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# THE $\mathrm{Sp}_n \times \mathrm{Sp}_n$-PERIOD OF A PSEUDO-EISENSTEIN SERIES ON $\mathrm{Sp}_{2n}$

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ABSTRACT. This is a report on our study of $\mathrm{Sp}_n \times \mathrm{Sp}_n$-period integrals on the automorphic spectrum of $\mathrm{Sp}_{2n}$. We announce our formula for the period integrals of pseudo-Eisenstein series.

This is a report on our study of period integrals over $\mathrm{Sp}_n \times \mathrm{Sp}_n$ of automorphic forms on $\mathrm{Sp}_{2n}$. In [LO] we suggest a notion of the $\mathrm{Sp}_n \times \mathrm{Sp}_n$-*distinguished automorphic spectrum* and provide an upper bound in terms of Langlands fine spectral expansion (cf. [MW95, §V]) of the automorphic spectrum of $\mathrm{Sp}_{2n}$. Roughly speaking, it is the orthogonal complement of the $\mathrm{Sp}_{2n}$-invariant space of pseudo-Eisenstein series with $\mathrm{Sp}_n \times \mathrm{Sp}_n$-vanishing period. The results of [AGR93] imply that the $\mathrm{Sp}_n \times \mathrm{Sp}_n$-distinguished automorphic spectrum contains no cuspidal automorphic functions. We study period integrals on the continuous spectrum. The technical heart of our work is a formula for the periods of pseudo-Eisenstein series that we explicitly describe below. Results that are mentioned below without reference are proved in [LO].

## 1. Notation

Let $F$ be a number field and let $\mathbb{A}$ be the ring of adeles of $F$. We denote $F$-varieties in bold letters such as $\mathbf{X}$ and write $X = \mathbf{X}(F)$ for the corresponding set of $F$-points.

For an algebraic group $\mathbf{Q}$ defined over $F$ we denote by $X^*(\mathbf{Q})$ the lattice of $F$-rational characters of $\mathbf{Q}$ and let $\mathfrak{a}_Q^* = X^*(\mathbf{Q}) \otimes \mathbb{R}$, $\mathfrak{a}_Q = \mathrm{Hom}(\mathfrak{a}_Q^*, \mathbb{R})$ the real vector space dual to $\mathfrak{a}_Q^*$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_Q$ the natural pairing between them. We view $\mathfrak{a}_Q^*$ and its dual as Euclidean spaces and denote the norm on either of them by $\|\cdot\|$. We denote by $\mathfrak{a}_{\mathbb{C}}$ the complexification of a real vector space $\mathfrak{a}$. We also set

$$\mathbf{Q}(\mathbb{A})^1 = \{q \in \mathbf{Q}(\mathbb{A}) : \forall \chi \in X^*(\mathbf{Q}),\ |\chi(q)| = 1\}.$$

There is an isomorphism

$$H_Q : \mathbf{Q}(\mathbb{A})^1 \backslash \mathbf{Q}(\mathbb{A}) \to \mathfrak{a}_Q$$

such that $e^{\langle \chi, H_Q(q) \rangle} = |\chi(q)|_{\mathbb{A}^*}$, $\chi \in X^*(\mathbf{Q})$, $q \in \mathbf{Q}(\mathbb{A})$.

Let $\delta_Q$ denote the modulus function of $\mathbf{Q}(\mathbb{A})$. It is a character of $\mathbf{Q}(\mathbb{A})^1 \backslash \mathbf{Q}(\mathbb{A})$ and therefore there exists $\rho_Q \in \mathfrak{a}_Q^*$ such that

$$\delta_Q(q) = e^{\langle 2\rho_Q, H_Q(q) \rangle}, \quad q \in \mathbf{Q}(\mathbb{A}).$$

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Let $\mathbf{G}$ be a reductive group and $\mathbf{P_0}$ a minimal parabolic subgroup of $\mathbf{G}$ both defined over $F$. In general we denote by $\mathfrak{S}_G$ a Siegel domain for $G\backslash\mathbf{G}(\mathbb{A})$ and by $\mathfrak{S}_G^1$ a Siegel domain for $G\backslash\mathbf{G}(\mathbb{A})^1$ (cf. [MW95, I.2.1]).

Let $K$ be a maximal compact subgroup of $\mathbf{G}(\mathbb{A})$ in good position with respect to $P_0$ so that we have the Iwasawa decomposition $\mathbf{G}(\mathbb{A}) = \mathbf{P}_0(\mathbb{A})K$. The map $H_0 = H_{P_0} : \mathbf{P_0}(\mathbb{A}) \to \mathfrak{a}_{P_0}$ is extended to $\mathbf{G}(\mathbb{A})$ via the Iwasawa decomposition, i.e., $H_0(pk) = H_0(p)$, $p \in \mathbf{P_0}(\mathbb{A})$ and $k \in K$. Let $\mathbf{T_G}$ be the split part of the identity connected component of the center of $\mathbf{G}$. Applying the imbedding $x \mapsto 1 \otimes x : \mathbb{R} \to F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ we imbed $\mathbf{T_G}(\mathbb{R})$ in $\mathbf{T_G}(F_\infty) \hookrightarrow \mathbf{T_G}(\mathbb{A})$ and denote by $A_G$ the image of the identity component $\mathbf{T_G}(\mathbb{R})^\circ$ in $\mathbf{T_G}(\mathbb{A})$. Then $H_G : A_G \to \mathfrak{a}_G$ is an isomorphism. Denote by $\nu \mapsto e^\nu$ its inverse.

For $n \in \mathbb{N}$ let $w_n = (\delta_{i,n+1-j}) \in \mathrm{GL}_n$, $J_n = \left(\begin{smallmatrix} 0 & w_n \\ -w_n & 0 \end{smallmatrix}\right)$ and

$$\mathbf{Sp}_n = \{g \in \mathbf{GL}_{2n} : {}^t g J_n g = J_n\}.$$

Let $^*$ be the automorphism of $\mathbf{GL}_n$ given by $g \mapsto g^* = w_n {}^t g^{-1} w_n$. The imbedding $g \mapsto \mathrm{diag}(g, g^*) : \mathbf{GL}_n \to \mathbf{Sp}_n$ identifies $\mathbf{GL}_n$ with the Siegel Levi subgroup of $\mathbf{Sp}_n$.

To every $\gamma = (n_1, \dots, n_k; r)$ where $k, r \geq 0$, $n_1, \dots, n_k > 0$, and $n_1 + \dots + n_k + r = n$ we associate the standard parabolic subgroup $\mathbf{P} = \mathbf{P}_\gamma = \mathbf{M} \ltimes \mathbf{U}$ consisting of block upper triangular matrices in $\mathbf{Sp}_n$ where

$$\mathbf{M} = \mathbf{M}_\gamma = \{\mathrm{diag}(g_1, \dots, g_k, h, g_k^*, \dots, g_1^*) : h \in \mathbf{Sp}_r,\ g_i \in \mathbf{GL}_{n_i},\ i = 1, \dots, k\}.$$

(We call such $\mathbf{M}$'s standard Levi subgroups.) In particular, $\mathbf{P}_{(n;0)}$ is the Siegel parabolic subgroup of $\mathbf{Sp}_n$ and $\mathbf{P}_\gamma \subseteq \mathbf{P}_{(n;0)}$ if and only if $r = 0$.

Let $\delta_n = \mathrm{diag}(1, -1, 1, \dots, (-1)^{n-1}) \in \mathrm{GL}_n$ and $\epsilon_n = \mathrm{diag}(\delta_n, \delta_n^*) \in \mathrm{Sp}_n$.

For the rest of this note fix $n \in \mathbb{N}$ and let $\mathbf{G} = \mathbf{Sp}_{2n}$, $\epsilon = \epsilon_{2n}$ and $\mathbf{H} = \mathbf{C_G}(\epsilon)$, the centraliser of $\epsilon$ in $\mathbf{G}$.

Let $\mathbf{B} = \mathbf{P}_{(1,\dots,1;0)} = \mathbf{T} \ltimes \mathbf{N}$ be the standard Borel subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{N}$ and $\mathbf{T} = \mathbf{M}_{(1,\dots,1;0)}$. We call a Levi subgroup of $\mathbf{G}$ semi-standard if it contains $\mathbf{T}$.

For a standard Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, we denote by $\mathcal{L}(\mathbf{M})$ the (finite) set of (semi-standard) Levi subgroups containing $\mathbf{M}$.

For a standard parabolic subgroup $P = M \ltimes U$ of $G$ let $\Sigma_M = R(T_M, G) \subseteq \mathfrak{a}_M^*$ be the root system of $G$ with respect to $T_M$ and $\Sigma_P^+$ the subset of positive roots in $\Sigma_M$ with respect to $P$.

We identify $\mathbf{G}/\mathbf{H}$ with the $\mathbf{G}$-conjugacy class $\mathbf{X}$ of $\epsilon$, a closed subvariety of $\mathbf{G}$, via $g\mathbf{H} \mapsto g\epsilon g^{-1}$. For $x \in X$ and a subgroup $Q$ of $G$ we denote by $Q \cdot x = \{qxq^{-1} : q \in Q\}$ the $Q$-conjugacy class of $x$ and by $Q_x = \{q \in Q : qxq^{-1} = x\}$ the centraliser of $x$ in $Q$.

## 2. Pseudo-Eisensteins series

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a standard parabolic subgroup of $\mathbf{G}$. For any $R > 0$ let $C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ be the space of continuous cuspidal functions $\phi$ on $\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A})$ such that for all $N > 1$ we have

$$\sup_{m \in \mathfrak{S}_M^1, a \in A_M, k \in K} |\phi(amk)| \, \|m\|^N e^{R\|H_P(a)\|} < \infty.$$

For any $\phi \in C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ define the pseudo-Eisenstein series $\theta_{P,\phi}$ on $G\backslash\mathbf{G}(\mathbb{A})$ by the absolutely convergent series

$$\theta_{P,\phi}(g) = \sum_{\gamma \in P\backslash G} \phi(\gamma g).$$

For $\phi \in C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ with $\|\mathrm{Re}\,\lambda + \rho_P\| < R$ we write

$$\phi[\lambda](g) = e^{-\langle \lambda, H_P(g)\rangle} \int_{A_M} e^{-\langle \lambda+\rho_P, H_P(a)\rangle} \phi(ag)\, da.$$

Let $C^\infty_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ be the smooth part of $C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$. For $\phi \in C^\infty_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ we have the inversion formula

$$\phi(g) = \int_{\lambda_0 + i\mathfrak{a}^*_M} \phi[\lambda]_\lambda(g)\, d\lambda$$

for any $\lambda_0 \in \mathfrak{a}^*_M$ with $\|\lambda_0 + \rho_P\| < R$. Moreover, for any $R' < R$ and $N > 0$ we have

$$\sup_{m \in \mathfrak{S}^1_M, k \in K, \lambda \in \mathfrak{a}^*_{M,\mathbb{C}} : \|\lambda + \rho_P\| \le R'} |\phi[\lambda](mk)| \left(\|m\| + \|\lambda\|\right)^N < \infty. \tag{1}$$

We wish to study the period integrals

$$\mathcal{P}_H(\theta_{P,\phi}) := \int_{H\backslash\mathbf{H}(\mathbb{A})} \theta_{P,\phi}(h)\, dh.$$

It is easy to see that for $R$ large enough $g \mapsto \sum_{\gamma\in P\backslash G} |\phi(\gamma g)|$ is bounded on $\mathbf{G}(\mathbb{A})^1$ for every $\phi \in C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ and therefore $\mathcal{P}_H(\theta_{P,\phi})$ is defined by an absolutely convergent integral for all $\phi \in C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$.

We express $\mathcal{P}_H(\theta_{P,\phi})$ as a sum parameterised by certain double cosets in $P\backslash G/H$ of certain $\mathbf{H}(\mathbb{A})$-invariant linear forms, called intertwining periods, on representations of $\mathbf{G}(\mathbb{A})$ induced from $\mathbf{P}(\mathbb{A})$. In order to formulate our results we explain in §3 the relevant results concerning the double coset decomposition and in §4 we define and state the convergence of the intertwining periods.

## 3. DOUBLE COSETS

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a standard parabolic subgroup of $\mathbf{G}$. The map $g \mapsto g\epsilon g^{-1}$ defines a bijection

$$P\backslash G/H \simeq P\backslash X.$$

Instead of double cosets we study $P$-conjugacy classes in $X$.

3.1. **Admissible orbits.** An element $x \in X$ (or $P\cdot x$) is called $M$-admissible if $N_G(M) \cap P \cdot x$ is not empty. We denote by $[P\backslash X]_{\mathrm{adm}}$ the set of $M$-admissible $P$-conjugacy classes in $X$.

If $x \in X$ is $M$-admissible and $y \in N_G(M) \cap P\cdot x$ then $N_G(M) \cap P \cdot x = M\cdot y$ is a unique $M$-orbit. The correspondence $P\cdot x \mapsto M\cdot y$ is a bijection

$$[P\backslash X]_{\mathrm{adm}} \simeq M\backslash(X \cap N_G(M))$$

between $M$-admissible $P$-conjugacy classes in $X$ and $M$-conjugacy classes in $N_G(M)\cap X$.

The group $N_G(M)$ acts on $\mathfrak{a}_M^*$ ($M$ acts trivially) and in particular, $x\in N_G(M)\cap X$ acts as an involution on $\mathfrak{a}_M^*$ and decomposes it into a direct sum of the $\pm1$-eigenspaces which we denote by $(\mathfrak{a}_M^*)_x^\pm$ respectively. (A similar decomposition applies to the dual space $\mathfrak{a}_M=(\mathfrak{a}_M)_x^+\oplus(\mathfrak{a}_M)_x^-$.) Any such $x$ defines

$$\mathbf{L}=\mathbf{L}(x)=\cap_{\mathbf{L}'\in\mathcal{L}(\mathbf{M}),\,x\in L'}\mathbf{L}'\in\mathcal{L}(\mathbf{M})$$

so that $(\mathfrak{a}_M^*)_x^+=\mathfrak{a}_L^*$ (cf. [Art82, p. 1299]).

We call $x\in N_G(M)\cap X$ (or $M\cdot x$) $M$-standard relevant if $M$ has the form

$$M=M_{(r_1,r_1,\dots,r_k,r_k,s_1,\dots,s_l,t_1,\dots,t_m;u)}$$

and

$$L(x)=M_{(2r_1,\dots,2r_k,s_1,\dots,s_l;v)}$$

(with $k$, $l$, $m$, $u$ or $v$ possibly zero) where $t_1,\dots,t_m$ are even and $v=u+t_1+\dots+t_m$.

The following lemma reduces the study of $M\backslash(N_G(M)\cap X)$ to $M$-standard relevant $M$-conjugacy classes.

**Lemma 3.1.** *Let $M$ be a standard Levi subgroup of $G$ and $x\in N_G(M)\cap X$. Then there exists $n\in N_G(T)$ such that $nMn^{-1}$ is a standard Levi subgroup of $G$, $nxn^{-1}$ is $nMn^{-1}$-standard relevant and $L(nxn^{-1})=nL(x)n^{-1}$.*

3.2. **Stabilizers and exponents.** Let $x\in N_G(M)\cap X$. Then $\mathbf{P}_x=\mathbf{M}_x\ltimes\mathbf{U}_x$. Set $\mathbf{M}_x(\mathbb{A})^{(1)}=\mathbf{M}_x(\mathbb{A})\cap\mathbf{M}(\mathbb{A})^1$ and note that $\mathbf{M}_x(\mathbb{A})^{(1)}$ contains (possibly strictly) $\mathbf{M}_x(\mathbb{A})^1$. The map $H_M$ defines an isomorphism

$$\mathbf{M}_x(\mathbb{A})^{(1)}\backslash\mathbf{M}_x(\mathbb{A})\simeq(\mathfrak{a}_M)_x^+.$$

Furthermore, $\mathbf{M}_x(\mathbb{A})=(\mathbf{M}_x(\mathbb{A})\cap A_M)\cdot\mathbf{M}_x(\mathbb{A})^{(1)}$ and $\mathbf{M}_x(\mathbb{A})\cap A_M=(A_M)_x^+$ where $(A_M)_x^+=e^{(\mathfrak{a}_M)_x^+}$.

Consequently, there exists a unique $\rho_x\in(\mathfrak{a}_M^*)_x^+$ such that

$$e^{\langle\rho_x,H(a)\rangle}=\delta_{P_x}(a)\delta_P(a)^{-\frac12}\quad\text{or equivalently}\quad\delta_{P_x}(a)=e^{\langle\rho_x+\rho_P,H(a)\rangle},\quad a\in(A_M)_x^+. \tag{2}$$

**Remark 3.2.** *The vector $\rho_x$ (with a slightly different convention) was encountered in the setup of* [Off06]. *It does not show up in the cases considered in* [LR03] *by [ibid., Proposition 4.3.2]. Note that in our case $\delta_{P_x}$ is non-trivial on $\mathbf{M}_x(\mathbb{A})^{(1)}$ in general. This is in contrast with the cases considered in* [LR03] *and* [Off06] *where $\mathbf{M}_x(\mathbb{A})^{(1)}=\mathbf{M}_x(\mathbb{A})^1$.*

3.3. **Cuspidal orbits.** Let $x\in N_G(M)\cap X$ be $M$-standard relevant and assume further that $M=M_{(r_1,r_1,\dots,r_k,r_k,s_1,\dots,s_l;0)}$ (i.e., $m=u=0$) and $L(x)=M_{(2r_1,\dots,2r_k,s_1,\dots,s_l;0)}$. Thus,

$$M\simeq\mathrm{GL}_{r_1}\times\mathrm{GL}_{r_1}\times\dots\times\mathrm{GL}_{r_k}\times\mathrm{GL}_{r_k}\times\mathrm{GL}_{s_1}\times\dots\times\mathrm{GL}_{s_l}.$$

The stabiliser $M_x$ can be described as follows. The element $x$ (in fact its $M$-conjugacy class) defines a decomposition $s_i=p_i+q_i$, $i=1,\dots,l$ so that

$$M_x\simeq\mathrm{GL}_{r_1}\times\dots\times\mathrm{GL}_{r_k}\times(\mathrm{GL}_{p_1}\times\mathrm{GL}_{q_1})\times\dots\times(\mathrm{GL}_{p_l}\times\mathrm{GL}_{q_l})$$

where $\mathrm{GL}_{r_i}$ is imbedded (twisted) diagonally in $\mathrm{GL}_{r_i} \times \mathrm{GL}_{r_i}$, $i = 1, \dots, k$ and $(\mathrm{GL}_{p_i} \times \mathrm{GL}_{q_i})$ is imbedded as the group of fixed points of an involution with signature $(p_i, q_i)$ in $\mathrm{GL}_{s_i}$, $i = 1, \dots, l$.

We call $x$ as above $M$-standard cuspidal if there exists $0 \le l_1 \le l$ such that $p_i = q_i$ for $i = 1, \dots, l_1$ (in particular, $s_1, \dots, s_{l_1}$ are even) and $s_i = 1$, $l_1 + 1 \le i \le l$.

More generally, we say that $x \in N_G(M) \cap X$ is $M$-cuspidal if there exists $n \in N_G(T)$ such that $nMn^{-1}$ is a standard Levi subgroup of $G$ and $nxn^{-1}$ is $nMn^{-1}$-standard cuspidal.

Let $[X]_{M,\mathrm{cusp}}$ be the set of $M$-cuspidal $M$-conjugacy classes in $N_G(M) \cap X$.

## 4. Intertwining periods

Our formula for the period integral of a psudo-Eisenstein series is in terms of certain $\mathbf{H}(\mathbb{A})$-invariant linear forms on induced representations of $\mathbf{G}(\mathbb{A})$ that we call intertwining periods. In this section we recall their definition for the pair $(G, H)$. They were introduced and studied in the Galois case in [JLR99] and [LR03].

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a parabolic subgroup of $\mathbf{G}$. Let $\mathcal{A}_P(G)$ be the space of continuous functions $\varphi$ on $\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A})$ such that $\varphi(ag) = e^{\langle \rho_P, H_0(a)\rangle}\varphi(g)$ for all $a \in A_M$, $g \in G(\mathbb{A})$ and $\varphi(g) \ll \|g\|^N$ for some $N$.

Note that $\phi[\lambda] \in \mathcal{A}_P(G)$ for every $R > 0$, $\phi \in C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ with $\|\mathrm{Re}\,\lambda + \rho_P\| < R$.

Denote by $\mathcal{A}_P^{rd}(G)$ the subspace of $\mathcal{A}_P(G)$ consisting of $\varphi$ such that for all $N > 0$

$$\sup_{m \in \mathfrak{S}^1_M, k \in K} |\varphi(mk)| \, \|m\|^N < \infty.$$

For instance, it follows from [MW95, Lemma I.2.10] that $\mathcal{A}_P^{rd}(G)$ contains the space of smooth functions $\varphi \in \mathcal{A}_P(G)$ of uniform moderate growth such that $m \mapsto \varphi(mg)$ is a cuspidal function on $\mathbf{M}(\mathbb{A})$ for all $g \in \mathbf{G}(\mathbb{A})$.

For $\varphi \in \mathcal{A}_P(G)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ let $\varphi_\lambda(g) = e^{\langle \lambda, H(g)\rangle}\varphi(g)$, $g \in \mathbf{G}(\mathbb{A})$.

For $\varphi \in \mathcal{A}_P(G)$ and $\lambda \in \rho_x + (\mathfrak{a}^*_{M,\mathbb{C}})^-_x$, whenever convergent, we define

$$J(\varphi, x, \lambda) = \int_{\mathbf{P}_x(\mathbb{A})\backslash\mathbf{G}_x(\mathbb{A})} \int_{M_x\backslash\mathbf{M}_x(\mathbb{A})^{(1)}} \delta^{-1}_{P_x}(m)\varphi_\lambda(mh\eta)\, dm\, dh$$

where $\eta \in G$ is such that $x = \eta\epsilon\eta^{-1}$. (Recall that $\mathbf{M}_x(\mathbb{A})^{(1)} = \mathbf{M}_x(\mathbb{A}) \cap \mathbf{M}(\mathbb{A})^1$ and $\rho_x$ is defined by (2).) Note that the expression does not depend on $\eta$, since $G_x\eta$ is determined by $x$. Furthermore, $J(\varphi, x, \lambda)$ only depends on the $M$-conjugacy class of $x$.

Let $\Sigma_{P,x} = \{\alpha \in \Sigma_P^+ : -x\alpha \in \Sigma_P^+\}$. For $\gamma > 0$ let

$$\mathfrak{D}_x(\gamma) = \{\lambda \in \rho_x + (\mathfrak{a}^*_{M,\mathbb{C}})^-_x : \mathrm{Re}\,\langle \lambda, \alpha^\vee\rangle > \gamma,\ \forall \alpha \in \Sigma_{P,x}\}.$$

**Theorem 4.1.** *There exists $\gamma > 0$ such that for any $M$-cuspidal $x = \eta\epsilon\eta^{-1}$ and $\varphi \in \mathcal{A}_P^{rd}(G)$ the integral defining $J(\varphi, x, \lambda)$ is absolutely convergent for $\lambda \in \mathfrak{D}_x(\gamma)$.*

## 5. The period of a pseudo-Eisenstein series

Fix $R > 0$ large enough so that $\mathcal{P}_H(\theta_{P,\phi})$ is defined by an absolutely convergent integral for all $\phi \in C_R(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$.

**Theorem 5.1.** *There exists $\gamma > 0$ such that for any $\phi \in C_R^\infty(\mathbf{U}(\mathbb{A})M\backslash\mathbf{G}(\mathbb{A}))$ we have*

$$\int_{H\backslash\mathbf{H}(\mathbb{A})} \theta_{P,\phi}(h)\, dh = \sum_{x\in[X]_{M,\mathrm{cusp}}} \int_{\lambda_x+i(\mathfrak{a}_M^*)_x^-} J(\phi[\lambda],x,\lambda)\, d\lambda$$

*where the integrals are absolutely convergent and for any $x \in [X]_{M,\mathrm{cusp}}$ we fix $\lambda_x \in \mathfrak{D}_x(\gamma)$ such that $\|\mathrm{Re}\,\lambda_x + \rho_P\| < R$. In particular, $\mathcal{P}_H(\theta_{P,\phi}) = 0$ if $[X]_{M,\mathrm{cusp}}$ is empty (and in particular, unless $M \subseteq M_{(n;0)}$).*

We briefly explain the main steps of the proof. All integrals involved are absolutely convergent. Expanding $\theta_{P,\phi}$ as a sum over $P\backslash G$ and unfolding, we get that

$$\int_{H\backslash\mathbf{H}(\mathbb{A})} \theta_\phi(h)\, dh = \sum_{x\in P\backslash X} I_x(\phi)$$

where

$$I_x(\phi) = \int_{P_x\backslash\mathbf{H}(\mathbb{A})} \phi(h\eta)\, dh$$

and $\eta \in G$ is such that $x = \eta\epsilon\eta^{-1}$. Unless $x$ is $M$-admissible, the integral $I_x$ factors through a constant term in $M$ and the cuspidality of $\phi$ implies that $I_x(\phi) = 0$. Our analysis of $M$-admissible orbits implies that

$$\int_{H\backslash\mathbf{H}(\mathbb{A})} \theta_\phi(h)\, dh = \sum_{x\in M\backslash(N_G(M)\cap X)} I_x(\phi).$$

Well-known vanishing results of periods of cuspidal functions (cf. [AGR93] and [JR92]) together with our study of the stabiliser $M_x$ imply that $I_x(\phi) = 0$ unless $x$ is $M$-cuspidal. The sum on the right hand side is therefore only over $[X]_{M,\mathrm{cusp}}$. For $M$-cuspidal $x$, a partial Fourier inversion formula with respect to the decomposition $\mathfrak{a}_M = (\mathfrak{a}_M)_x^+ \oplus (\mathfrak{a}_M)_x^-$ implies that

$$I_x(\phi) = \int_{\mathbf{P}_x(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \int_{M_x\backslash\mathbf{M}_x(\mathbb{A})^{(1)}} \left( \int_{\lambda_x+i(\mathfrak{a}_M^*)_x^-} \phi[\lambda]_\lambda(mh\eta_x)\, d\lambda \right) \delta_{P_x}^{-1}(m)\, dm\, dh$$

for any $\lambda_x \in \rho_x + (\mathfrak{a}_{M,\mathbb{C}}^*)_x^-$ such that $\|\rho_P + \mathrm{Re}\,\lambda_x\| < R$. By Theorem 4.1 and (1) the triple integral converges provided that $\lambda_x \in \mathfrak{D}_x(\gamma)$ for suitable $\gamma$. Changing the order of integration we obtain

$$I_x(\phi) = \int_{\lambda_x+i(\mathfrak{a}_M^*)_x^-} J(\phi[\lambda],x,\lambda)\, d\lambda.$$

The theorem follows.

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