lambda = \kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi') \, d\lambda + \sum_{i=1}^{n} \cdots,
\]

where the last sum vanishes under another mild condition of \( f \). The upshot is

\[
\int_{\Re \lambda = \kappa} f(\lambda) P_{\text{reg}}^G(E(\varphi, \lambda) \otimes \varphi'_s) \, d\lambda = \int_{\Re \lambda = \kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi') \, d\lambda
\]

for sufficiently many \( f \) which allows us to extract the desired identity.

REFERENCES


DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA OIWAKE-CHO, SAKYO-KU, KYOTO 606-8502, JAPAN

E-mail address: ichino@math.kyoto-u.ac.jp