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WHITTAKER-FOURIER COEFFICIENTS OF CUSPIDAL REPRESENTATIONS ON CLASSICAL GROUPS

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Let G be a quasi-split reductive group defined over a number field F. Fix a Borel subgroup B of G defined over F and let N be its unipotent radical. Fix a non-degenerate character of $N(\mathbb{A})$, trivial on $N(F)$. The ψ_N-Whittaker coefficient of an automorphic form φ on $G(F)\backslash G(\mathbb{A})$ is defined by

$$\mathcal{W}^{\psi_N}(\varphi)=\int_{N(F)\backslash N(\mathbb{A})}\varphi(u)\psi_N^{-1}(u)\,du.$$

When φ belongs to a ψ_N-generic automorphic cuspidal representation π we would like to compare $\mathcal{W}^{\psi_N}(\varphi)$ with the Petersson inner product. More precisely, we have a relation

$$|\mathcal{W}(\varphi)|^2=c_\pi\lim_{s=1}\frac{\Delta_F^S(s)}{L^S(s,\pi,\mathrm{Ad})}\int_{N(F_S)}^{\mathrm{st}}(\pi(u)\varphi,\varphi)_{G(F)\backslash G(\mathbb{A})^1}\psi_N(u)^{-1}\,du$$

where

- S is a sufficiently large finite set of places of F containing all archimedean ones and the places where either G, π or ψ_N are ramified.
- $\Delta_F^S(s)$ is a certain partial L-function which depends only on G.
- $L^S(s,\pi,\mathrm{Ad})$ is the partial L-function of π with respect to the adjoint L-function of ${}^LG$.
- The local integral makes sense by a suitable regularization. (In the p-adic case it is simply the integral over a sufficiently large compact open subgroup of $N(F_v)$.)

The constant c_π depends on the automorphic realization of π as well as the Haar measures chosen. It follows from the Casselman–Shalika formula [CS80] that c_π does not depend on S. It is convenient to choose the Haar measures by $\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)=\mathrm{vol}(N(F)\backslash N(\mathbb{A}))=1$. The measure on $N(F_S)$ is chosen so that under the decomposition $N(\mathbb{A})=N(F_S)\times N(F^S)$, $\mathrm{vol}(N(F^S)\cap K^S)=1$ where K^S is a suitable maximal compact subgroup of $G(\mathbb{A}^S)$. Implicit here is the assumption that the limit $\lim_{s=1}\frac{\Delta_F^S(s)}{L^S(s,\pi,\mathrm{Ad})}$ exists and is non-zero.

In the case where $G=\mathrm{GL}_n$ the theory of Rankin-Selberg integrals for $\mathrm{GL}_n\times\mathrm{GL}_n$ developed by Jacquet–Piatetski-Shapiro–Shalika (cf. [Jac01, §2]) together with local unfolding shows that $c_\pi=1$ for any cuspidal representation π. In the case where $G=\mathrm{SL}_n$ it easily follows that $c_\pi=|X(\tilde{\pi})|^{-1}$ if π is the ψ_N-generic irreducible constituent of the restriction of functions from a cuspidal representation $\tilde{\pi}$ of $\mathrm{GL}_n(\mathbb{A})$ to $\mathrm{SL}_n(F)\backslash\mathrm{SL}_n(\mathbb{A})$. Here $X(\tilde{\pi})$ is the finite group of Hecke character χ such that $\tilde{\pi}\otimes\chi=\tilde{\pi}$.

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Our work concerns the value of c_π for the identity component of classical groups. By the works of Ginzburg–Rallis–Soudry [GRS11] and Cogdell–Kim–Piatetski-Shapiro–Shahidi [CKPSS04] the generic cuspidal representations of a classical group G are parameterized by isobraic representations $\Pi = \pi_1 \boxplus \cdots \boxplus \pi_k$ of GL_N (with N determined by G) where $\pi_1, \dots, \pi_k$ are distinct cuspidal representations of $\mathrm{GL}_{n_1}, \dots, \mathrm{GL}_{n_k}$ of a certain type. Moreover, Ginzburg–Rallis–Soudry give an automorphic realization of such π's in terms of Π (namely, π is the descent of Π). We conjecture that for this π we have $c_\pi = 2^{1-k}$ unless $G = SO(2n)$ and all the local components of π_v are invariant under $O(2n)$, in which case $c_\pi = 2^{2-k}$. This relation is analogous to a conjecture of Ichino–Ikeda [II10]. There is also an analogous conjecture for the metaplectic two-fold cover for $\mathrm{Sp}_n(\mathbb{A})$. In this case we expect that

$$|\mathcal{W}(\varphi)|^2 = 2^{-k} \prod_{i=1}^{n} \zeta_F^S(2i) \frac{L^S(\frac{1}{2}, \Pi)}{L^S(1, \Pi, \mathrm{sym}^2)} \int_{N(F_S)}^{\mathrm{st}} (\pi(u)\varphi, \varphi)_{G(F)\backslash G(\mathbb{A})^1} \psi_N(u)^{-1} \, du$$

where Π is as before with all π_i's satisfying $L(\frac{1}{2}, \pi_i) L^S(1, \pi_i, \wedge^2) = \infty$ and π is the descent of Π. The case $n = 1$ is essentially a reformulation of a well-known result of Waldspurger [Wal81].

In the cases of odd orthogonal, unitary and metaplectic groups we reduce the conjecture to a local statement. We also have partial results toward the local statement.

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