THE $\text{Sp}_n \times \text{Sp}_n$-PERIOD OF A PSEUDO-EISENSTEIN SERIES ON $\text{Sp}_{2n}$

EREZ LAPID AND OMER OFFEN

ABSTRACT. This is a report on our study of $\text{Sp}_n \times \text{Sp}_n$-period integrals on the automorphic spectrum of $\text{Sp}_{2n}$. We announce our formula for the period integrals of pseudo-Eisenstein series.

This is a report on our study of period integrals over $\text{Sp}_n \times \text{Sp}_n$ of automorphic forms on $\text{Sp}_{2n}$. In [LO] we suggest a notion of the $\text{Sp}_n \times \text{Sp}_n$-distinguished automorphic spectrum and provide an upper bound in terms of Langlands fine spectral expansion (cf. [MW95, § V]) of the automorphic spectrum of $\text{Sp}_{2n}$. Roughly speaking, it is the orthogonal complement of the $\text{Sp}_{2n}$-invariant space of pseudo-Eisenstein series with $\text{Sp}_n \times \text{Sp}_n$-vanishing period. The results of [AGR93] imply that the $\text{Sp}_n \times \text{Sp}_n$-distinguished automorphic spectrum contains no cuspidal automorphic functions. We study period integrals on the continuous spectrum.

The technical heart of our work is a formula for the periods of pseudo-Eisenstein series that we explicitly describe below. Results that are mentioned below without reference are proved in [LO].

1. NOTATION

Let $F$ be a number field and let $\mathbb{A}$ be the ring of adeles of $F$. We denote $F$-varieties in bold letters such as $X$ and write $X = X(F)$ for the corresponding set of $F$-points.

For an algebraic group $Q$ defined over $F$ we denote by $X^*(Q)$ the lattice of $F$-rational characters of $Q$ and let $a_Q^* = X^*(Q) \otimes \mathbb{R}$, $a_Q = \text{Hom}(a_Q^*, \mathbb{R})$ the real vector space dual to $a_Q^*$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_Q$ the natural pairing between them. We view $a_Q^*$ and its dual as Euclidean spaces and denote the norm on either of them by $\|\cdot\|$. We denote by $a_\mathbb{C}$ the complexification of a real vector space $a$. We also set

$$Q(\mathbb{A})^1 = \{q \in Q(\mathbb{A}) : \forall \chi \in X^*(Q), |\chi(q)| = 1\}.$$

There is an isomorphism

$$H_Q : Q(\mathbb{A})^1 \backslash Q(\mathbb{A}) \rightarrow a_Q$$

such that $e^{\langle \chi, H_Q(q) \rangle} = |\chi(q)|_\mathbb{A}^*$, $\chi \in X^*(Q), q \in Q(\mathbb{A})$.

Let $\delta_Q$ denote the modulus function of $Q(\mathbb{A})$. It is a character of $Q(\mathbb{A})^1 \backslash Q(\mathbb{A})$ and therefore there exists $\rho_Q \in a_Q^*$ such that

$$\delta_Q(q) = e^{\langle 2\rho_Q, H_Q(q) \rangle}, \quad q \in Q(\mathbb{A}).$$


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Let $G$ be a reductive group and $P_0$ a minimal parabolic subgroup of $G$ both defined over $F$. In general we denote by $\mathfrak{S}_G$ a Siegel domain for $G\backslash G(\mathbb{A})$ and by $\mathfrak{S}_G^1$ a Siegel domain for $G^1\backslash G(\mathbb{A})$ (cf. [MW95, I.2.1]).

Let $K$ be a maximal compact subgroup of $G(\mathbb{A})$ in good position with respect to $P_0$ so that we have the Iwasawa decomposition $G(\mathbb{A}) = P_0(\mathbb{A})K$. The map $H_0 = H_{P_0} : P_0(\mathbb{A}) \to \alpha_{P_0}$ is extended to $G(\mathbb{A})$ via the Iwasawa decomposition, i.e., $H_0(pk) = H_0(p)$, $p \in P_0(\mathbb{A})$ and $k \in K$. Let $T_G$ be the split part of the identity connected component of the center of $G$. Applying the imbedding $x \mapsto 1 \otimes x : \mathbb{R} \to F_{\infty} = F \otimes_{\mathbb{Q}} \mathbb{R}$ we imbed $T_G(\mathbb{R})$ in $T_G(F_{\infty}) \hookrightarrow T_G(\mathbb{A})$ and denote by $A_G$ the image of the identity component $T_G(\mathbb{R})$ in $T_G(\mathbb{A})$. Then $H_G : A_G \to G$ is an isomorphism. Denote by $\nu \mapsto e^\nu$ its inverse.

For $n \in \mathbb{N}$ let $w_n = (\delta_{n+1-j}) \in GL_n$, $J_n = \left( \begin{array}{cc} 0 & w_n \\ -w_n & 0 \end{array} \right)$ and

$$Sp_n = \{g \in GL_{2n} : gJ_n g^{-1} \}.$$ 

Let $^*$ be the automorphism of $GL_n$ given by $g \mapsto g^* = w_n^t g^{-1} w_n$. The imbedding $g \mapsto \text{diag}(g, g^*) : GL_n \to Sp_n$ identifies $GL_n$ with the Siegel Levi subgroup of $Sp_n$.

To every $\gamma = (n_1, \ldots, n_k; r)$ where $k, r \geq 0$, $n_1, \ldots, n_k > 0$, and $n_1 + \cdots + n_k + r = n$ we associate the standard parabolic subgroup $P = P_{\gamma} = M \ltimes U$ consisting of block upper triangular matrices in $Sp_n$ where

$$M = M_\gamma = \{\text{diag}(g_1, \ldots, g_k, h, g_k^*, \ldots, g_1^*) : h \in Sp_r, g_i \in GL_{n_i}, i = 1, \ldots, k\}.$$ (We call such $M$'s standard Levi subgroups.) In particular, $P_{(n,0)}$ is the Siegel parablic subgroup of $Sp_n$ and $P_{(n,0)} \subseteq P_{(n,0)}$ if and only if $r = 0$.

Let $\delta_n = \text{diag}(1, -1, 1, \ldots, (-1)^{n-1}) \in GL_n$ and $\epsilon_n = \text{diag}(\delta_n, \delta_n^*) \in Sp_n$.

For the rest of this note fix $n \in \mathbb{N}$ and let $G = Sp_{2n}$, $\epsilon = \epsilon_{2n}$ and $H = C_G(\epsilon)$, the centraliser of $\epsilon$ in $G$.

Let $B = P_{(1, \ldots, 1; 0)} = T \ltimes N$ be the standard Borel subgroup of $G$ with unipotent radical $N$ and $T = M_{(1, \ldots, 1; 0)}$. We call a Levi subgroup of $G$ semi-standard if it contains $T$.

For a standard Levi subgroup $M$ of $G$, we denote by $\mathcal{L}(M)$ the (finite) set of (semi-standard) Levi subgroups containing $M$.

For a standard parabolic subgroup $P = M \ltimes U$ of $G$ let $\Sigma_M = R(T_M, G) \subseteq \mathfrak{a}_M^+$ be the root system of $G$ with respect to $T_M$ and $\Sigma_p^+$ the subset of positive roots in $\Sigma_M$ with respect to $P$.

We identify $G/H$ with the $G$-conjugacy class $X$ of $\epsilon$, a closed subvariety of $G$, via $gH \mapsto g\epsilon g^{-1}$. For $x \in X$ and a subgroup $Q$ of $G$ we denote by $Q \cdot x = \{qxq^{-1} : q \in Q\}$ the $Q$-conjugacy class of $x$ and by $Q_x = \{q \in Q : qxq^{-1} = x\}$ the centraliser of $x$ in $Q$.

2. PSEUDO-EISENSTEINS SERIES

Let $P = M \ltimes U$ be a standard parabolic subgroup of $G$. For any $R > 0$ let $C_R(U(\mathbb{A})M \backslash G(\mathbb{A}))$ be the space of continuous cuspidal functions $\phi$ on $U(\mathbb{A})M \backslash G(\mathbb{A})$ such that for all $N > 1$ we have

$$\sup_{m \in \Sigma'_U, \alpha \in \Sigma_M, k \in K} |\phi(amk)| \|m\|^N e^{R\|H_P(\alpha)\|} < \infty.$$
For any \( \phi \in C_{R}(U(\mathbb{A})M \backslash G(\mathbb{A})) \) define the pseudo-Eisenstein series \( \theta_{P,\phi} \) on \( G \backslash G(\mathbb{A}) \) by the absolutely convergent series
\[
\theta_{P,\phi}(g) = \sum_{\gamma \in P \backslash G} \phi(\gamma g).
\]

For \( \phi \in C_{R}(U(\mathbb{A})M \backslash G(\mathbb{A})) \) and \( \lambda \in \mathfrak{a}_{M,\mathbb{C}}^{*} \) with \( \| \lambda + \rho_{P} \| < R \) we write
\[
\phi[\lambda](g) = e^{-\langle \lambda, H_{P}(g) \rangle} \int_{A_{M}} e^{-\langle \lambda + \rho_{P}, H_{P}(a) \rangle} \phi(ag) \, da.
\]

Let \( C_{R}^\infty(U(\mathbb{A})M \backslash G(\mathbb{A})) \) be the smooth part of \( C_{R}(U(\mathbb{A})M \backslash G(\mathbb{A})) \). For \( \phi \in C_{R}^\infty(U(\mathbb{A})M \backslash G(\mathbb{A})) \) we have the inversion formula
\[
\phi(g) = \int_{\lambda_{0} + i \mathfrak{a}_{M}} \phi[\lambda]_{\lambda}(g) \, d\lambda
\]
for any \( \lambda_{0} \in \mathfrak{a}_{M}^{*} \) with \( \| \lambda_{0} + \rho_{P} \| < R \).

Moreover, for any \( R' < R \) and \( N > 0 \) we have
\[
(1) \quad \sup_{m \in \mathfrak{S}_{M}^{1}, k \in K, \lambda \in \mathfrak{a}_{M,\mathbb{C}}^{*}, \| \lambda + \rho_{P} \| \leq R'} |\phi[\lambda](mk)| (\| m \| + \| \lambda \|)^{N} < \infty.
\]

We wish to study the period integrals
\[
\mathcal{P}_{H}(\theta_{P,\phi}) := \int_{H \backslash H(\mathbb{A})} \theta_{P,\phi}(h) \, dh.
\]
It is easy to see that for \( R \) large enough \( g \mapsto \sum_{\gamma \in P \backslash G} |\phi(\gamma g)| \) is bounded on \( G(\mathbb{A}) \) for every \( \phi \in C_{R}(U(\mathbb{A})M \backslash G(\mathbb{A})) \) and therefore \( \mathcal{P}_{H}(\theta_{P,\phi}) \) is defined by an absolutely convergent integral for all \( \phi \in C_{R}(U(\mathbb{A})M \backslash G(\mathbb{A})) \).

We express \( \mathcal{P}_{H}(\theta_{P,\phi}) \) as a sum parameterised by certain double cosets in \( P \backslash G/H \) of certain \( H(\mathbb{A}) \)-invariant linear forms, called intertwining periods, on representations of \( G(\mathbb{A}) \) induced from \( P(\mathbb{A}) \). In order to formulate our results we explain in §3 the relevant results concerning the double coset decomposition and in §4 we define and state the convergence of the intertwining periods.

### 3. DOUBLE COSETS

Let \( P = M \ltimes U \) be a standard parabolic subgroup of \( G \). The map \( g \mapsto g e g^{-1} \) defines a bijection
\[
P \backslash G/H \simeq P \backslash X.
\]
Instead of double cosets we study \( P \)-conjugacy classes in \( X \).

#### 3.1. Admissible orbits.

An element \( x \in X \) (or \( P \cdot x \)) is called \( M \)-admissible if \( N_{G}(M) \cap P \cdot x \) is not empty. We denote by \( [P \backslash X]_{\text{adm}} \) the set of \( M \)-admissible \( P \)-conjugacy classes in \( X \).

If \( x \in X \) is \( M \)-admissible and \( y \in N_{G}(M) \cap P \cdot x \) then \( N_{G}(M) \cap P \cdot x = M \cdot y \) is a unique \( M \)-orbit. The correspondence \( P \cdot x \mapsto M \cdot y \) is a bijection
\[
[P \backslash X]_{\text{adm}} \simeq M \backslash (X \cap N_{G}(M))
\]
between $M$-admissible $P$-conjugacy classes in $X$ and $M$-conjugacy classes in $N_G(M) \cap X$.

The group $N_G(M)$ acts on $\mathfrak{a}_M^*$ ($M$ acts trivially) and in particular, $x \in N_G(M) \cap X$ acts as an involution on $\mathfrak{a}_M^*$ and decomposes it into a direct sum of the ±1-eigenspaces which we denote by $(\mathfrak{a}_M^*)^\pm_x$ respectively. (A similar decomposition applies to the dual space $\mathfrak{a}_M = (\mathfrak{a}_M)_x^- \oplus (\mathfrak{a}_M)_x^+$.) Any such $x$ defines

$$L = L(x) = \cap \mathcal{U} \in \mathcal{L}(M), x \in L \mathcal{U}' \in \mathcal{L}(M)$$

so that $(\mathfrak{a}_M)_x^\pm = \mathfrak{a}_L^*$ (cf. [Art82, p. 1299]).

We call $x \in N_G(M) \cap X$ (or $M \cdot x$) $M$-standard relevant if $M$ has the form

$$M = M_{(r_1,r_1,\ldots,r_k,r_k,s_1,\ldots,s_l,t_1,\ldots,t_m;u)}$$

and

$$L(x) = M_{(2r_1,2r_k,s_1,\ldots,s_l,u)}$$

(with $k, l, m, u$ or $v$ possibly zero) where $t_1, \ldots, t_m$ are even and $v = u + t_1 + \cdots + t_m$.

The following lemma reduces the study of $M \setminus (N_G(M) \cap X)$ to $M$-standard relevant $M\!$-conjugacy classes.

**Lemma 3.1.** Let $M$ be a standard Levi subgroup of $G$ and $x \in N_G(M) \cap X$. Then there exists $n \in N_G(T)$ such that $nMn^{-1}$ is a standard Levi subgroup of $G$, $nMn^{-1}$ is $M$-standard relevant and $L(nx^n) = nL(x)n^{-1}$.

3.2. Stabilizers and exponents. Let $x \in N_G(M) \cap X$. Then $P_x = M_x \ltimes U_x$. Set $M_x(\mathfrak{a})^{(1)} = M_x(\mathfrak{a}) \cap M(\mathfrak{a})^1$ and note that $M_x(\mathfrak{a})^{(1)}$ contains (possibly strictly) $M_x(\mathfrak{a})^1$. The map $H_M$ defines an isomorphism

$$M_x(\mathfrak{a})^{(1)} \setminus M_x(\mathfrak{a}) \simeq (\mathfrak{a}_M)_x^+.$$

Furthermore, $M_x(\mathfrak{a}) = (M_x(\mathfrak{a}) \cap A_M) \cdot M_x(\mathfrak{a})^{(1)}$ and $M_x(\mathfrak{a}) \cap A_M = (A_M)_x^+$ where $(A_M)_x^+ = e^{(\mathfrak{a}_M_x^+)}$.

Consequently, there exists a unique $\rho_x \in (\mathfrak{a}_M)_x^+$ such that

$$e^{\langle \rho_x, H(a) \rangle} = \delta_{P_x}(a)\delta_P(a)^{-\frac{1}{2}} \quad \text{or equivalently} \quad \delta_{P_x}(a) = e^{\langle \rho_x + pp, H(a) \rangle}, \quad a \in (A_M)_x^+.$$

**Remark 3.2.** The vector $\rho_x$ (with a slightly different convention) was encountered in the setup of [Off06]. It does not show up in the cases considered in [LR03] by [ibid., Proposition 4.3.2]. Note that in our case $\delta_{P_x}$ is non-trivial on $M_x(\mathfrak{a})^{(1)}$ in general. This is in contrast with the cases considered in [LR03] and [Off06] where $M_x(\mathfrak{a})^{(1)} = M_x(\mathfrak{a})^1$.

3.3. Cuspidal orbits. Let $x \in N_G(M) \cap X$ be $M$-standard relevant and assume further that $M = M_{(r_1,r_1,\ldots,r_k,r_k,s_1,\ldots,s_l,t_1,\ldots,t_m;0)}$ (i.e., $m = u = 0$) and $L(x) = M_{(2r_1,2r_k,s_1,\ldots,s_l;0)}$.

Thus,

$$M \simeq \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k} \times \mathrm{GL}_{s_1} \times \cdots \times \mathrm{GL}_{s_l} \times \cdots \times \mathrm{GL}_{s_l}.$$

The stabiliser $M_x$ can be described as follows. The element $x$ (in fact its $M$-conjugacy class) defines a decomposition $s_i = p_i + q_i$, $i = 1, \ldots, l$ so that

$$M_x \simeq \mathrm{GL}_{p_1} \times \cdots \times \mathrm{GL}_{p_l} \times \mathrm{GL}_{q_1} \times \cdots \times (\mathrm{GL}_{p_1} \times \mathrm{GL}_{q_l})$$
where $\text{GL}_{r_{i}}$ is imbedded (twisted) diagonally in $\text{GL}_{r_{1}} \times \text{GL}_{r_{i}}$, $i = 1, \ldots, k$ and $(\text{GL}_{p_{i}} \times \text{GL}_{q_{i}})$ is imbedded as the group of fixed points of an involution with signature $(p_{i}, q_{i})$ in $\text{GL}_{s_{l}}$, $i = 1, \ldots, l$. We call $x$ as above $M$-standard cuspidal if there exists $0 \leq l_{1} \leq l$ such that $p_{i} = q_{i}$ for $i = 1, \ldots, l_{1}$ (in particular, $s_{1}, \ldots, s_{l_{1}}$ are even) and $s_{i} = 1, l_{1} + 1 \leq i \leq l$.

More generally, we say that $x \in N_{G}(M) \cap X$ is $M$-cuspidal if there exists $n \in N_{G}(T)$ such that $n M n^{-1}$ is a standard Levi subgroup of $G$ and $n x n^{-1}$ is $n M n^{-1}$-standard cuspidal.

Let $[X]_{M,\text{cusp}}$ be the set of $M$-cuspidal $M$-conjugacy classes in $N_{G}(M) \cap X$.

4. INTERTWINEIG PERIODS

Our formula for the period integral of a pseudo-Eisenstein series is in terms of certain $H(\mathbb{A})$-invariant linear forms on induced representations of $G(\mathbb{A})$ that we call intertwining periods. In this section we recall their definition for the pair $(G, H)$. They were introduced and studied in the Galois case in [JLR99] and [LR03]. Let $P = M \times U$ be a parabolic subgroup of $G$. Let $\mathcal{A}_{P}(G)$ be the space of continuous functions $\varphi$ on $U(\mathbb{A}) M \backslash G(\mathbb{A})$ such that $\varphi(ag) = e^{\rho_{P} \langle H_{0}(a), \lambda \rangle} \varphi(g)$ for all $a \in A_{M}$, $g \in G(\mathbb{A})$ and $\varphi(g) \ll \Vert g \Vert^{N}$ for some $N$.

Note that $\varphi[\lambda] \in \mathcal{A}_{P}(G)$ for every $R > 0$, $\phi \in C_{R}(U(\mathbb{A}) M \backslash G(\mathbb{A}))$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^{*}$ with $\|\text{Re} \lambda + \rho_{P}\| < R$.

Denote by $\mathcal{A}_{P}^{*}(G)$ the subspace of $\mathcal{A}_{P}(G)$ consisting of $\varphi$ such that for all $N > 0$

$$
\sup_{m \in \mathfrak{a}_{M,\mathbb{C}}^{*}} \|\varphi(mk)\|^{N} < \infty.
$$

For instance, it follows from [MW95, Lemma I.2.10] that $\mathcal{A}_{P}^{*}(G)$ contains the space of smooth functions $\varphi \in \mathcal{A}_{P}(G)$ of uniform moderate growth such that $m \mapsto \varphi(mg)$ is a cuspidal function on $M(\mathbb{A})$ for all $g \in G(\mathbb{A})$.

For $\varphi \in \mathcal{A}_{P}(G)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^{*}$ let $\varphi_{\lambda}(g) = e^{\langle \lambda, H(g) \rangle} \varphi(g), g \in G(\mathbb{A})$.

For $\varphi \in \mathcal{A}_{P}(G)$ and $\lambda \in \rho_{\pi} + (\mathfrak{a}_{M,\mathbb{C}}^{*})_{\pi}^{-}$, whenever convergent, we define

$$
J(\varphi, x, \lambda) = \int_{P_{x}(\mathbb{A}) \backslash G_{x}(\mathbb{A})} \int_{M_{x}(\mathbb{A})^{(1)}} \delta_{P_{x}}^{-1}(m) \varphi_{\lambda}(mh\eta) \, dm \, dh
$$

where $\eta \in G$ is such that $x = \eta \eta^{-1}$. (Recall that $M_{x}(\mathbb{A})^{(1)} = M_{x}(\mathbb{A}) \cap M(\mathbb{A})^{1}$ and $\rho_{\pi}$ is defined by (2).) Note that the expression does not depend on $\eta$, since $G_{x} \eta$ is determined by $x$. Furthermore, $J(\varphi, x, \lambda)$ only depends on the $M$-conjugacy class of $x$.

Let $\Sigma_{P_{x}} = \{ \alpha \in \Sigma_{P}^{+} : -x \alpha \in \Sigma_{P}^{+} \}$. For $\gamma > 0$ let

$$
\mathfrak{D}_{x}(\gamma) = \{ \lambda \in \rho_{\pi} + (\mathfrak{a}_{M,\mathbb{C}}^{*})_{\pi}^{-} : \text{Re} \langle \lambda, \alpha' \rangle > \gamma, \forall \alpha \in \Sigma_{P_{x}} \}.
$$

**Theorem 4.1.** There exists $\gamma > 0$ such that for any $M$-cuspidal $x = \eta \eta^{-1}$ and $\varphi \in \mathcal{A}_{P}^{*}(G)$ the integral defining $J(\varphi, x, \lambda)$ is absolutely convergent for $\lambda \in \mathfrak{D}_{x}(\gamma)$. 


5. THE PERIOD OF A PSEUDO-EISENSTEIN SERIES

Fix $R > 0$ large enough so that $\mathcal{P}_H(\theta_{P,\phi})$ is defined by an absolutely convergent integral for all $\phi \in C^\infty_R(U(A)M\setminus G(A))$.

**Theorem 5.1.** There exists $\gamma > 0$ such that for any $\phi \in C^\infty_R(U(A)M\setminus G(A))$ we have

$$\int_{H\backslash H(A)} \theta_{P,\phi}(h) \, dh = \sum_{x \in \mathcal{X}_{M,cusp}} \int_{\lambda_x+i(\mathfrak{a}_M)_{\overline{x}}} J(\phi[\lambda], x, \lambda) \, d\lambda$$

where the integrals are absolutely convergent and for any $x \in \mathcal{X}_{M,cusp}$ we fix $\lambda_x \in \mathfrak{D}_x(\gamma)$ such that $||\text{Re} \lambda_x + \rho_P|| < R$. In particular, $\mathcal{P}_H(\theta_{P,\phi}) = 0$ if $\mathcal{X}_{M,cusp}$ is empty (and in particular, unless $M \subseteq M_{(n,0)}$).

We briefly explain the main steps of the proof. All integrals involved are absolutely convergent. Expanding $\theta_{P,\phi}$ as a sum over $P \backslash G$ and unfolding, we get that

$$\int_{H\backslash H(A)} \theta_{\phi}(h) \, dh = \sum_{x \in P \backslash X} I_x(\phi)$$

where

$$I_x(\phi) = \int_{P_x \backslash H(A)} \phi(h\eta) \, dh$$

and $\eta \in G$ is such that $x = \eta \epsilon \eta^{-1}$. Unless $x$ is $M$-admissible, the integral $I_x$ factors through a constant term in $M$ and the cuspidality of $\phi$ implies that $I_x(\phi) = 0$. Our analysis of $M$-admissible orbits implies that

$$\int_{H\backslash H(A)} \theta_{\phi}(h) \, dh = \sum_{x \in M \backslash (N_G(M) \cap \mathcal{X})} I_x(\phi).$$

Well-known vanishing results of periods of cuspidal functions (cf. [AGR93] and [JR92]) together with our study of the stabiliser $M_x$ imply that $I_x(\phi) = 0$ unless $x$ is $M$-cuspidal. The sum on the right hand side is therefore only over $\mathcal{X}_{M,cusp}$. For $M$-cuspidal $x$, a partial Fourier inversion formula with respect to the decomposition $\mathfrak{a}_M = (\mathfrak{a}_M)^{+}_x \oplus (\mathfrak{a}_M)^{-}_x$ implies that

$$I_x(\phi) = \int_{P_x \backslash H(A)} \int_{M_x \backslash M_x(A)} \left( \int_{\lambda_x+i(\mathfrak{a}_M)_{\overline{x}}} \phi[\lambda](m\eta \mathfrak{n}) \, d\lambda \right) \delta_{P_x}^{-1}(m) \, dm \, dh$$

for any $\lambda_x \in \rho_x + (\mathfrak{a}_M)_{\overline{x}}^{-}$ such that $||\rho_P + \text{Re} \lambda_x|| < R$. By Theorem 4.1 and (1) the triple integral converges provided that $\lambda_x \in \mathfrak{D}_x(\gamma)$ for suitable $\gamma$. Changing the order of integration we obtain

$$I_x(\phi) = \int_{\lambda_x+i(\mathfrak{a}_M)_{\overline{x}}} J(\phi[\lambda], x, \lambda) \, d\lambda.$$

The theorem follows.
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INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM 91904, ISRAEL
E-mail address: erezlanmath.huji.ac.il

DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA 32000, ISRAEL
E-mail address: offen@tx.technion.ac.il