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Supercuspidal representations in the cohomology of the Rapoport-Zink space for the unitary group in three variables

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1. Introduction

This is a summary of the author's talk at the RIMS workshop "Automorphic Representations and Related Topics" on January 23, 2013. We report on a recent joint work with Yoichi Mieda on supercuspidal representations appearing in the ℓ-adic cohomology of the Rapoport-Zink space for the unramified unitary similitude group in three variables over $\mathbb{Q}_p$ for $p \neq 2$. Details will appear elsewhere ([IM2]).

Rapoport-Zink spaces are certain formal schemes $\mathscr{M}$ parameterizing quasi-isogenies of p-divisible groups with additional structures introduced by M. Rapoport and Th. Zink in the 1990's ([RZ], [Ra]). These spaces are generalizations of Lubin-Tate spaces and Drinfeld upper half spaces. They play an important role in the theory of p-adic uniformization of Shimura varieties, which has many striking applications to number theory and automorphic forms. It is widely believed that the ℓ-adic cohomology of the Rapoport-Zink spaces realize the local Langlands and Jacquet-Langlands correspondences in a rather mysterious way.

Let us explain a rough outline of the story. For the background on Lubin-Tate spaces and Drinfeld upper half spaces, see Carayol's paper [Ca]. (Note that the definition of general Rapoport-Zink spaces was not known at that time.) Let $\mathrm{M} := \mathscr{M}^{\mathrm{rig}}$ be the rigid analytic space associated to the generic fiber of the formal scheme $\mathscr{M}$. We have a tower of finite étale coverings $\mathrm{M}_r \longrightarrow \mathrm{M}$ defined by the level p^r-structures on the universal p-divisible group on M. The pro-object $\mathrm{M}_\infty = \{\mathrm{M}_r\}_r$ is sometimes called the *Rapoport-Zink tower* or the *Rapoport-Zink space at infinite level.* If the linear algebra datum (*Rapoport-Zink datum*) defining the Rapoport-Zink space satisfies certain technical conditions, we have a p-adic reductive group G, an inner form J of G, and a finite extension E of $\mathbb{Q}_p$ (*local reflex field*). We have a natural action of the product of three groups $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_E$, where W_E is the Weil group of E, on the ℓ-adic cohomology with compact support

$$H_c^i(\mathrm{M}_\infty, \overline{\mathbb{Q}}_\ell) := \varinjlim_r H_c^i(\mathrm{M}_r, \overline{\mathbb{Q}}_\ell).$$

Everybody working in this area believes that this $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_E$-representation is very interesting.

So far, many beautiful results are obtained for Lubin-Tate spaces and Drinfeld upper half spaces, where G or J is isomorphic to GL_n (e.g. [Ca], [HT], [Ha1], [Bo], [Far2]). We

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would like to study more general Rapoport-Zink spaces. However, when the group G is not an inner form of GL_n, we confront a fundamental problem — the local Langlands and Jacquet-Langlands correspondences are not bijective for general G. They are bijections between certain representations of the Weil group (*L-parameters*) and certain finite sets of irreducible smooth representations of $G(\mathbb{Q}_p)$ (*L-packets*). In order to understand the description of the ℓ-adic cohomology of Rapoport-Zink spaces, we need to understand the structure of L-packets (and A-packets) in detail.

In this note, we study *supercuspidal* representations of $\mathrm{GU}_{1,2}(\mathbb{Q}_p)$ appearing in the ℓ-adic cohomology of the Rapoport-Zink space for $\mathrm{GU}_{1,2}$. The main results are summarized in Theorem 4.2. Fortunately, thanks to Rogawski, we have enough tools in representation theory and in geometry ([Ro1], [Ro2]). We have a satisfactory classification of L-packets and A-packets for $\mathrm{GU}_{1,2}$ which enables us to state the main results clearly. We hope our results shed a new light on the study of the ℓ-adic cohomology in each degree of general Rapoport-Zink spaces.

Our results might be considered as a confirmation of a refinement of Kottwitz's conjecture on the alternating sum of the ℓ-adic cohomology of the Rapoport-Zink spaces ([Ra, Conjecture 5.1], [Ha2, Conjecture 5.3]). The alternating sum of the supercuspidal part of the ℓ-adic cohomology of the Rapoport-Zink space for $\mathrm{GU}_{1,2}$ was studied by Fargues in his thesis ([Far1, Théorème 8.2.2]). Note that, in our theorem (Theorem 4.2), we study the ℓ-adic cohomology in each degree rather than the alternating sum. We also treat supercuspidal representations whose L-parameters have *nontrivial* $\mathrm{SL}_2(\mathbb{C})$-part. We discovered peculiar phenomena for such supercuspidal representations. For example, they appear both in H_c^2 (middle degree) and H_c^3. This reflects the fact that such supercuspidal representations can be obtained as local components of non-tempered cuspidal automorphic representations. On the other hand, we expect supercuspidal representations whose L-parameters have *trivial* $\mathrm{SL}_2(\mathbb{C})$-part appear only in H_c^2. Our results may suggest a kind of "duality" or "mirror symmetry" between the degree of cohomology (Lefschetz's SL_2) and the $\mathrm{SL}_2(\mathbb{C})$-part in the L-parameter or A-parameter (see [Gr, Corollary 8.2] for an archimedean analogue). We also have similar results for the Rapoport-Zink space for $\mathrm{GSp}_4/\mathbb{Q}_p$.

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2. The local Langlands correspondence for GL_n and the ℓ-adic cohomology of Lubin-Tate spaces

We recall the local Langlands correspondence for GL_n and its realization in the ℓ-adic cohomology of Lubin-Tate spaces. In this case, the group G is GL_n and the group J is the multiplicative group of a central division algebra of invariant $1/n$. Most of the results explained in this section are obtained by Harris-Taylor and Boyer ([HT], [Bo]). Prior to [HT], Harris obtained similar results for the Drinfeld upper half spaces ([Ha1]), where the role of G and J are interchanged (i.e. the group G is the multiplicative group of a central division algebra of invariant $1/n$ and the group J is GL_n). For a relation between Lubin-Tate spaces and Drinfeld upper half spaces at infinite level, see [Far2] (and also [Fal], [SW] for recent developments).

Fix a prime number p and a finite extension F of $\mathbb{Q}_p$. Denote the residue field of F by $\mathbb{F}_q$. Let W_F be the Weil group of F. We have the following exact sequence of topological groups:

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \langle \mathrm{Frob}_q \rangle \cong \mathbb{Z} \longrightarrow 1,$$

where I_F is an open subgroup of W_F called the *inertia group*, and $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the geometric Frobenius element (i.e. the *inverse* of the q-th power map). *Local class field theory* gives us a canonical isomorphism of topological groups (*local reciprocity isomorphism*)

$$\mathrm{Art}_F \colon F^\times \xrightarrow{\cong} W_F^{\mathrm{ab}}$$

such that the uniformizers on the left hand side correspond to the lifts of Frob_q on the right hand side. Using the local reciprocity isomorphism Art_F, we identify continuous characters $\chi \colon F^\times \longrightarrow \mathbb{C}^\times$ and one dimensional continuous representations $\phi \colon W_F \longrightarrow \mathbb{C}^\times$.

The *local Langlands correspondence for* GL_n/F is a non-abelian generalization of local class field theory. Let $\mathrm{Irr}(\mathrm{GL}_n(F))$ denote the set of equivalence classes of irreducible smooth representations of the topological group $\mathrm{GL}_n(F)$. (The set $\mathrm{Irr}(\mathrm{GL}_n(F))$ is also denoted by $\Pi(\mathrm{GL}_n(F))$ by some authors.) Let $\Phi(\mathrm{GL}_n/F)$ denote the set of $\mathrm{GL}_n(\mathbb{C})$-conjugacy classes of L-parameters for GL_n/F. Recall that an *L-parameter* for GL_n/F is a continuous homomorphism

$$\phi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C}) \times W_F$$

such that the second factor of $\phi(\sigma, x)$ is equal to σ for all $(\sigma, x) \in W_F \times \mathrm{SL}_2(\mathbb{C})$, the first factor of $\phi(\sigma)$ is a semisimple element of $\mathrm{GL}_n(\mathbb{C})$ (i.e. ϕ is *Frobenius semisimple*), the image of $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is contained in $\mathrm{GL}_n(\mathbb{C})$, and the induced map $\phi|_{\mathrm{SL}_2(\mathbb{C})} \colon \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is a homomorphism of algebraic groups over $\mathbb{C}$. The group ${}^L\mathrm{GL}_n$ is called the *L-group* of GL_n/F. *The local Langlands correspondence for* GL_n is a canonical bijection

$$\mathrm{LLC}\colon\ \mathrm{Irr}(\mathrm{GL}_n(F)) \xleftrightarrow{1:1} \Phi(\mathrm{GL}_n/F)$$

characterized in terms of L-factors and ε-factors for pairs ([HT], [He]). Under the local Langlands correspondence, *supercuspidal representations* of $\mathrm{GL}_n(F)$ correspond to irreducible n-dimensional representations of W_F, and *(essentially) discrete series representations* of $\mathrm{GL}_n(F)$ correspond to irreducible n-dimensional representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$. When two L-parameters $\phi_1 \in \Phi(\mathrm{GL}_{n_1}/F)$, $\phi_2 \in \Phi(\mathrm{GL}_{n_2}/F)$ correspond to $\pi_1 \in \mathrm{Irr}(\mathrm{GL}_{n_1}(F))$, $\pi_2 \in \mathrm{Irr}(\mathrm{GL}_{n_2}(F))$ respectively, the direct sum $\phi_1 \oplus \phi_2$ corresponds to an irreducible smooth representation $\pi_1 \boxplus \pi_2 \in \Phi(\mathrm{GL}_{n_1+n_2}/F)$ called the *Langlands sum* of π_1 and π_2.

The Lubin-Tate space LT is an $(n-1)$-dimensional rigid analytic space over $\widehat{F^{\mathrm{ur}}}$, the p-adic completion of the maximal unramified extension of F. This space is defined by the deformation theory of one dimensional formal p-divisible groups with $\mathcal{O}_F$-action. We do not give a precise definition of LT here, but we only note that LT is non-canonically isomorphic to the countable disjoint union of open unit disks:

$$\mathrm{LT} \cong \coprod_{i\in\mathbb{Z}} \left(\operatorname{Spf} \mathcal{O}_{\widehat{F^{\mathrm{ur}}}}[[T_1,\ldots,T_{n-1}]]\right)^{\mathrm{rig}}$$

By putting the level p^r-structure on the universal p-divisible group on LT, we have a pro-étale Galois covering (*Lubin-Tate tower*) : $\mathrm{LT}_\infty = \{\mathrm{LT}_r\}_r \longrightarrow \mathrm{LT}$. The Galois group of the Lubin-Tate tower is $\mathrm{GL}_n(\mathcal{O}_F)$. On the ℓ-adic cohomology with compact support

$$H_c^i(\mathrm{LT}_\infty, \overline{\mathbb{Q}}_\ell) := \varinjlim_r H_c^i(\mathrm{LT}_r, \overline{\mathbb{Q}}_\ell),$$

we have a natural action of the product of three groups $\mathrm{GL}_n(\mathcal{O}_F) \times \mathcal{O}_D^\times \times I_F$, where D is a central division algebra over F of invariant $1/n$, and I_F is the inertia group of F. It is a nontrivial but important fact that this action naturally extends to an action of $\mathrm{GL}_n(F) \times D^\times \times W_F$ using Hecke correspondences and the Weil descent datum ([Ca], [HT], [RZ]).

Using local and global methods, Harris-Taylor and Boyer obtained the following fantastic results.

Theorem 2.1 (Harris-Taylor, Boyer ([HT], [Bo])). *Let $\tau \in \mathrm{Irr}(D^\times)$ be an irreducible smooth representation of $D^\times$, and let $\mathrm{JL}(\tau) \in \mathrm{Irr}(\mathrm{GL}_n(F))$ be the discrete series representation of $\mathrm{GL}_n(F)$ corresponding to τ by the local Jacquet-Langlands correspondence. By Zelevinsky's classification, $\mathrm{JL}(\tau) \cong \mathrm{Sp}_s(\pi)$, where $n = st$ and π is a supercuspidal representation of $\mathrm{GL}_t(F)$. Then, we have an isomorphism as $\mathrm{GL}_n(F) \times W_F$-representations:*

$$\left(\varinjlim_r \mathrm{Hom}_{D^\times}\left(H_c^{n-1+i}(\mathrm{LT}_r, \overline{\mathbb{Q}}_\ell), \tau\right)\right)^{\text{Frob-ss}}$$
$$\cong \begin{cases} \left(\mathrm{Sp}_{s-i}(\pi) \boxplus \pi|\det|^{s-i} \boxplus \cdots \boxplus \pi|\det|^{s-1}\right) \otimes \mathrm{LLC}(\pi^\vee)(\frac{n-s+2i}{2}) & 0 \le i \le s-1 \\ 0 & \text{otherwise,} \end{cases}$$

where Frob-ss *denotes the Frobenius semisimplification,* $\mathrm{LLC}(\pi^\vee)$ *denotes the local Langlands correspondence composed with contragredient, and* $(\frac{n-s+2i}{2})$ *denotes the Tate twist.*

Precisely speaking, Harris-Taylor proved the equality of the alternating sum of the cohomology groups, and Boyer calculated the cohomology in each degree. They use *vanishing cycle cohomology* (or *nearby cycle cohomology*) which is dual to the ℓ-adic cohomology of Lubin-Tate spaces. For an interpretation of the results of Harris-Taylor and Boyer in terms of the ℓ-adic cohomology of Lubin-Tate spaces, see the proof of Proposition 2.2 in [S]. Historically, when Harris-Taylor studied the Lubin-Tate spaces, they in fact *proved* the local Langlands correspondence for GL_n/F and (an alternating sum version of) Theorem 2.1 simultaneously by a rather indirect inductive argument.

Let us observe the statement of Theorem 2.1 a little more. Assume that $\mathrm{JL}(\tau)$ is *supercuspidal*. Then, we have $\mathrm{JL}(\tau) = \pi$ and $t = n$, $s = 1$. The left hand side of Theorem

2.1 survives only when $i = 0$. When $i = 0$, we have

$$\left(\varinjlim_{r} \mathrm{Hom}_{D^\times}\left(H_c^{n-1}(\mathrm{LT}_r, \overline{\mathbb{Q}}_\ell), \tau\right) \right)^{\text{Frob-ss}} \cong \pi \otimes \mathrm{LLC}(\pi^\vee)(\tfrac{n-1}{2}).$$

We see that the local Jacquet-Langlands correspondence $\mathrm{JL}\colon \tau \mapsto \pi$ and the local Langlands correspondence LLC are encoded in the ℓ-adic cohomology of the Lubin-Tate space. Since the right hand side of Theorem 2.1 is not supercuspidal unless $i = 0$ (in fact, it is not discrete series), we have the following observation: *supercuspidal representations of* $\mathrm{GL}_n(F)$ *appear only in the middle degree cohomology* H_c^{n-1} *of Lubin-Tate spaces.* A local elegant proof of this non-supercuspidality result was obtained by Mieda ([Mi]). Next, assume that $\mathrm{JL}(\tau)$ is *not supercuspidal.* We have $s > 1$. We also have the following observation: *When i becomes larger, the cohomology H_c^{n-1+i} becomes farther away from the middle degree, and the representation $\mathrm{Sp}_{s-i}(\pi) \boxplus \pi|\det|^{s-i} \boxplus \cdots \boxplus \pi|\det|^{s-1}$ becomes "farther away" from the discrete series $\mathrm{Sp}_s(\pi)$.* In some sense, the distance of the cohomological degree from the middle degree measures the "distance" of the representation from the discrete series. It seems interesting to pursue it from the viewpoint of the derived category version of Theorem 2.1 established by Dat ([D]).

We would like to generalize Theorem 2.1 to general Rapoport-Zink spaces. Our knowledge is very limited for the moment. There are several difficulties both in representation theory and geometry. We can overcome the difficulties when the group G is $\mathrm{GU}_{1,2}/\mathbb{Q}_p$ and the $G(\mathbb{Q}_p)$-representation is supercuspidal.

3. The local Langlands correspondence for the unitary similitude groups in three variables (after Rogawski)

Let p be a prime number, F a p-adic field, and E/F a quadratic extension. We recall Rogawski's results on the local Langlands correspondence for the unitary similitude group $\mathrm{GU}_{1,2}/F$. Of course, our references are [Ro1] and [Ro2].

Let us consider the unitary similitude group in three variables defined by

$$\mathrm{GU}_{1,2}(R) := \left\{ (g,\lambda) \in \mathrm{GL}_3(R \otimes_F E) \times R^\times \;\middle|\; g\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} {}^t\overline{g} = \lambda \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \right\}$$

for an F-algebra R, where $g \mapsto \overline{g}$ denotes the action of the nontrivial element of $\mathrm{Gal}(E/F)$. Let G' be another unitary similitude group in three variables with respect to E/F. By Landherr's theorem, there are exactly two isomorphism classes of hermitian forms in three variables with respect to E/F, and the unitary similitude groups defined by the two hermitian forms are isomorphic. Hence G' is (non-canonically) isomorphic to $\mathrm{GU}_{1,2}/F$, and isomorphisms between them are unique up to inner automorphisms. Therefore, we can canonically identify $\mathrm{Irr}(\mathrm{GU}_{1,2}(F))$ and $\mathrm{Irr}(G'(F))$. Hence we need only to consider $\mathrm{GU}_{1,2}/F$ in this section.

The local Langlands correspondence for $\mathrm{GU}_{1,2}/F$ was established by Rogawski. Let

$${}^L\mathrm{GU}_{1,2} := \left(\mathrm{GL}_3(\mathbb{C}) \times \mathbb{C}^\times\right) \rtimes W_F$$

be the L-group of $\mathrm{GU}_{1,2}/F$. Let $\Phi(\mathrm{GU}_{1,2}/F)$ be the set of $(\mathrm{GL}_3(\mathbb{C}) \times \mathbb{C}^\times)$-conjugacy classes of L-parameters

$$\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L\mathrm{GU}_{1,2}$$

(for the definition of L-groups and L-parameters, see [Ro1], [Ro2]). Rogawski defined a surjective map with finite fibers:

$$\mathrm{LLC}\colon\ \mathrm{Irr}(\mathrm{GU}_{1,2}(F)) \longrightarrow \Phi(\mathrm{GU}_{1,2}/F)\ .$$

For an L-parameter $\phi \in \Phi(\mathrm{GU}_{1,2}/F)$, the fiber $\Pi_\phi := \mathrm{LLC}^{-1}(\phi)$ is called the *L-packet.* Unlike the case of GL_n/F, the map LLC is not bijective. The cardinality of the L-packet Π_ϕ is either 1,2 or 4 depending on ϕ. The elements of Π_ϕ are parameterized by characters of a finite abelian group S_ϕ, which is isomorphic to either 0, $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$.

Rogawski also defined the *A-packets* for $\mathrm{GU}_{1,2}/F$. The A-packets are finite subsets of $\mathrm{Irr}(\mathrm{GU}_{1,2}(F))$ parameterized by A-parameters

$$\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L\mathrm{GU}_{1,2}\,.$$

The cardinality of an A-packet for $\mathrm{GU}_{1,2}/F$ is either 1,2 or 4. In most cases, L-packets are the same as A-packets. But there are few exceptions. In general, A-packets are not L-packets, and two A-packets may have nontrivial intersection. The notion of A-packets are important when we study global automorphic representations. The multiplicity formula for global automorphic representations is described in terms of global and local A-packets rather than L-packets (see [Ro2]). See also [BR1], [BR2], where the ℓ-adic cohomology of Shimura varieties (Picard modular surfaces) was studied in terms of A-packets.

Precisely speaking, in [Ro1], Rogawski defined L-packets and A-packets for the unitary group $\mathrm{U}_{1,2}/F$ rather than the unitary similitude group $\mathrm{GU}_{1,2}/F$ using endoscopic character relations. The definition of the L-packets for $\mathrm{GU}_{1,2}/F$ is given in [Ro2, §2]. Fortunately, the representation theory of $\mathrm{GU}_{1,2}(F)$ is almost identical to that of $\mathrm{U}_{1,2}(F)$ because $\mathrm{GU}_{1,2}(F)$ is generated by its center and $\mathrm{U}_{1,2}(F) \subset \mathrm{GU}_{1,2}(F)$. We define L-packets (resp. A-packets) for $\mathrm{GU}_{1,2}/F$ as follows: a finite set of irreducible smooth representations of $\mathrm{GU}_{1,2}(F)$ is an L-packet (resp. A-packet) if and only if they have the same central character, and the restriction of them to $\mathrm{U}_{1,2}(F)$ forms an L-packet (resp. A-packet) of $\mathrm{U}_{1,2}/F$.

Rogawski classified L-packets for $\mathrm{U}_{1,2}/F$ and $\mathrm{GU}_{1,2}/F$ into 9 types according to the structure of L-parameters. See the list (1)–(9) in page 174 of [Ro1, §12.2], where the list is written for $\mathrm{U}_{1,2}/F$. The list for $\mathrm{GU}_{1,2}/F$ is essentially the same. Among 9 types of L-packets, the following 4 types of L-packets contain supercuspidal representations (for unexplained notation on endoscopic transfer, see [Ro1]).

Type (2) $\Pi(\mathrm{St}_H(\xi)) = \{\pi^2(\xi),\ \pi^s(\xi)\}$

ξ is a one-dimensional representation of the elliptic endoscopic group $\mathrm{U}_{1,1}(F) \times \mathrm{U}_1(F)$. $\pi^2(\xi)$ is non-supercuspidal discrete series, and $\pi^s(\xi)$ is supercuspidal.

Type (4) $\Pi(\rho) = \{\pi_0,\ \pi_1\}$

Both π_0, π_1 are supercuspidal. ρ is a supercuspidal representation of $\mathrm{U}_{1,1}(F) \times \mathrm{U}_1(F)$, and not contained in any L-packet obtained from $\mathrm{U}_1(F) \times \mathrm{U}_1(F) \times \mathrm{U}_1(F)$.

Type (5) $\Pi(\rho(\theta)) = \{\pi_0,\ \pi_1,\ \pi_2,\ \pi_3\}$

All of $\pi_0, \pi_1, \pi_2, \pi_3$ are supercuspidal. θ is a regular character of $\mathrm{U}_1(F) \times \mathrm{U}_1(F) \times \mathrm{U}_1(F)$.

Type (9) $\Pi = \{\pi_0\}$

π_0 is supercuspidal. π_0 is not contained in any L-packet obtained from an L-packet of $\mathrm{U}_{1,1}(F) \times \mathrm{U}_1(F)$.

In this list, all representations except for $\pi^2(\xi)$ in Type (2) are supercuspidal. For L-packets of Type (4),(5),(9), the L-parameters have trivial $\mathrm{SL}_2(\mathbb{C})$-part. But for L-packets of Type (2), the L-parameters have nontrivial $\mathrm{SL}_2(\mathbb{C})$-part.

All of the L-packets of Type (2),(4),(5),(9) are also A-packets. There is another type of A-packets of the form $\Pi(\xi) = \{\, \mathrm{Zel}(\pi^2),\ \pi^s(\xi)\,\}$ consisting of a non-tempered unitary representation $\mathrm{Zel}(\pi^2)$ and a supercuspidal representation $\pi^s(\xi)$ in an L-packet of Type (2) ([Ro1, §13.1]). The non-tempered representation $\mathrm{Zel}(\pi^2(\xi))$ is the Zelevinsky dual to the discrete series $\pi^2(\xi)$. In [Ro1], $\mathrm{Zel}(\pi^2(\xi))$ is denoted by $\pi^n(\xi)$. Therefore, a supercuspidal representation of the form $\pi^s(\xi)$ is contained in two different A-packets. According to [Ro1], a supercuspidal representation not of the form $\pi^s(\xi)$ is contained in exactly one A-packet.

The *(standard) base change map* is a natural map from the set of L-packets of $\mathrm{GU}_{1,2}/F$ to the set of L-packets of GL_3/E. The L-group ${}^L\mathrm{GU}_{1,2}$ is a semidirect product of $\mathrm{GL}_3(\mathbb{C}) \times \mathbb{C}^\times$ and W_F, which is split when it is restricted to $W_E \subset W_F$. For an L-parameter $\phi \in \Phi(\mathrm{GU}_{1,2}/F)$, the restriction of ϕ to $W_E \times \mathrm{SL}_2(\mathbb{C})$ composed with $\mathrm{GL}_3(\mathbb{C}) \times \mathbb{C}^\times \longrightarrow \mathrm{GL}_3(\mathbb{C}),\ (g,\lambda) \mapsto \lambda g$ gives the following homomorphism

$$\phi_E\colon W_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_3(\mathbb{C}) \times \mathbb{C}^\times \longrightarrow \mathrm{GL}_3(\mathbb{C})\ .$$

The map ϕ_E is an L-parameter for GL_3/E. By the local Langlands correspondence for GL_3/E, ϕ_E corresponds to an irreducible smooth representation $\pi_E \in \mathrm{Irr}(\mathrm{GL}_3(E))$. The map $\Pi_\phi \mapsto \{\pi_E\}$ is called the *(standard) base change map.* There is a variant of this map, called the *non-standard base change map* or *variant base change map*, which is useful when we study a relation between base change and endoscopic transfer ([Ro2, §2.4]).

4. Supercuspidal representations in the ℓ-adic cohomology of the Rapoport-Zink space for $\mathrm{GU}_{1,2}$

From now on, we assume the following:

Assumption 4.1. $p \neq 2$, $F = \mathbb{Q}_p$, and E is a unramified quadratic extension of $\mathbb{Q}_p$.

The main reason why we need such a technical assumption is geometric. We use Vollaard-Wedhorn's results on the underlying space of the Rapoport-Zink space for $\mathrm{GU}_{1,2}/\mathbb{Q}_p$. Their papers [V], [VW] are written under this assumption. (In fact, Vollaard-Wedhorn obtained similar results for $\mathrm{GU}_{1,n-1}/\mathbb{Q}_p$ for any n ([VW]). In [Z], Wei Zhang studied the Rapoport-Zink space for $\mathrm{GU}_{1,2}/F$ when $F \neq \mathbb{Q}_p$ (still assuming $p \neq 2$ and E/F is unramified). But a cautious reader will note that the details of proofs are not written in [Z]. Instead, [VW] is cited in that paper.)

Let M be the Rapoport-Zink space for $\mathrm{GU}_{1,2}/F$. This is the rigid analytic space associated with the moduli space of quasi-isogenies of 3-dimensional p-divisible groups with $\mathcal{O}_E$-action satisfying certain conditions on the Lie algebra (for the precise definition, see [V], [VW]). Let

$$\mathrm{M}_\infty = \{\mathrm{M}_r\}_r \longrightarrow \mathrm{M}$$

be the Rapoport-Zink tower on M. On the ℓ-adic cohomology with compact support

$$H_c^i(\mathrm{M}_\infty, \overline{\mathbb{Q}}_\ell) := \varinjlim_r H_c^i(\mathrm{M}_r, \overline{\mathbb{Q}}_\ell),$$

we have a natural action of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_E$. We would like to study $H_c^i(\mathrm{M}_\infty, \overline{\mathbb{Q}}_\ell)$ as a representation of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_E$. The p-adic reductive groups G and J are (non-canonically) isomorphic to the unitary similitude group $\mathrm{GU}_{1,2}/\mathbb{Q}_p$. Hence we can use results on the local Langlands correspondence for $\mathrm{GU}_{1,2}/\mathbb{Q}_p$ as in §3.

By several technical reasons, we cannot study $H_c^i(\mathrm{M}_\infty, \overline{\mathbb{Q}}_\ell)$ directly. Instead, we study the following space. For $\tau \in \mathrm{Irr}(J(\mathbb{Q}_p))$, we define a $G(\mathbb{Q}_p) \times W_E$-representation $M^i(\tau)$ by

$$M^i(\tau) := \left(\varinjlim_r \mathrm{Hom}_{J(\mathbb{Q}_p)}\left(H_c^i(\mathrm{M}_r, \overline{\mathbb{Q}}_\ell), \tau\right)\right)^{\text{Frob-ss}, \, G(\mathbb{Q}_p)\text{-supercusp}},$$

where "Frob-ss" denotes the Frobenius semisimplification and "$G(\mathbb{Q}_p)$-supercusp" denotes the $G(\mathbb{Q}_p)$-supercuspidal part. We would like to determine $M^i(\tau)$ as a representation of $G(\mathbb{Q}_p) \times W_E$.

Now we give the statement of our main results. As you may imagine, the structure of $M^i(\tau)$ depends on the type of the L-packet containing τ. Recall that there are 4 types of L-packets of $G \cong J \cong \mathrm{GU}_{1,2}/\mathbb{Q}_p$ (see §3) containing supercuspidal representations. There is another type of A-packet containing both supercuspidal representations and non-tempered representations. Recall that, for an L-parameter $\phi \in \Phi(\mathrm{GU}_{1,2}/\mathbb{Q}_p)$,

$$\phi_E \colon W_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_3(\mathbb{C})$$

denotes the base change of ϕ.

Theorem 4.2. *Let $\tau \in \mathrm{Irr}(J(\mathbb{Q}_p))$ be an irreducible smooth representation of $J(\mathbb{Q}_p)$ with L-parameter $\phi \in \Phi(\mathrm{GU}_{1,2}/\mathbb{Q}_p)$. Assume that τ belongs to an L-packet or A-packet containing a supercuspidal representation. (Note that τ itself need not be supercuspidal.) Then, we calculate the $G(\mathbb{Q}_p) \times W_E$-representation $M^i(\tau)$ as follows.*

- *Assume that τ belongs to an L-packet of Type (9) (i.e. τ is supercuspidal and $\{\pi\}$ forms an L-packet). We consider $\tau \in \mathrm{Irr}(G(\mathbb{Q}_p))$ via an isomorphism $G(\mathbb{Q}_p) \cong J(\mathbb{Q}_p)$. Then, we have*

$$M^i(\tau) = \begin{cases} \tau \otimes \phi_E(1) & i = 2 \\ 0 & i \neq 2 \end{cases}$$

Here, (1) denotes the Tate twist. In this case, ϕ_E is an irreducible 3-dimensional representation of W_E.

- *Assume that τ belongs to an L-packet of Type (4) (i.e. τ belongs to an L-packet of the form $\{\pi_0, \pi_1\}$, where π_0 is generic supercuspidal and π_1 is non-generic supercuspidal.). Then, τ is either π_0 or π_1. In this case, ϕ_E is a direct sum of a character of W_E and an irreducible 2-dimensional representation of W_E. We decompose it as $\phi_E = \phi_1 \oplus \phi_2$, where $\dim \phi_i = i$. Then, for $k = 0, 1$, we have*

$$M^i(\pi_k) = \begin{cases} \big(\pi_k \otimes \phi_1(1)\big) \oplus \big(\pi_{1-k} \otimes \phi_2(1)\big) & i = 2 \\ 0 & i \neq 2 \end{cases}$$

- *Assume that τ belongs to an L-packet of Type (5) (i.e. τ belongs to an L-packet of the form $\{\pi_0, \pi_1, \pi_2, \pi_3\}$, where π_0 is generic supercuspidal and π_1, π_2, π_3 are non-generic supercuspidal.) Then, τ is either π_0, π_1, π_2 or π_3. In this case, ϕ_E is a direct sum of three different characters, i.e. $\phi_E = \theta_1 \oplus \theta_2 \oplus \theta_3$. For each $k = 0, 1, 2, 3$, we have a bijection*

$$\sigma_k \colon \{0,1,2,3\} \backslash \{k\} \xrightarrow{1:1} \{1,2,3\}$$

Then, for $k = 0, 1, 2, 3$, we have

$$M^i(\pi_k) = \begin{cases} \bigoplus_{j \in \{0,1,2,3\} \backslash \{k\}} \big(\pi_j \otimes \theta_{\sigma_k(j)}(1)\big) & i = 2 \\ 0 & i \neq 2 \end{cases}$$

(Note that π_k does not appear in $M^i(\pi_k)$. We do not explain how to specify $\theta_1, \theta_2, \theta_3$ and how to define σ_k. They can be defined explicitly in terms of characters of $S_\phi \cong (\mathbb{Z}/2\mathbb{Z})^2$.)

- *Assume that τ belongs to an L-packet of Type (2) (i.e. τ belongs to an L-packet of the form $\{\pi^2, \pi^s\}$, where π^2 is non-supercuspidal discrete series and π^s is supercuspidal.) In this case, $\phi_E|_{\mathrm{SL}_2(\mathbb{C})}$ is nontrivial. As a representation of $W_E \times \mathrm{SL}_2(\mathbb{C})$,*

$$\phi_E = \big(\nu \otimes \mathrm{std}\big) \oplus \big(\xi \otimes \mathrm{triv}\big),$$

where ν, ξ are characters of W_E, and std *(resp.* triv*) denotes the standard (resp. trivial) representation of $\mathrm{SL}_2(\mathbb{C})$. Then, we have*

$$M^i(\pi^2) = \begin{cases} \pi^s \otimes \nu(-\tfrac{1}{2}) & i = 2 \\ 0 & i \neq 2 \end{cases}, \qquad M^i(\pi^s) = \begin{cases} \pi^s \otimes \xi & i = 2 \\ 0 & i \neq 2 \end{cases}$$

(Note that we take the $G(\mathbb{Q}_p)$-supercuspidal part in the definition of M^i. Hence π^2 does not appear in $M^i(\pi^2), M^i(\pi^s)$. It seems natural to expect that π^2 also appears in the space $\varinjlim_r \mathrm{Hom}_{J(\mathbb{Q}_p)}\big(H^2_c(\mathrm{M}_r, \overline{\mathbb{Q}}_\ell), \tau\big)$.)

- *Assume that τ is non-tempered, and τ belongs to an A-packet containing a supercuspidal representation. Then, there is an L-packet $\{\pi^2, \pi^s\}$ of Type (2) such that $\tau = \mathrm{Zel}(\pi^2)$. Then, $\{\tau = \mathrm{Zel}(\pi^2), \pi^s\}$ is an A-packet containing τ and a supercuspidal representation π^s. Let ϕ' be the L-parameter of the L-packet $\{\pi^2, \pi^s\}$, and denote the base change of ϕ' as $\phi'_E = \big(\nu \otimes \mathrm{std}\big) \oplus \big(\xi \otimes \mathrm{triv}\big)$. Then, we have*

$$M^i\big(\mathrm{Zel}(\pi^2)\big) = \begin{cases} \pi^s \otimes \nu(\tfrac{1}{2}) & i = 3 \\ 0 & i \neq 3 \end{cases}$$

(Note that π^s appears in H_c^3 (not in H_c^2) in this case. The L-parameter ϕ_E can be obtained from ϕ'_E by the same method as in the definition of ϕ_ψ in page 19 of [A1]*.)*

Interested reader may compare Theorem 4.2 with Theorem 2.1. Note that all supercuspidal representations of $G(\mathbb{Q}_p)$ appears in the middle degree cohomology H_c^2 of the Rapoport-Zink space. Supercuspidal representations of $G(\mathbb{Q}_p)$ whose L-parameters have nontrivial $\mathrm{SL}_2(\mathbb{C})$-part (i.e. those belonging to L-packets of Type (2) in Rogawski's list) appear both in H_c^2 and H_c^3.

We explain the outline of the proof. In short, our proof is a combination of the methods of Harris-Taylor for Lubin-Tate spaces (so-called "Boyer's trick"), and the methods of Harris for Drinfeld upper half spaces using p-adic uniformization and the Hochshild-Serre spectral sequence ([HT], [Ha1]). We use the Hochschild-Serre spectral sequence constructed by Fargues ([Far1, Corollaire 4.5.21]):

$$E_2^{i,j} = \varinjlim_{r} \mathrm{Ext}^i_{J(\mathbb{Q}_p)\text{-smooth}}\big(H_c^{4-j}(\mathrm{M}_r, \overline{\mathbb{Q}}_\ell), \mathscr{A}\big) \implies H^{i+j}(\mathrm{Sh}^{\mathrm{rig}}_{\mathrm{basic}}, \overline{\mathbb{Q}}_\ell),$$

where $\mathscr{A}$ denotes a space of automorphic forms on an inner form I of $\mathrm{GU}_{1,2}/\mathbb{Q}$ such that $I(\mathbb{R})$ is compact modulo center. This spectral sequence is $\mathrm{GU}_{1,2}(\mathbb{A}_f) \times W_E$-equivariant, and it connects the ℓ-adic cohomology of the Rapoport-Zink space and the ℓ-adic cohomology of the rigid analytic space associated with the formal completion along the *basic locus* (*supersingular locus* in the notation of [V], [VW]) of the Shimura variety (Picard modular surface). Since the split semisimple rank of $J \cong \mathrm{GU}_{1,2}/\mathbb{Q}_p$ is equal to 1, this spectral sequence degenerates at E_2-terms ([SS, Corollary III 3.3]). Hence we can obtain information on the ℓ-adic cohomology of the Rapoport-Zink space from that of the Shimura variety. The calculation of the ℓ-adic cohomology of unitary Shimura varieties was finally completed by Shin ([S]). In order to isolate the $G(\mathbb{Q}_p)$-supercuspidal part, we use non-supercuspidality results as in [IM1]. In order to obtain the information of $M^i(\tau)$ for each i, we globalize τ to an automorphic representation of $\mathrm{GU}_{1,2}(\mathbb{A})$ appropriately. Rogawski's multiplicity formula for global A-packets plays an important role ([Ro1], [Ro2]).

5. Concluding remarks and some speculations

Theorem 4.2 seems one of the first results on the endoscopic decomposition of the ℓ-adic cohomology in each degree of the Rapoport-Zink spaces where the group G (or J) is not an inner form of GL_n. Of course, it is a natural question to generalize Theorem 4.2 to more general Rapoport-Zink spaces.

Results similar to Theorem 4.2 can be obtained for $\mathrm{GSp}_4/\mathbb{Q}_p$. The geometry of the supersingular locus of the Siegel threefolds is classically known (cf. [LO]). We have non-supercuspidality results ([IM1]). Thanks to the work of Gan, Takeda, Tantono, Chan, we have fairly complete information about the local Langlands and Jacquet-Langlands correspondences for GSp_4 and its inner forms ([GTak1], [GTan], [GC]). In fact, understanding the results of [GTak1], [GTan] was a source of inspiration and motivation of our work.

For more general Rapoport-Zink spaces, very little is known. There are many difficulties both in representation theory and geometry. Representation theoretically, we do not yet have satisfactory results on the local Langlands and Jacquet-Langlands correspondences in general. Thanks to the recent results of Arthur and Mok, we can now understand L-packets and A-packets better than before ([A2], [Mo]). We would like to understand more. We would like to know which members in L-packets (and A-packets) are supercuspidal (see also [Mœ]). They treat quasi-split semisimple groups (such as Sp_{2n}), but we need to work with inner forms with similitudes (such as GSp_{2n} or an inner form of it). For the comparison between L-packets for Sp_4 and GSp_4, see [GTak2]. Geometrically, the situation seems more serious and we need new ideas. The geometry of Rapoport-Zink spaces seems more complicated for higher rank groups such as $\mathrm{GU}_{r,s}$ $(r+s\geq 4)$ and GSp_{2n} $(n\geq 3)$. For example, for $G=\mathrm{GSp}_{2n}$, the dimension of the underlying space of the Rapoport-Zink space is $\lfloor n^2/4\rfloor$ ([LO]), which is much larger than the semisimple rank of GSp_{2n}. The analysis of the Hochshild-Serre spectral sequence would become more difficult for higher rank groups.

Nevertheless, it seems natural to expect that a supercuspidal representation π of $G(\mathbb{Q}_p)$ appears outside the middle degree cohomology of the Rapoport-Zink space if and only if the L-parameter of π has nontrivial $\mathrm{SL}_2(\mathbb{C})$-part. Perhaps, it might be helpful to study particular supercuspidal representations as a motivating example. For example, in [HKS], Harris-Kudla-Sweet constructed supercuspidal representations of unitary groups over a p-adic field whose (conjectural) L-parameter have large $\mathrm{SL}_2(\mathbb{C})$-part ([HKS, Speculation 7.7]).

Of course, there are many other problems on the geometry and the cohomology of Rapoport-Zink spaces which are not discussed in this note. It is interesting to study the contribution of *non-supercuspidal* representations as in Boyer's work ([Bo]). Except for Lubin-Tate spaces and Drinfeld upper half spaces, very little is known so far.

References

[A1] Arthur, J., *Unipotent automorphic representations: conjectures*, Orbites unipotentes et représentations, II. Astérisque No. 171-172 (1989), 13-71.

[A2] Arthur, J., *The endoscopic classification of representations: orthogonal and symplectic groups*, preprint, to appear as a Colloquium Publication of the American Mathematical Society, 2012.

[BR1] Blasius, D., Rogawski, J. D., *Tate classes and arithmetic quotients of the two-ball*, The zeta functions of Picard modular surfaces, 421-444, Univ. Montréal, Montreal, QC, 1992.

[BR2] Blasius, D., Rogawski, J. D., *Motives for Hilbert modular forms*, Invent. Math. 114 (1993), no. 1, 55-87.

[Bo] Boyer, P., *Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples*, Invent. Math. 177 (2009), no. 2, 239-280.

[Ca] Carayol, H., *Nonabelian Lubin-Tate theory*, in Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), 15-39, Perspect. Math., 11, Academic Press, Boston, MA, 1990.

[D] Dat, J.-F., *Théorie de Lubin-Tate non-abélienne et représentations elliptiques*, Invent. Math. 169 (2007), no. 1, 75-152.

[Fal] Faltings, G., *Coverings of p-adic period domains*, J. Reine Angew. Math. 643 (2010), 111-139.

[Far1] Fargues, L., *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, in Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, Astérisque No. 291 (2004), 1-199.

[Far2] Fargues, L., *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques*, in L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld, 1-325, Progr. Math., 262, Birkhäuser, Basel, 2008.

[GC] Gan, W. T., Chan, P. S., *The local Langlands conjecture for* GSp(4) *III: stability and twisted endoscopy*, preprint.

[GTak1] Gan, W. T., Takeda, S., *The local Langlands conjecture for* GSp(4), Ann. of Math. (2) 173 (2011), no. 3, 1841-1882.

[GTak2] Gan, W. T., Takeda, S., *The local Langlands conjecture for* Sp(4), Int. Math. Res. Not. 2010, no. 15, 2987-3038.

[GTan] Gan, W. T., Tantono, W., *The local Langlands conjecture for* GSp(4) *II: the case of inner forms*, preprint.

[Gr] Gross, B. H., *On minuscule representations and the principal* SL_2, Represent. Theory 4 (2000), 225-244.

[HKS] Harris, M., Kudla, S. S., Sweet, W. J., *Theta dichotomy for unitary groups*, J. Amer. Math. Soc. 9 (1996), no. 4, 941-1004.

[Ha1] Harris, M., *Supercuspidal representations in the cohomology of Drinfel'd upper half spaces; elaboration of Carayol's program*, Invent. Math. 129 (1997), no. 1, 75-119.

[Ha2] Harris, M., *Local Langlands correspondences and vanishing cycles on Shimura varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 407-427, Progr. Math., 201, Birkhäuser, Basel, 2001.

[HT] Harris, M., Taylor, R., *The geometry and cohomology of some simple Shimura varieties*, With an appendix by Vladimir G. Berkovich., Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ, 2001.

[He] Henniart, G., *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. 139 (2000), no. 2, 439-455.

[IM1] Ito, T., Mieda, Y., *Cuspidal representations in the l-adic cohomology of the Rapoport-Zink space for* GSp(4), preprint, arXiv:1005.5619.

[IM2] Ito, T., Mieda, Y., *Supercuspidal representations in the cohomology of the Rapoport-Zink space for the unitary group in three variables*, in preparation.

[LO] Li, K.-Z., Oort, F., *Moduli of supersingular abelian varieties*, Lecture Notes in Mathematics, 1680. Springer-Verlag, Berlin, 1998.

[Mi] Mieda, Y., *Non-cuspidality outside the middle degree of ℓ-adic cohomology of the Lubin-Tate tower*, Adv. Math. 225 (2010), no. 4, 2287-2297.

[Mœ] Mœglin, C., *Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques*, Pacific J. Math. 233 (2007), no. 1, 159-204.

[Mo] Mok, C.-P., *Endoscopic classification of representations of quasi-split unitary groups*, preprint.

[Ra] Rapoport, M., *Non-Archimedean period domains*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 423-434, Birkhäuser, Basel, 1995.

[RZ] Rapoport, M., Zink, Th., *Period spaces for p-divisible groups*, Annals of Mathematics Studies, 141. Princeton University Press, Princeton, NJ, 1996.

[Ro1] Rogawski, J. D., *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, 123. Princeton University Press, Princeton, NJ, 1990.

[Ro2] Rogawski, J. D., *The multiplicity formula for A-packets*, The zeta functions of Picard modular surfaces, 395-419, Univ. Montréal, Montreal, QC, 1992.

[SS] Schneider, P., Stuhler, U., *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. No. 85 (1997), 97-191.

[SW] Scholze P., Weinstein, J., *Moduli of p-divisible groups*, preprint, arXiv:1211.6357

[S] Shin, S. W., *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) 173 (2011), no. 3, 1645-1741.

[V] Vollaard, I., *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1,s)$, Canad. J. Math. 62 (2010), no. 3, 668-720.

[VW] Vollaard, I., Wedhorn, T., *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1,n-1)$ *II*, Invent. Math. 184 (2011), no. 3, 591-627.

[Z] Zhang, W., *On arithmetic fundamental lemmas*, Invent. Math. 188 (2012), no. 1, 197-252.