THE TWISTED SATAKE ISOMORPHISM AND CASSELMAN-SHALIKA FORMULA

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ABSTRACT. For an arbitrary split adjoint group we identify the unramified Whittaker space with the space of skew-invariant functions on the lattice of coweights and deduce from it the Casselman-Shalika formula.

1. INTRODUCTION

The Casselman-Shalika formula is a beautiful formula relating values of special functions on a $p$-adic group to the values of finite dimensional complex representations of its dual group. Further, the formula is particularly useful in the theory of automorphic forms for studying $L$-functions.

In this note we provide a new approach to (and a new proof of) Casselman-Shalika formula for the value of spherical Whittaker functions.

To state our results we fix some notations. Let $G$ be a split adjoint group over a local field $F$. We fix a Borel subgroup $B$ with unipotent radical $N$, and consider its Levi decomposition $B = NT$, where $T$ is the maximal split torus. Furthermore, we fix a maximal compact subgroup $K$.

Let $\Psi$ be a non-degenerate complex character of $N$. For an irreducible representation $\pi$ of $G$ it is well known that $\dim \text{Hom}_G(\pi, \text{Ind}_N^G \Psi) \leq 1$ and in case it is non-zero we say that the representation $\pi$ is generic. The Whittaker model of such a generic irreducible representation $\pi$ of $G$ is the image of an embedding

$$W : \pi \hookrightarrow \text{Ind}_N^G \Psi.$$ 

Let now $\pi$ be a generic irreducible representation. Recall that $\pi$ is called unramified if $\pi^K \neq \{0\}$ and that in this case $\pi^K = \mathbb{C} \cdot v_0$. Here $v_0$ is a spherical vector. The explicit formula for the function $W(v_0)$ was given in [CS] and is commonly called the Casselman-Shalika formula.

Recall that there is a bijection between irreducible unramified representations of $G$ and the spectrum of the spherical Hecke algebra $H_K = C_c(K \backslash G/K)$. This commutative algebra admits the following description. Let $\Lambda$ be the coweight lattice of $G$. Recall that $\Lambda$ is canonically identified with $T/(T \cap K)$. The Weyl group $W$ acts naturally on the lattice $\Lambda$. Denote by $\mathbb{C}[\Lambda]^W$ the algebra of $W$-invariant elements in $\mathbb{C}[\Lambda]$.

The Satake isomorphism

$$S : H_K \simeq (\text{Ind}_T^{T \cap K} 1)^W = \mathbb{C}[\Lambda]^W.$$
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is defined by

\[ S(f)(t) = \delta_B^{-1/2}(t) \int_N f(nt) \, dn \]

The main result of the paper is a description of the Whittaker spherical space \((\text{Ind}_N^G \Psi)^K\) as a concrete \(H_K \simeq \mathbb{C}[\Lambda]^W\)-module. Namely, it is identified with the space \(\mathbb{C}[\Lambda]^{W, -}\) of functions on the lattice of coweights, that are skew-invariant under the action of the Weyl group \(W\). More formally,

**Theorem 1.1.** There is a canonical isomorphism

\[ j : (\text{Ind}_N^G \Psi)^K \to \mathbb{C}[\Lambda]^{W, -}, \]

compatible with Satake isomorphism \(S : H_K \simeq \mathbb{C}[\Lambda]^W\).

From this result it easily follows that the twisted Satake map

\[ S_{\Psi} : C_c(G/K) \to (\text{Ind}_N^G \Psi)^K, \quad S_{\Psi}(f)(t) = \int_N f(nt) \overline{\Psi(n)} \, dn \]

sends the spectral basis of the spherical Hecke algebra \(H_K = C_c(K \backslash G/K)\) to the basis of characteristic functions of \((\text{ind}_N^G \Psi)^K\). In [FGKV], it is explained that this latter result is equivalent to the Casselman-Shalika formula in [CS] (see section 6 for a full account). Thus, we obtain a proof of the Casselman-Shalika formula that does not use the uniqueness of the Whittaker model.

Let us now quickly describe our proof of the main result. It was inspired by a new simple proof [S] by Savin of the Satake isomorphism of algebras

\[ S : H_K \simeq (\text{ind}_{I \cap K}^T 1)^W = \mathbb{C}[\Lambda]^W. \]

Savin has observed that the Satake map \(S : C_c(G/K) \to (\text{ind}_N^G 1)^K\) restricted to

\[ C_c(I \backslash G/K) = (\text{ind}_I^G 1)^K, \]

where \(I\) is an Iwahori subgroup, defines an explicit isomorphism

\[ (\text{ind}_I^G 1)^K \simeq \text{ind}_{T \cap K}^T 1 = \mathbb{C}[\Lambda]. \]

Restricting \(S_{\Psi}\) to \((\text{ind}_I^G 1)^K\), we prove there there exists an isomorphism

\[ j : (\text{ind}_N^G \Psi)^K \to \mathbb{C}[\Lambda]^{W, -} \]

of \(H_K \simeq \mathbb{C}[\Lambda]^W\)-modules, making the following diagram

\[
\begin{array}{ccc}
(\text{ind}_I^G 1)^K & \xrightarrow{S_{\Psi}} & (\text{ind}_N^G \Psi)^K \\
\downarrow S & & \downarrow j \\
\mathbb{C}[\Lambda] & \xrightarrow{\text{alt}} & \mathbb{C}[\Lambda]^{W, -}
\end{array}
\]
commutative. Here the space $\mathbb{C}[\Lambda]^{W^-}$ is a space of $W$ skew-invariant elements of $\mathbb{C}[\Lambda]$ and the alternating map $alt$ are defined in the section 3.

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2. Notations

Let $F$ be a local non-archimedean field and let $q$ be a characteristic of its residue field. Let $G$ be a split adjoint group defined over $F$. Denote by $B$ a Borel subgroup of $G$, by $N$ its unipotent radical, by $\bar{N}$ the opposite unipotent radical, by $T$ the maximal split torus and by $W$ the Weyl group.

Denote by $R$ the set of positive roots of $G$ and by $\Delta$ the set of simple roots. For each $\alpha \in R$ let $x_\alpha : F \to N$ denote the one parametric subgroup corresponding to the root $\alpha$ and $N_\alpha^k = \{x_\alpha(r) : |r| \leq q^{-k}\}$.

Let $\Psi$ be a non-degenerate complex character of $N$ of conductor 1, i.e. for any $\alpha \in \Delta$

$$\Psi|_{N_\alpha^0} \neq 1, \Psi|_{N_\alpha^1} = 1.$$  

Let $K$ be a maximal compact subgroup of $G$. Then $T_K = T \cap K$ is a maximal compact subgroup of $T$. Choose an Iwahori subgroup $I \subset K$ such that $I \cap N = N_\alpha^1$ for all $\alpha \in R$. In particular $\Psi_{N \cap I} = 1$, but $\Psi|_{N_\alpha^0} \neq 1$ for any $\alpha \in \Delta$.

We fix a Haar measure on $G$ normalized such that the measure of $I$ is one.

The coweight lattice $\Lambda$ of $G$ is canonically identified with $T/T_K$. For any $\lambda \in \Lambda$ denote by $t_\lambda \in T$ its representative. The coweight $\rho$ denotes the half of all the positive coroots. Since $G$ is adjoint one has $\rho \in \Lambda$. We denote by $\Lambda^+$ the set of dominant coweights.

Let $L G$ be the complex dual group of $G$. Then $\Lambda$ is also identified with the lattice of weights of $L G$. For a dominant weight $\lambda$ we denote by $V_\lambda$ the highest weight module of $L G$ and by $wt(V_\lambda)$ the multiset of all the weights of this module.

3. Functions on Lattices

Consider the algebra $\mathbb{C}[\Lambda] = \text{Span}\{e^\nu : \nu \in \Lambda\}$. The Weyl group $W$ acts naturally on the lattice $\Lambda$. We denote by $\mathbb{C}[\Lambda]^W$ the algebra of $W$-invariant elements in $\mathbb{C}[\Lambda]$. The character map defines an isomorphism of algebras

$$\text{Rep}(L G) \simeq \mathbb{C}[\Lambda]^W.$$
For an irreducible module $V_\lambda$ denote

$$a_\lambda = \text{char}(V_\lambda) = \sum_{\nu \in \text{wt}(V_\lambda)} e^\nu.$$  

The elements \(\{a_\lambda | \lambda \in \Lambda^+\}\) form a basis of \(\mathbb{C}[\Lambda]^W\). The algebra \(\mathbb{C}[\Lambda]^W\) acts on the space \(\mathbb{C}[\Lambda]\) by multiplication.

The element \(f \in \mathbb{C}[\Lambda]\) is called skew-invariant if \(w(f) = (-1)^{l(w)}f\), where \(l(w)\) is the length of the element \(w\). Denote by \(\mathbb{C}[\Lambda]^{W,-}\) the space of \(W\) skew-invariant elements. The algebra \(\mathbb{C}[\Lambda]^W\) acts on \(\mathbb{C}[\Lambda]^{W,-}\) by multiplication. Note that the action is torsion free.

Define the alternating map

$$\text{alt} : \mathbb{C}[\Lambda] \to \mathbb{C}[\Lambda]^{W,-}, \quad \text{alt}(e^\mu) = \sum_{w \in W} (-1)^{l(w)} e^{w\mu} : \mu \in \Lambda$$

It is a map of \(\mathbb{C}[\Lambda]^W\) modules. The elements

$$\{r_{\mu+\rho} = \sum_{w \in W} (-1)^{l(w)} e^{w(\mu+\rho)} : \mu \in \Lambda^+\}$$

form a basis of \(\mathbb{C}[\Lambda]^{W,-}\). Note that for any \(\lambda \in \Lambda^+\)

$$\text{alt}(e^{\lambda+\rho}) = r_{\lambda+\rho} = r_{\rho} \cdot a_\lambda = \text{alt}(a_\lambda),$$

where the second equality is the Weyl character formula.

## 4. Hecke algebras

### 4.1. The spherical Hecke algebra

The spherical Hecke algebra \(H_K = C_c(K \backslash G/K)\) is the algebra of locally constant compactly supported bi-\(K\) invariant functions with the multiplication given by convolution \(*\). It has identity element \(1_K\) - the characteristic function of \(K\) divided by \([K : I]\).

Consider the Satake map

$$S : C_c(G/K) \to C(N \backslash G/K) = C(T/T_K)$$

defined by

$$S(f)(t) = \delta_B^{-1/2}(t) \int_{N} f(nt)dn.$$  

The famous Satake theorem claims that the restriction of \(S\) to \(H_K\) defines an isomorphism of algebras \(S : H_K \simeq \mathbb{C}[\Lambda]^W\). Denote by \(A_\lambda\) the element of \(H_K\) corresponding to \(a_\lambda\) under this map. Thus \(H_K = \text{Span}\{A_\lambda : \lambda \in \Lambda^+\}\).

### 4.2. The Iwahori-Hecke algebra

The Iwahori-Hecke algebra \(H_I = C_c(I \backslash G/I)\) is the algebra of locally constant compactly supported bi-\(I\) invariant functions with the multiplication given by convolution. Below we remind the list of properties of \(H_I\), all can be found in [HKP].
(1) The algebra $H_I$ contains a commutative algebra $A \cong \mathbb{C}[\Lambda].$

$$A = \text{Span}\{\theta_\mu| \mu \in \Lambda\},$$

where

$$\theta_\mu = \begin{cases} \delta_B^{1/2} 1_{I \mu I} & \mu \in \Lambda^+; \\
\theta_{\mu_1} * \theta_{\mu_2}^{-1} & \mu = \mu_1 - \mu_2, \quad \mu_1, \mu_2 \in \Lambda^+
\end{cases}$$

The center $Z_I$ of the algebra $H_I$ is $A^W \cong \mathbb{C}[\Lambda]^W.$

(2) The finite dimensional Hecke algebra $H_f = C(I \backslash K/I)$ is a subalgebra of $H_I$. The elements $t_w = 1_{IwI},$ where $w \in W$ form a basis of $H_f$. Multiplication in $H$ induces a vector space isomorphism

$$H_f \otimes_{\mathbb{C}} A \rightarrow H_I$$

In particular the elements $t_w \theta_\mu$ where $w \in W, \mu \in \Lambda$ form a basis of $H_I$.

(3) The algebra $H_K$ is embedded naturally in $H_I$. One has $H_K = Z_I * 1_K$ and

$$A_\lambda = \left( \sum_{\nu \in \text{wt}(V_\lambda)} \theta_\nu \right) * 1_K$$

(4) For the simple reflection $s \in W$ corresponding to a simple root $\alpha$ and a coweight $\mu$ one has

$$t_s \theta_\mu = \theta_{s\mu} t_s + (1 - q) \frac{\theta_{s\mu} - \theta_\mu}{1 - \theta_{-\alpha}}.$$

In particular, $t_s$ commutes with $\theta_{k\alpha} + \theta_{-k\alpha}$ for any $k \geq 0$. In addition $t_s$ commutes with $\theta_\mu$ whenever $s\mu = \mu$.

4.3. The intermediate algebra. Finally consider the space $H_{I,K}$ defined by

$$H_K \subset H_{I,K} = H_I * 1_K = C_c(I \backslash G/K) \subset H_I.$$

It has a structure of right $H_K$ module. The space $H_{I,K}$ plays a crucial role in Savin’s paper [S]. The Satake map restricted to it is the isomorphism of $H_K \cong \mathbb{C}[\Lambda]^W$ modules:

$$S : H_{I,K} \cong \mathbb{C}[\Lambda], \quad S(\theta_\mu * 1_K) = e^\mu.$$

In particular, it is shown that the elements $\{\theta^K_\mu = \theta_\mu * 1_K, \mu \in \Lambda\}$ form a basis of $H_{I,K}$.

5. The Whittaker space $(\text{ind}_N^G \Psi)^K$

Let $\Psi$ be a non-degenerate character of conductor 1. Consider the space $(\text{ind}_N^G \Psi)^K$ of complex valued functions on $G$ that are ($N, \Psi$)-equivariant on the left, right $K$-invariant functions and are compactly supported modulo $N$. 

The space \((\text{ind}^G_N \Psi)^K\) has a structure of right \(H_K\) module by
\[
(\phi * f)(x) = \int_G \phi(xy^{-1})f(y)dy, \quad \phi \in (\text{ind}^G_N \Psi)^K, f \in H_K.
\]

Any function \(\phi\) on \((\text{ind}^G_N \Psi)^K\) is determined by its values on \(t_\lambda : \lambda \in \Lambda\) and \(\phi(t_\lambda) = 0\) unless \(\lambda \in \Lambda^+ + \rho\).

The space \((\text{ind}^G_N \Psi)^K\) has a basis of characteristic functions \(\{\phi_\lambda : \lambda \in \Lambda^+ + \rho\}\) where
\[
\phi_\lambda(ntk) = \begin{cases} 
\delta_B^{1/2}(t)\Psi(n) & t \in Nt_\lambda K, \lambda \in \Lambda^+ + \rho; \\
0 & \text{otherwise}
\end{cases}
\]

The main theorem of this paper is the description of \((\text{ind}^G_N \Psi)^K\) as \(H_K\) module.

**Theorem 5.1.** Let \(\Psi\) be a character of conductor 1. Then there is an isomorphism
\[
j : (\text{ind}^G_N \Psi)^K \simeq \mathbb{C}[\Lambda]^{W,-}
\]
compatible with \(H_K \simeq \mathbb{C}[\Lambda]^{W}\).

### 5.1. The twisted Satake isomorphism.
For a fixed character \(\Psi\) of \(N\), consider a twisted Satake map
\[
S_\Psi : C_c(G/I) \to (\text{ind}^G_N \Psi)^I
\]
defined by
\[
S_\Psi(f)(t) = \int_N f(nt)\overline{\Psi(n)}dn.
\]

**Corrolary 5.2.** The restriction of \(S_\Psi\) to the right \(H_K\) submodule \(\theta^K_\rho \ast H_K\) defines an isomorphism
\[
S_\Psi : \theta^K_\rho \ast H_K \simeq (\text{ind}^G_N \Psi)^K
\]
such that \(S_\Psi(\theta^K_\rho \ast A_\lambda) = \phi_{\lambda+\rho}\).

**Proof.** By Weyl character formula \(r_{\lambda+\rho} = r_\rho \cdot a_\lambda\). Hence
\[
j(\phi_{\lambda+\rho}) = r_{\lambda+\rho} = r_\rho \cdot a_\lambda = j(\phi_\rho \ast A_\lambda),
\]
and thus \(\phi_\rho \ast A_\lambda = \phi_{\lambda+\rho}\).

Restricting \(S_\Psi\) to \(H_{I,K}\), we obtain
\[
S_\Psi(\theta^K_\rho \ast A_\lambda) = S_\Psi(\theta^K_\rho) \ast A_\lambda = \phi_\rho \ast A_\lambda = \phi_{\lambda+\rho}.
\]

Since \(A_\lambda\) and \(\phi_{\lambda+\rho}\) are bases of \(H_K\) and \((\text{ind}^G_N \Psi)^K\) respectively, the map \(S_\Psi\) is an isomorphism.

To prove the theorem we shall need two lemmas. The first one ensures surjectivity of the map \(S_\Psi\) and the second one describes its kernel.
Lemma 5.3. $S_{\Psi}(\theta_{\mu}^{K}) = \phi_{\mu}$ for all $\mu \in \Lambda^{+} + \rho$. In particular the map $S_{\Psi} : H_{I,K} \rightarrow \left( \text{ind}_{N}^{G} \Psi \right)^{K}$ is surjective.

Proof. It is enough to compute $S_{\Psi}(\theta_{\mu}^{K} * 1_{K})(t_{\gamma})$ for $\gamma \in \Lambda^{+}$.

Since $\mu$ is dominant one has

$$\theta_{\mu}^{K} = \delta_{B}^{1/2}(t_{\mu})1_{It_{\mu}K}$$

and hence

$$S_{\Psi}(\theta_{\mu}^{K})(t_{\gamma}) = \delta_{B}^{1/2}(t_{\gamma}) \int_{N_{\gamma,\mu}} \overline{\Psi(n)} dn,$$

where

$$N_{\gamma,\mu} = \{ n \in N : nt_{\gamma} \in It_{\mu}K \}.$$

The set

$$N_{\gamma,\mu} = \begin{cases} \emptyset & \text{if } \gamma \neq \mu \\ N \cap K & \text{if } \gamma = \mu \end{cases}$$

Indeed, since $\mu \in \Lambda^{+}$ one has $It_{\mu}K = (N \cap I)t_{\mu}K$. One inclusion is obvious. For another inclusion use the Iwahori factorization

$$I = (I \cap N)T_{K}(I \cap \overline{N})$$

to represent any $g \in It_{\mu}K$ as

$$g = na_{0}\overline{n}t_{\mu}k = nt_{\mu}a_{0}(t_{\mu}^{-1}\overline{n}t_{\mu})k,$$

where $n \in N \cap I, a_{0} \in T_{K}, \overline{n} \in \overline{N}, k \in K$. Since $\mu$ is dominant one has $(t_{\mu}^{-1}\overline{n}t_{\mu}) \in K$. So $g \in (N \cap I)t_{\mu}K$. Hence $N_{\gamma,\mu} = \emptyset$ unless $\gamma = \mu$ and $N_{\mu,\mu} = (N \cap I)t_{\mu}(N \cap K)t_{\mu}^{-1} = N \cap I$ since $\mu \in \Lambda^{+} + \rho$. In particular $\Psi|_{N_{\mu,\mu}} = 1$. Hence

$$S_{\Psi}(\theta_{\mu} * 1_{K}) = \phi_{\mu}.$$

\[ \square \]

Lemma 5.4. Let $\alpha \in \Delta, s$ be a simple reflection corresponding to $\alpha$ and $\iota_{\alpha} = 1_{I} + t_{s}$ be the characteristic function of a parahoric subgroup $I_{\alpha}$ corresponding to $\alpha$.

1) $S_{\Psi}(\iota_{\alpha}) = 0$.

2) $S_{\Psi}(\theta_{\mu}^{K} + \theta_{s,\mu}^{K}) = 0$ for all $\mu \in \Lambda$.

Proof. 1) $S_{\Psi}(\iota_{\alpha})(tw) = \int_{N} \iota_{\alpha}(ntw)\overline{\Psi(n)}dn = \int_{N \cap I_{\alpha}(tw)^{-1}} \overline{\Psi(n)}dn$

The set $N \cap I_{\alpha}(tw)^{-1}$ is empty unless $w \in \{e, s \}$ and $t \in T_{K}$, in which case

$$S_{\Psi}(\iota_{\alpha})(tw) = \int_{N \cap I_{\alpha}} \overline{\Psi(n)}dn = 0$$
since the integral contains an inner integral over $N_{\alpha}^{0}$ on which $\Psi$ is not trivial.

2) Let us represent any $\mu = \mu' + k\alpha$ where $(\mu', \alpha) = 0$. Then $s\mu = \mu' - k\alpha$. In particular
\[ \theta_{\mu}^{K} + \theta_{s\mu}^{K} = \theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})i_{\alpha}1_{K} \]
By the results in 4.2 the element $i_{\alpha}$ commutes with $\theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})$ and hence the above equals
\[ \iota_{\alpha}\theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})1_{K} = \iota_{\alpha}(\theta_{\mu}^{K} + \theta_{s\mu}^{K}) \]
By part (1) it follows that $S_{\Psi}(\theta_{\mu}^{K} + \theta_{s\mu}^{K}) = 0$. \hfill \square

**Proof.** of 5.1. We have shown that the map $S_{\Psi}$ is surjective onto $(\text{ind}_{N}^{G}\Psi)^{K}$ and
\[ \text{Ker} \ S_{\Psi} = \text{Span}\{\theta_{\mu} - (-1)^{l(w)}\theta_{w\mu} | \mu \in \Lambda, w \in W\} \]
Another words
\[ (\text{ind}_{N}^{G}\Psi)^{K} \simeq H_{I,K}/\text{Ker} \ S_{\Psi} = \mathbb{C}[\Lambda]^{W,-} \]
as $H_{K} \simeq \mathbb{C}[\Lambda]^{W}$-modules. \hfill \square

6. **Casselman-Shalika Formula**

Let $(\pi, G, V)$ be an irreducible smooth generic unramified representation and denote by $\gamma \in LT/W$ its Satake conjugacy class. Choose a spherical vector $v_{0}$ and normalize the Whittaker functional $W_{\gamma} \in \text{Hom}_{G}(\pi, \text{Ind}_{N}^{G}\overline{\Psi})$ such that $W_{\gamma}(t_{\rho}v_{0}) = 1$.

The Casselman-Shalika formula reads as follows:

**Theorem 6.1.**
\[ W_{\gamma}(v_{0})(t_{\lambda+\rho}) = \begin{cases} \delta_{B}^{1/2}(t_{\lambda+\rho}) \text{tr} \ V_{\lambda}(t_{\gamma}) & \lambda \in \Lambda^{+} \\ 0 & \text{otherwise} \end{cases} \]

It is shown in [FGKV], that Theorem 5.2 that the formula (6.1) implies the Corollary 5.2 and it is mentioned that the two statements are equivalent. Let us now prove the other direction.

**Proof.** We deduce the formula 6.1 from 5.2. Let $\pi$ be a generic unramified representation with the Satake parameter $\gamma \in LT$ and a spherical vector $v_{0}$ and the Whittaker model $W_{\gamma} : \pi_{\gamma} \rightarrow \text{Ind}_{N}^{G}\overline{\Psi}$ such that $W_{\gamma}(v_{0})(t_{\rho}) = 1$.

Define the map $\chi_{\gamma} : H_{K} \rightarrow \mathbb{C}$ by
\[ \pi(f)v_{0} = \int_{G} f(g)\pi(g)v_{0} \, dg = \chi_{\gamma}(f)v_{0} \]
and the map $r_{\gamma} : \text{ind}_{N}^{G}\Psi \rightarrow \mathbb{C}$ by
\[ r_{\gamma}(\phi) = \int_{N\backslash G} W_{\gamma}(v_{0})(g)\phi(g) \, dg. \]
Then
\[ r_\gamma(S_\Psi(\theta^K_\rho * A_\lambda)) = \int_{N \backslash G} \int_N (\theta^K_\rho * A_\lambda)(ng) \bar{\Psi}(n) W_\gamma(v_0)(g) \, dn \, dg = \]
\[ \int_G \int_G (\theta^K_\rho * A_\lambda)(g) W_\gamma(v_0)(g) \, dg = \]
\[ \int_G \int_G \theta^K_\rho (gx^{-1}) A_\lambda(x) W_\gamma(gx^{-1} \cdot x \cdot v_0)(1) \, dx \, dg = \chi_\gamma(A_\lambda) W_\gamma(v_0)(t_\rho). \]

Under the identification \( H_K \simeq \text{Rep}^L(G) \) the homomorphism \( \chi_\gamma \) sends an irreducible representation \( V \) to \( \text{tr} \, V(\gamma) \). In particular \( \chi_\gamma(A_\lambda) = \text{tr} \, V_\lambda(\gamma) \).

\[ \text{tr} \, V_\lambda(\gamma) = \chi_\gamma(A_\lambda) W(v_0)(t_\rho) = \]
\[ r_\gamma(S_\Psi(\theta^K_\rho * A_\lambda)) = r_\gamma(\phi_{\lambda+\rho}) = \delta_B^{-1/2}(t_{\lambda+\rho}) W_\gamma(v_0)(t_{\lambda+\rho}) \]

Hence
\[ W_\gamma(v_0)(t_{\lambda+\rho}) = \delta_B^{1/2}(t_{\lambda+\rho}) \text{tr} \, V_\lambda(\gamma). \]

\[ \square \]

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