ARCHIMEDEAN $L$-FACTORS FOR STANDARD $L$-FUNCTIONS ATTACHED TO NON-HOLOMORPHIC SIEGEL MODULAR FORMS OF DEGREE 2

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INTRODUCTION

Bump, Friedberg and Ginzburg [1] introduced a zeta integral which contains two complex variables $s_1$, $s_2$ and interpolates the standard and the spinor $L$-functions for generic cuspidal representation $\pi = \otimes_v' \pi_v$ of $\text{GSp}(2, \mathbf{A})$. Actually they carried out the unramified computation to show that the local zeta integral coincides with the product $L(s_1, \pi_v, \text{std})L(s_2, \pi_v, \text{spin})$ of the local $L$-factors at the unramified place $v$. We compute the (real) archimedean zeta integral by using the explicit formulas of the Whittaker functions on $\text{GSp}(2, \mathbf{R})$ developed by Oda, Miyazaki, Moriyama and the author. When $\pi_\infty$ is isomorphic to a large discrete series representations, for an appropriate choice of Whittaker function and sections for Eisenstein series, we show that the archimedean zeta integral is equal to the product of two archimedean $L$-factors.

1. ZETA INTEGRALS

We recall the zeta integral discovered by Bump, Friedberg and Ginzburg [1]. Let $G$ be the symplectic group with similitude of degree two defined over $\mathbf{Q}$:

$$G = \text{GSp}(2) = \{g \in \text{GL}(4) \mid {}^tCJg = \nu(g) J \text{ for some } \nu(g) \in \text{GL}(1), \quad J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \}.$$  

Let $\mathbf{A}$ be the ring of adeles of $\mathbf{Q}$. Let $\pi = \otimes_v' \pi_v$ be a cuspidal automorphic representation of $G(\mathbf{A})$. For simplicity, we assume that the central character of $\pi$ is trivial. We take a maximal unipotent subgroup $N_0$ of $G$ by

$$N_0 = \{n(x_0, x_1, x_2, x_3) = \begin{pmatrix} 1 & x_0 \\ 1 & 1 \\ -x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ 1 & x_2 & x_3 \\ 1 & 1 \end{pmatrix} \in G \}.$$  

We fix a nontrivial additive character $\psi = \prod_v \psi_v : \mathbf{A}/\mathbf{Q} \rightarrow \mathbf{C}^{(1)}$ and define a non-degenerate unitary character $\psi_{N_0}$ of $N_0(\mathbf{A})$ by $\psi_{N_0}(n(x_0, x_1, x_2, x_3)) = \psi(-x_0 - x_3)$. For a cusp form $\varphi \in \pi$, the global Whittaker function $W_\varphi$, attached to $\varphi$ is defined by

$$W_\varphi(g) = \int_{N_0(\mathbf{Q}) \backslash N_0(\mathbf{A})} \varphi(ng) \psi_{N_0}(n^{-1}) \, dn, \quad g \in G(\mathbf{A}).$$

Thoughout this paper we assume that $\pi$ is generic, that is, $W_\varphi$ does not vanish for some $\varphi$ in $\pi$. 
Let $P_1$ and $P_2$ be the Siegel and Jacobi (Klingen) parabolic subgroups of $G$, respectively:

$$P_1 = \left\{ \begin{pmatrix} * & * \\ 0_2 & * \end{pmatrix} \in G \right\}, \quad P_2 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in G \right\}.$$

The unipotent radical $N_i$ of $P_i$ is given by

$$N_1 = \{ n(0, x_1, x_2, x_3) \in G \}, \quad N_2 = \{ n(x_0, x_1, 0, x_3) \in G \}.$$

The Levi part of $P_i$ is isomorphic to $GL(2) \times GL(1)$ embedded via the maps $\iota_i$:

$$\iota_1(\alpha, g) = \left( \begin{array}{cc} \alpha g \end{array} \right), \quad \iota_2(\alpha, g) = \left( \begin{array}{cc} \alpha \alpha^{-1} \det g \end{array} \right),$$

where $\alpha \in GL(1)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$. The modulus characters $\delta_i$ of $P_i$ are given by

$$\delta_1(\iota_1(\alpha, g)) = |\det g|^{3}|\alpha|^{3}, \quad \delta_2(\iota_2(\alpha, g)) = |\det g|^{-2}|\alpha|^{4}.$$

For a complex number $s$, we denote by $Ind_{P_i(A)}^{G(A)}(\delta_i^s)$ the space of smooth functions $f_i(s, g)$ on $G(A)$ satisfying $f_i(s, pg) = \delta_i^s(p)f_i(s, g)$ for all $p \in P_i(A)$ and $g \in G(A)$. For complex numbers $s_1$ and $s_2$, we take a global sections $f_1 \in Ind_{P_1(A)}^{G(A)}(\delta_1^{s_1 + 1/3})$ and $f_2 \in Ind_{P_2(A)}^{G(A)}(\delta_2^{s_2/2 + 1/4})$. We define Eisenstein series $E_i(s_i, f_i, g)$ as usual manner:

$$E_i(s_i, f_i, g) = \sum_{\gamma \in P_i(Q) \backslash G(Q)} f_i(s_i, \gamma g).$$

For a generic cusp form $\varphi \in \pi$, the global zeta integral is defined by

$$Z(s_1, s_2, \varphi, f_1, f_2) = \int_{Z(A)G(Q) \backslash G(A)} \varphi(g)E_1(s_1, f_1, g)E_2(s_2, f_2, g) dg.$$

Here we denote by $Z$ the center of $G$. Unfolding two Eisenstein series, one can find the basic identity:

$$Z(s_1, s_2, \varphi, f_1, f_2) = \int_{Z(A)N_{12}(A) \backslash G(A)} W_{\varphi}(g) f_1(s_1, w_2 g) f_2(s_2, w_1 g) dg$$

for $\text{Re}(s_1)$ and $\text{Re}(s_2)$ sufficiently large. Here $N_{12} = N_1 \cap N_2 = \{ n(0, x_1, x_2, 0) \in G \}$,

$$w_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Suppose that $\pi$, $f_1$ and $f_2$ are factorizable. Then the global zeta integral is the product of local zeta integrals

$$Z_v(s_1, s_2, W_v, f_{1,v}, f_{2,v}) = \int_{Z(Q_v)N_{12}(Q_v) \backslash G(Q_v)} W_v(g) f_{1,v}(s_1, w_2 g) f_{2,v}(s_2, w_1 g) dg,$$

where the subscripts denote the local analogues. The unramified computation is the following:
Theorem 1.1. [1, Theorem 1.2] For \( v = p < \infty \), we suppose that \( \pi_v \) is an unramified principal series representation of \( G(\mathbb{Q}_v) \). Let \( \text{diag}(\alpha_0, \alpha_0 \alpha_1, \alpha_0 \alpha_2, \alpha_0 \alpha_1 \alpha_2) \in \text{GSp}(2, \mathbb{C}) \) be the Satake parameter of \( \pi_v \). If all data are unramified, then we have

\[
Z_v(s_1, s_2, W_v, f_{1,v}, f_{2,v}) = \frac{L(s_1, \pi_v, \text{std}) L(s_2, \pi_v, \text{spin})}{((1-p^{-1}(s_1+1))(1-p^{-2s_1})(1-p^{-2s_2}))^{-1}},
\]

where the local \( L \)-factors are given by

\[
L(s, \pi_v, \text{std}) = ((1-p^{-s})(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})(1-\alpha_1^{-1} p^{-s})(1-\alpha_2^{-1} p^{-s}))^{-1},
\]

\[
L(s, \pi_v, \text{spin}) = ((1-\alpha_0 p^{-s})(1-\alpha_0 \alpha_1 p^{-s})(1-\alpha_0 \alpha_2 p^{-s})(1-\alpha_0 \alpha_1 \alpha_2 p^{-s}))^{-1}.
\]

2. Representation Theory of \( \text{GSp}(2, \mathbb{R}) \)

We introduce some representations of \( G := \text{GSp}(2, \mathbb{R}) \). Main references are [6] and [7]. We denote by \( G_0 = \text{Sp}(2, \mathbb{R}) \), \( P_i = P_i(\mathbb{R}) \), \( N_i = N_i(\mathbb{R}) \). Since we have assumed \( \pi = \mathbb{C}_v \pi \) is generic, each local component \( \pi_v \) is also generic. In particular, by a theorem of Kostant [2], the representation \( \pi_{\infty} \) of \( G \) must be large in the sense of Vogan [9]. An irreducible large representation \( \pi_{\infty} \) of \( G \) is equivalent to one of the following:

(i) a (limit of) large discrete series representation of \( G \);

(ii) an irreducible (generalized) principal series representation induced from the parabolic subgroup \( P_i \) (\( i = 0, 1, 2 \), \( P_0 \) : Borel).

We mainly treat the case (i).

Fix a maximal compact subgroup \( K \) of \( G \) by \( K = G \cap O(4) \). Let \( K_0 = G_0 \cap O(4) \). Then \( K_0 \) is isomorphic to the unitary group \( U(2) \) via the isomorphism \( u : K_0 \ni \left( \begin{array}{cc} A & B \\ -B & A \end{array} \right) \mapsto A + \sqrt{-1}B \in U(2) \). For \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \) with \( \lambda_1 \geq \lambda_2 \), let \( V_\lambda = \{ f \in \mathbb{C}[x_1, x_2] \mid \text{homogeneous, deg}(f) = \lambda_1 - \lambda_2 \} \). For \( f \in V_\lambda \) and \( k \in K_0 \), we set \( (\tau_\lambda(k)f)(x_1, x_2) = (\det u(k))^{\lambda_1}f((x_1, x_2) \cdot u(k)) \). Here \( (x_1, x_2) \cdot u(k) \) means the ordinal product of matrices. Then \( (\tau_\lambda, V_\lambda) \) is an irreducible \((\lambda_1 - \lambda_2 + 1)\)-dimensional representation of \( K_0 \) with highest weight \( \lambda \). We take a basis of \( V_\lambda \) as \( \{ v_i \equiv v_i^\lambda := x_1^i x_2^{\lambda_1 - \lambda_2 - i} \mid 0 \leq i \leq \lambda_1 - \lambda_2 \} \).

For \( (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \), we denote by \( D_{(\lambda_1, \lambda_2)} \) the (limit of) large discrete series representation of \( G_0 \) with Blattner parameter \((\lambda_1, \lambda_2)\). Since \( D_{(\lambda_1, \lambda_2)} \) is large, \( (\lambda_1, \lambda_2) \) satisfies \( 1 - \lambda_1 \leq \lambda_2 \leq 0 \) or \( 1 + \lambda_2 \leq -\lambda_1 \leq 0 \). For \( c \in \mathbb{C} \), we denote by \( D_{(\lambda_1, \lambda_2)}[c] \) the irreducible admissible representation of \( G \) characterized by \( D_{(\lambda_1, \lambda_2)}[c] = D_{(\lambda_1, \lambda_2)} \oplus D_{(-\lambda_2, -\lambda_1)} \) and \( D_{(\lambda_1, \lambda_2)}[c](z) = z^c \ (z > 0) \).

Let \( \Gamma_R(s) = \pi^{-s/2} \Gamma(s/2) \) and \( \Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s) \). We fix \( \psi_\infty(x) = \exp(2\pi \sqrt{-1}x) \ (x \in \mathbb{R}) \). When \( \pi_{\infty} \cong D_{(\lambda_1, \lambda_2)}[c] \) with \( 1 - \lambda_1 \leq \lambda_2 \leq 0 \), the \( L \)- and \( \varepsilon \)-factors at the real places via the Landau parameter are the following (see [7, §4]):

\[
L(s, \pi_{\infty}, \text{spin}) = \Gamma_C(s + \frac{c + \lambda_1 - \lambda_2}{2}) \Gamma_C(s + \frac{c + \lambda_1 - \lambda_2 - 1}{2}),
\]

\[
L(s, \pi_{\infty}, \text{std}) = \Gamma_R(s) \Gamma_C(s + \lambda_1 - 1) \Gamma_C(s - \lambda_2),
\]

\[
\varepsilon(s, \pi_{\infty}, \text{spin}, \psi_\infty) = (-1)^{\lambda_1},
\]

\[
\varepsilon(s, \pi_{\infty}, \text{std}, \psi_\infty) = (-1)^{\lambda_1 - \lambda_2}.
\]
3. WHITTAKER FUNCTIONS ON GSp(2, R)

We recall the explicit formulas for Whittaker functions on $G$ for the large discrete series representations. A nondegenerate unitary character $\psi_{N_0}$ of $N_0$ is of the form $\psi_{N_0}(n(x_0, x_1, x_2, x_3)) = \exp\{2\pi \sqrt{-1}(c_0 x_0 + c_3 x_3)\}$ with nonzero real numbers $c_0$ and $c_3$. We introduce the space

$$C^\infty(N_0 \backslash G; \psi_{N_0}) := \{W \in C^\infty(G, C) \mid W(ng) = \psi_{N_0}(n)W(g), \forall (n, g) \in N_0 \times G\}$$

on which the group $G$ acts by right translation. The restriction of a global Whittaker function to $G$ is of moderate growth. Then we consider the subspace $C^\infty_{mg}(N_0 \backslash G; \psi_{N_0}) := \{W \in C^\infty(N_0 \backslash G; \psi_{N_0}) \mid W \text{ is of moderate growth}\}$ of $C^\infty(N_0 \backslash G; \psi_{N_0})$. Let $g$ and $g_0$ be the Lie algebra of $G$ and $G_0$, respectively. Wallach’s multiplicity one theorem asserts that for an irreducible $(g, K)$-module $\pi_\infty$, $\dim_{C}\text{Hom}_{(g, K)}(\pi_\infty, C^\infty_{mg}(N_0 \backslash G; \psi_{N_0})) \leq 1$. If there is a nonzero intertwining operator $\Psi \in \text{Hom}_{(g, K)}(\pi_\infty, C^\infty_{mg}(N_0 \backslash G; \psi_{N_0}))$, then we call the image $W(v; \ast) = \Psi(v)$ of $v \in \pi_\infty$ the Whittaker function corresponding to $v \in \pi_\infty$.

**Theorem 3.1.** [8], [6] Assume that $1 - \lambda_1 \leq \lambda_2 \leq 0$.

(i) We have

$$\dim_{C}\text{Hom}_{(g_0, K)}(D(\lambda_1, \lambda_2), C^\infty(N_0 \backslash G_0; \psi_{N_0})) = \begin{cases} 1 & \text{if } c_3 > 0; \\ 0 & \text{if } c_3 < 0. \end{cases}$$

$$\dim_{C}\text{Hom}_{(g_0, K)}(D(-\lambda_2, -\lambda_1), C^\infty(N_0 \backslash G_0; \psi_{N_0})) = \begin{cases} 0 & \text{if } c_3 > 0; \\ 1 & \text{if } c_3 < 0. \end{cases}$$

(ii) Let $c_3 < 0$. We take a standard basis $\{v_{l}^{(-\lambda_2, -\lambda_1)} \mid 0 \leq l \leq \lambda_1 - \lambda_2\}$ of the minimal $K_0$-type $\tau_{(-\lambda_2, -\lambda_1)}$ of $D(-\lambda_2, -\lambda_1)$. Then, up to a constant multiple, the radial part of Whittaker function corresponding to $v_{l}^{(-\lambda_2, -\lambda_1)}$ is given by

$$W(v_{l}^{(-\lambda_2, -\lambda_1)}, \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})) = (2\sqrt{-1})^{-l}(|c_0|^{a_1}/a_2^{\lambda_1-l+1})(|c_3|a_2^{(\lambda_1+\lambda_2+1)/2}) \exp(-2\pi|c_3|a_2^2)$$

$$\times \frac{1}{(2\pi \sqrt{-1})^{2}} \int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty} \int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty} (\pi|c_0|^{a_1}/a_2^{\lambda_1-l+1})^{-2s_1}(4\pi|c_3|a_2^{2})^{-s_2}$$

$$\times (2\pi \Gamma(s_1)\Gamma(s_2))^{-1} \Gamma(s_2 - \lambda_2 + 1/2) \Gamma(s_2 - \lambda_2 + 1/2) ds_1 ds_2,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, and real numbers $(\sigma_1, \sigma_2)$ are chosen such that $\sigma_1 > \sigma_2 > 0$.

4. COMPUTATION OF ARCHIMEDEAN ZETA INTEGRALS

Using the explicit formulas for Whittaker functions, we can show the following.

**Theorem 4.1.** Assume that $\pi_\infty \cong D(\lambda_1, \lambda_2)[0]$ with $1 - \lambda_1 \leq \lambda_2 \leq 0$. We take Whittaker function $W_\infty$ for $D(\lambda_1, \lambda_2)[0]$ and sections $f_{1,\infty}, f_{2,\infty}$ as follows:

$$W_\infty(g) = W(v_{-\lambda_2, -\lambda_1}^{(-\lambda_2, -\lambda_1)}; g), \quad g \in G,$$

$$f_{1,\infty}(k) = 1, \quad k \in K_0,$$

$$f_{2,\infty}(k) = \langle \tau_{(-\lambda_2, -\lambda_1)}(k)v_{-\lambda_2, -\lambda_1}^{(-\lambda_2, -\lambda_1)}, v_{-\lambda_2, -\lambda_1}^{(-\lambda_2, -\lambda_1)} \rangle, \quad k \in K_0.$$
Then we have
\[ Z_{\infty}(s_1, s_2, W_{\infty}, f_{1,\infty}, f_{2,\infty}) = c \cdot (\sqrt{-1})^{-\lambda_1} \frac{L(s_1, \pi_\infty, std)L(s_2, \pi_\infty, spin)}{\Gamma_R(s_1 + 1)\Gamma_R(2s_1)\Gamma_R(2s_2 + \lambda - \lambda_2 + 1)} . \]

Here \( c = 2^{-2}(2\pi)^{-\lambda_2} \frac{\lambda_1!(-\lambda_2)!}{(-\lambda_2 + 1)!} \).

(Outline of proof) Recall that
\[ Z_{\infty}(s_1, s_2, W_{\infty}, f_{1,\infty}, f_{2,\infty}) = \int_{Z(R)N_{12}(R)\backslash G(R)} W_{\infty}(g)f_{1,\infty}(s_1, w_{2}g)f_{2,\infty}(s_2, w_{1}g) \, dg. \]

We denote by \( Z(R)N_{12}(R)\backslash G(R) \ni g = n(x_0, 0, 0, 0)n(0, 0, 0, x_3) \cdot \text{diag}(ab, a, b^{-1}, 1) \cdot \kappa \) \((x_0, x_3 \in R, a \in R^\times, b > 0, \kappa \in K_0)\) and consider the Iwasawa decomposition of \( w_ig \).

Then we have
\[ Z_{\infty}(s_1, s_2, W_{\infty}, f_{1,\infty}, f_{2,\infty}) = \int_{R_+^{x}}\int_{R^\times}\int_{K_0}\int_{R^{2}} W_{\infty}(diag(ab, a, b^{-1}, 1)k) \times f_{1,\infty}(s_1, u^{-1}(\frac{1}{\sqrt{x_0^2 + b^2}}(-x_0 \ b \ x_0))) \times f_{2,\infty}(s_2, u^{-1}(\frac{1}{\sqrt{x_0^2 + b^2}})) \times \exp\{-2\pi\sqrt{-1}(x_0 + x_3)\} \, dx_0dx_3dk \frac{da}{a} \frac{db}{b}. \]

We choose the data \((W_{1,\infty}, f_{1,\infty}, f_{2,\infty})\) as above and abbreviate \( \tau = \tau(-\lambda_2, -\lambda_1) \) and \( v_\ell = v_\ell^{(-\lambda_2, -\lambda_1)} \).

For \( \kappa \in K_0 \), we have
\[ W(v_{-\lambda_2};gk) = W(\tau(k)v_{-\lambda_2};g) = \sum_{0 \leq i \leq \lambda_1 - \lambda_2} \frac{\langle \tau(k)v_{-\lambda_2}, v_i \rangle}{\langle v_i, v_i \rangle} W(v_i;g), \]
\[ f_{2,\infty}(s_2, pk) = \sum_{0 \leq j \leq \lambda_1 - \lambda_2} \frac{\langle \tau(k)v_{-\lambda_2}, v_j \rangle}{\langle v_j, v_j \rangle} \langle \tau(p)v_j, v_\ell \rangle, \]
where \( p = p^{-1} = u^{-1}(\frac{1}{\sqrt{x_0^2 + b^2}}(-x_0 \ b \ x_0)) \in K_0 \). Then Schur’s orthogonal relation implies that
\[ Z_{\infty}(s_1, s_2, W_{\infty}, f_{1,\infty}, f_{2,\infty}) = \frac{\lambda_1!(-\lambda_2)!}{(\lambda_1 - \lambda_2 + 1)!} \sum_{0 \leq i \leq \lambda_1 - \lambda_2} \int_{(R^+)^2} \int_{R^d} W_{\infty}(v_i; \text{diag}(ab, a, b^{-1}, 1)) \frac{\langle \tau(p)v_i, v_{-\lambda_2} \rangle}{\langle v_i, v_i \rangle} \frac{a^{3}b^{4}}{(x_3^{2} + a^2)^{(s_1+1)/2}(x_0^{2} + b^2)^{s_2+1/2}} \exp\{-2\pi\sqrt{-1}(x_0 + x_3)\} \, dx_0dx_3 \frac{da}{a} \frac{db}{b}. \]

We substitute the explicit formula for \( W_{\infty} \) and compute the integrations with respect to \( x_0, x_3, a, b \):
\[ Z_{\infty}(s_1, s_2, W_{\infty}, f_{1,\infty}, f_{2,\infty}) = \frac{c'}{(2\pi\sqrt{-1})^2} \int_{(R^+)^2} \int_{(R^d)^2} \frac{\Gamma(s_1 + s_2 - t_2 + \lambda_1 + \lambda_2 - 2)\Gamma(s_2 - t_2 + \lambda_1 + \lambda_2 - 2)}{\Gamma(\frac{1}{2} + \lambda_1 + \lambda_2 - 1)\Gamma(\frac{1}{2} + \lambda_1 + \lambda_2 - 1)} \times \sum_{\epsilon \in \{0,1\}}\sum_{\substack{0 \leq k \leq \lambda_1 - \lambda_2 \\ k \equiv \lambda_1 + \epsilon(\text{mod } 2)}} (-1)^{m + (\lambda_1 + \epsilon)/2} (-\sqrt{-1})^{\epsilon} \sum_{j,m} \left( \begin{array}{c} \lambda_1 \\ j \end{array} \right) \frac{\lambda_1}{j + \lambda_1 + \epsilon - i/2} \frac{a^{3}b^{4}}{(x_3^{2} + a^2)^{(s_1+1)/2}(x_0^{2} + b^2)^{s_2+1/2}} \exp\{-2\pi\sqrt{-1}(x_0 + x_3)\} \, dx_0dx_3 \frac{da}{a} \frac{db}{b}. \]
Here \( c' = \frac{\lambda_1(-\lambda_2)}{(\lambda_1 - \lambda_2 + 1)!} 2^{-(s_1 + 2s_2) - \lambda_1 - \lambda_2 - 2\pi i - s_1 - s_2 - (\lambda_1 + \lambda_2)/2 + 7/2} \). We use Barnes’ first lemma for the integration over \( t_1 \), and collect the terms to find that

\[
Z_\infty(s_1, s_2, W_\infty, f_1, f_2, \infty) = \frac{\Gamma(\frac{s_1 + \lambda_1}{2})\Gamma(\frac{s_1 + \lambda_1 - 1}{2})}{\Gamma(\frac{s_1 + \lambda_1 - 1}{2})\Gamma(\frac{s_1 + \lambda_1 - 1}{2})} \times \frac{c'}{(2\pi \sqrt{-1})} \int_{t_2}^{\tau_1 + \sqrt{-1} \infty} \chi(s_2 - t_2 + \lambda_1 + \lambda_2 - 2) \Gamma(t_2 - \lambda_2 + 1) dt_2
\]

\[
\begin{aligned}
&= \frac{1}{\pi^{s_1}} \sum_{q=0}^{-\lambda_2/2} \frac{(\sqrt{-1})^{-\lambda_1 + \lambda_2 + 2q + 2q + \lambda_2} (-\lambda_2)!}{(2q)!(-\lambda_2^2 - q)!} \Gamma(s_1/2 - t_2 + \lambda_2 + 1/2 + q) dt_2 \quad \text{if } -\lambda_2 \text{ is even; } \\
&= \frac{1}{\pi^{s_1}} \sum_{q=0}^{-\lambda_2/2} \frac{(\sqrt{-1})^{-\lambda_1 + \lambda_2 + 2q + 2q + \lambda_2} (-\lambda_2)!}{(2q + 1)!(-\lambda_2^2 - q)!} \Gamma(s_1/2 - t_2 + \lambda_2 + 1/2 + q) dt_2 \quad \text{if } -\lambda_2 \text{ is odd.}
\end{aligned}
\]

Using Barnes’ first lemma for \( \int_{t_2} \), again, and taking the summation over \( q \), we can get the assertion. \( \square \)

**Remark 1.** When \( \pi_\infty \) is isomorphic to the class one principal series, coincidence of archimedean zeta integral and the product of local \( L \)-factors (divided by normalizing factors of two Eisenstein series) is shown ([3]).

**References**


