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SOME REMARKS ON THE GL(2) CONVERSE THEOREM

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These notes are a transcript of my lecture at the conference on Automorphic Forms and Related Topics at RIMS, describing my joint work with Krishnamurthy on the GL(2) converse theorem. For a more formal account of our results, see [2, 3, 4].

Over the years, many people have worked on reducing the twisting set in the various converse theorems for automorphic representations, though it would be fair to say that the theory has reached a sort of plateau, with few substantial improvements over the last decade. The approach that Krishnamurthy and I have been exploring is a bit different, in that we are trying to reduce the analytic properties required of the twists rather than the twisting set itself, though in some cases it has the same effect, as we will see.

1. The classical converse theorem

$$\begin{tikzcd} \text{modular forms} \arrow[r, "\text{Hecke}"] & L\text{-functions} \\ f \in M_k(\Gamma_1(N)) & \\ f(z) = \sum_{n=0}^{\infty} f_n e^{2\pi i n z} & L_f(s) = \sum_{n=1}^{\infty} f_n n^{-s} \arrow[l, dashed, bend left, "?"] \end{tikzcd}$$

Let me begin with a brief history of the subject. The theory of the *converse theorem* began with Hecke, who showed how to associate L-functions with nice analytic properties (analytic continuation, functional equation) to modular forms. His recipe is very simple: Given a modular form f, as in the above diagram, once simply takes the coefficients a_n from the Fourier expansion and puts them into a Dirichlet series $L_f(s)$. Using the Mellin transform, one can show that $L_f(s)$ continues to a meromorphic function and satisfies a functional equation.

Hecke conceived the idea that the logic of this derivation should be invertible, i.e. it should be possible to characterize modular forms in terms of the analytic properties of their L-functions. When $N = 1$ (and for a few other small levels), he showed that this is indeed the case, but he could not justify his intuition in general.

This is where the theory stood for several decades until the 1960s, when two things happened nearly simultaneously. One was the Langlands conjectures, which brought representation theory to the forefront of the theory of L-functions, and described a sort of Grand Unified Theory governing the relationships between them. In this language, Hecke's idea of the converse theorem found its proper place, and it became possible to imagine that one knew all of the L-functions.

The second was the theorem by Weil [9], who understood the missing piece of the puzzle in Hecke's converse theorem: the notion of "twisting". The point is that the L-function $L_f(s)$ does not exist in isolation, but can be embedded in a family of L-functions $L_f(s, \chi) = \sum_{n=1}^{\infty} f_n \chi(n) n^{-s}$, where χ ranges over primitive Dirichlet characters of modulus

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co-prime to N, all of which have similar analytic properties. Here is the version of Weil's result that you will find in the excellent book by Miyake [8, Theorem 4.3.15]:

Theorem (Weil, [9]). *Let ψ be a Dirichlet character (mod N), k a positive integer with $\psi(-1) = (-1)^k$, $\{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty$ sequences of complex numbers satisfying $f_n, g_n = O(n^\sigma)$ for some $\sigma > 0$,*

$$\Lambda_f(s,\chi) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} f_n\chi(n)n^{-s}, \quad \Lambda_g(s,\overline{\chi}) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} g_n\overline{\chi(n)}n^{-s}$$

for primitive Dirichlet characters χ, and $\Lambda_f(s) = \Lambda_f(s, \mathbf{1})$, where $\mathbf{1}$ is the character of modulus 1.

Let $\mathcal{P}$ be a set of primes such that $\{p \in \mathcal{P} : p \equiv u \pmod v\}$ is infinite $\forall u, v \in \mathbb{Z}_{>0}$ with $(u, v) = 1$ and $p \nmid N$ for any $p \in \mathcal{P}$. For every χ mod $q \in \mathcal{P} \cup \{1\}$, assume that $\Lambda_f(s, \chi)$ and $\Lambda_g(s, \overline{\chi})$:

- *continue to entire functions of finite order, except possibly for simple poles at $s \in \{0, k\}$ if $\chi = \mathbf{1}$,*
- *satisfy the functional equation*

$$\Lambda_f(s,\chi) = \epsilon\psi(q)\chi(N)\frac{\tau(\chi)^2}{q}(q^2N)^{\frac{k}{2}-s}\Lambda_g(k-s,\overline{\chi}),$$

where $\tau(\chi)$ is the Gauss sum and $\epsilon \in \mathbb{C}^\times$ is fixed.

Set $f_0 = -\operatorname{Res}_{s=0}\Lambda_f(s)$, $f(z) = \sum_{n=0}^\infty f_n e^{2\pi i n z}$.
Then $f \in M_k(\Gamma_0(N), \psi)$.

As one can see from the statement, it is still the case that a special role is played by the L-function with no twist. Precisely, this is the only one allowed to have poles (which can occur when f is an Eisenstein series).

Relaxing the analytic data in Weil's theorem. Our first result uses exactly the same twisting set as in Weil's theorem, but reduces the strength of the analytic input for all but the L-function with no twist, $L_f(s)$:

Theorem (B.-Krishnamurthy [4]). *Let notation be as in Weil's theorem. For each $q \in \mathcal{P} \cup \{1\}$, suppose that $\Lambda_f(s, \chi)$ and $\Lambda_g(s, \overline{\chi})$ continue to* meromorphic *functions on $\mathbb{C}$ and satisfy the same functional equation as before. Assume further that there is a non-zero polynomial P such that $P(s)\Lambda_f(s)$ continues to an entire function of finite order. Then*

- *if $k \neq 2$ or ψ is non-trivial, $f \in M_k(\Gamma_0(N), \psi)$;*
- *if $k = 2$ and ψ is trivial, $\exists c \in \mathbb{C}$ such that $f - cE_2 \in M_2(\Gamma_0(N))$, where $E_2(z) = 1 - 24\sum_{n=1}^\infty \frac{ne^{2\pi i n z}}{1-e^{2\pi i n z}}$ is the Eisenstein series of weight 2 and level 1.*

Note that with these relaxed hypotheses with pick up some *quasi*-modular forms, such as the Eisenstein series E_2. Curiously, those arise not because of the very relaxed conditions for the twists $L_f(s, \chi)$ but because we allow finitely many poles for $L_f(s)$ itself. (The complete L-function of E_2 has three poles instead of two!)

This result is closer in spirit to the one envisioned by Hecke. I personally think that this is the right way to view the classical converse theorem, and perhaps if there is ever a new edition of Miyake's book, this is the version that it will include. That said, this result, as with Weil's theorem before it, has few applications (it does have some, one of which I will mention at the end), because it is a bit too classical. Put another way, we now understand classical holomorphic modular forms too well (the biggest outstanding result was Serre's conjecture, and that is now a theorem!). The theorem should rather be viewed as a prototype for more general settings.

2. Generalizations

Our next result is to the Jacquet–Langlands converse theorem [5, Theorem 11.3] what the first result was to Weil's. In other words, we generalize to automorphic representations of GL(2) over a number field.

Theorem (B.-Krishnamurthy [2]). *Let F be a number field with adèle ring $\mathbb{A}_F$, $\pi = \bigotimes_v \pi_v$ an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central idèle class quasicharacter ω_π, and*

$$\Lambda(s, \pi \otimes \omega) = \prod_v L(s, \pi_v \otimes \omega_v), \quad \Lambda(s, \widetilde{\pi} \otimes \omega^{-1}) = \prod_v L(s, \widetilde{\pi}_v \otimes \omega_v^{-1}),$$

for unitary idèle class characters ω. Suppose that $\Lambda(s, \pi \otimes \omega)$ and $\Lambda(s, \widetilde{\pi} \otimes \omega^{-1})$:

- *converge absolutely and define analytic functions in some right half-plane, $\Re(s) > \sigma$;*
- *continue to meromorphic functions on $\mathbb{C}$ and satisfy the functional equation*

  $$\Lambda(s, \pi \otimes \omega) = \epsilon(s, \pi \otimes \omega)\Lambda(1 - s, \widetilde{\pi} \otimes \omega^{-1}),$$

  where $\epsilon(s, \pi \otimes \omega)$ is the product of local ϵ-factors defined by Jacquet and Langlands;
- *are entire of finite order whenever ω is unramified at every non-archimedean place.*

Then π is an automorphic representation.

With this result we get our first glimpse of how the method generalizes: It allows us to reduce ramification in the twisting set. Unfortunately, over a number field other than $\mathbb{Q}$, there are infinitely many unramified Grossencharakters, so we do not get down to a single L-function in general. This is in line with other analytic results. Philosophically speaking, it seems that one should regard a given π as inseparable from its twists $\pi \otimes \omega$ by unramified Grossencharakters.

Applications. As predicted, with a more general statement, we get some interesting applications:

Corollary. *Let $\rho : W_F \to \mathrm{GL}_2(\mathbb{C})$ be a representation of the Weil group of F. Suppose that the associated L-functions $\Lambda(s, \rho \otimes \omega)$ are entire for all idèle class characters ω that are unramified at every non-archimedean place. Then ρ is automorphic, i.e., there is an automorphic representation π such that π_v corresponds to ρ_v under the local Langlands correspondence for each place v.*

In particular:

Corollary. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ be a continuous Galois representation. If Artin's conjecture is true for ρ then ρ is modular.*

Note that there are still 2-dimensional complex Galois representations over $\mathbb{Q}$ for which the Artin conjecture is not known! Namely, the proof Serre's conjecture settles the matter only for the odd representations (those which turn out to be associated to a holomorphic modular form), and not the even ones (associated to Maass forms). The Langlands–Tunnell theorem handles all cases with solvable image, but that still leaves the even icosahedral representations.

It should be noted that the above corollary does not prove any new cases of Artin's conjecture. What it does show is that the Langlands philosophy for resolving the conjecture is the right one, i.e. even if we manage to find some way of attacking the Artin conjecture without recourse to automorphic forms, we conclude Langlands' stronger conjecture anyway.

Refinements. The alert reader will recall that we allowed finitely many poles for $L_f(s)$ in our generalization of Weil's theorem, but not in the last result generalizing the Jacquet–Langlands theorem. In more recent work [3], we have shown that one can indeed weaken the conditions on the twists by unramified idèle class characters ω. Precisely, it is enough to know that $D(s,\omega)\Lambda(s,\pi\otimes\omega)$ is entire of finite order, where

$$D(s,\omega)=\sum_{\mathfrak{a}\supset\mathfrak{m}}\lambda(\mathfrak{a})\chi_\omega(\mathfrak{a})N(\mathfrak{a})^{-s}$$

is a twisted Dirichlet polynomial and $L(s,\omega)=\sum_{\mathfrak{a}}\chi_\omega(\mathfrak{a})N(\mathfrak{a})^{-s}$.

In some ways this is actually stronger than the classical result (e.g. it allows the unramified twists to have some thin infinite sets of poles), thanks to the extra formality of the adelic language. Some other features include the following:

- The result yields a clean statement of the converse theorem including all cases of Eisenstein series. There was previously a version of the GL(2) converse theorem including Eisenstein series, due to Winnie Li [7], but the statement is complicated by the need to specify the locations of all poles, just as in Weil's theorem (though significantly more complicated in general). Our result can accommodate any finite collection of poles for finitely many ω.
- The result applies to partial L-functions, i.e. it is enough to know the analytic properties of $\Lambda^S(s,\pi\otimes\omega)=\prod_{v\notin S}L(s,\pi_v\otimes\omega_v)$ for a fixed finite set S of non-archimedean places.
- Along the way, we obtained a new characterization of the generic representations over an archimedean local field, in terms of L-factors. (The characterization was new to us, though like many things, it was implicit in Jacquet–Langlands if ones reads deeply enough.)

Related results and work in progress. Before sketching the proofs of our results, let me mention some related results and where we are headed with this method. First, using the same ideas that go into the proof of the classical theorem, one can show:

Theorem (B. [1]). *Let $f\in S_k(\Gamma_1(N))^{\mathrm{new}}$ be a normalized Hecke eigenform. Then the complete L-function $\Lambda(s,f)=\int_0^\infty f(iy)y^{s-1}\,dy$ has infinitely many simple zeros.*

This result was previously known for *finitely many* modular forms, beginning with the proof for $f=\Delta$ given by Conrey and Ghosh about 25 years ago. Actually, their method is general, but they found themselves in a situation reminiscent of the proof of Siegel's theorem, in that they could prove infinitely many simple zeros once there is at least one, but they couldn't rule out the possibility that every zero is multiple!

In a sense, our proof takes their leading order analysis and refines it to all orders, and this is eventually enough to break the cycle of circular logic. We will see the same effect in the proof of the classical converse theorem shortly.

Work in progress:

- Reducing further to twists by finite-order Hecke characters when π is algebraic.

- Converse theorems for $GL(n)$, following Cogdell and Piatetski-Shapiro. The most natural version uses twists by unramified $GL(n-1)$ representations, and in fact this is already a theorem for fields of class number 1. We expect that our method will generalize to remove the class number restriction, and hopefully lead to new applications in higher rank.

3. Main ideas of the proof

Hecke's argument. We turn now to the proof of our version of the classical converse theorem. For the sake of this discussion, we will make several simplifying assumptions along the way, but retain enough to see the general flavor of the method. First, let us assume that $\Lambda_f(s)$ is entire and $\Lambda_f(s,\chi)$ is meromorphic and satisfies a functional equation for every primitive character χ.

The key point in Hecke's theory is that $\Lambda_f(s)$ is related to the Fourier series f by the Mellin transform:

$$\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} f_n n^{-s} = \sum_{n=1}^{\infty}\int_0^{\infty} f_n e^{-2\pi n y} y^s \frac{dy}{y} = \int_0^{\infty} f(iy) y^s \frac{dy}{y}.$$

Applying Mellin inversion, the analytic properties of $\Lambda_f(s)$ (continuation to a function of finite order, functional equation) are equivalent to the modularity relation

$$f(z) = \epsilon\left(\frac{i}{\sqrt{N}z}\right)^k g\left(-\frac{1}{Nz}\right).$$

When $N=1$, this and the invariance under $z\mapsto z+1$ are enough to generate the full modular group, which gives Hecke's converse theorem.

Additive twists. In general we don't get enough to generate the full congruence group, so we need more information. Weil showed that this can be obtained via the twists by Dirichlet characters, and in many ways these are the natural objects to consider. However, at its core, Weil's proof actually uses the so-called *additive twists*:

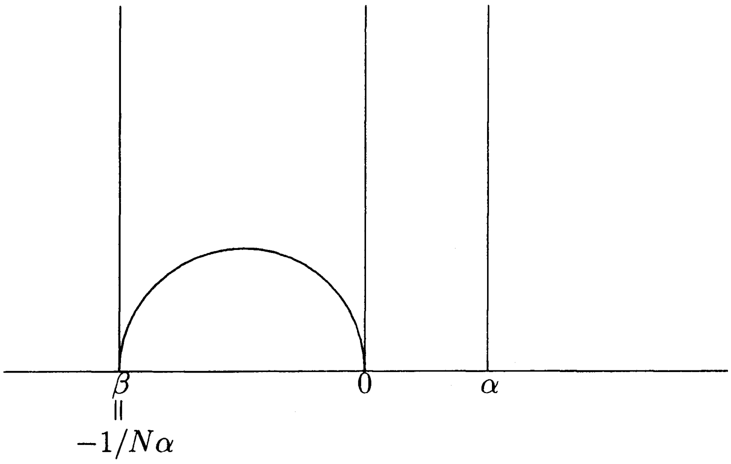
$$\Lambda_f(s,\alpha) = (2\pi|\alpha|)^{-s}\Gamma(s)\sum_{n=1}^{\infty} f_n e^{2\pi i\alpha n} n^{-s}, \quad \alpha\in\mathbb{Q}^\times.$$

These are related to the multiplicative twists $\Lambda_f(s,\chi)$ by Fourier analysis on $\mathbb{Z}/q\mathbb{Z}$ and $(\mathbb{Z}/q\mathbb{Z})^\times$. Note that it is easier to go from additive twists to multiplicative twists than vice versa, because of complications with imprimitive characters. However, if we assume, for instance, that $\Lambda_f(s)$ is given by an Euler product, then it follows from the properties of the multiplicative twists that $\Lambda_f(s,\alpha)$ continues meromorphically to $\mathbb{C}$ with all poles confined to a vertical strip $\Re(s)\in[\sigma_1,\sigma_2]$.

The additive twists are relevant because they also occur as Mellin transforms of f, but along the vertical line passing through the cusp at α rather than the imaginary axis:

$$\Lambda_f(s,\alpha) = \sum_{n=1}^{\infty} f_n e^{2\pi i\alpha n}\int_0^{\infty} e^{-2\pi|\alpha| n y} y^s \frac{dy}{y} = \int_0^{\infty} f(\alpha + i|\alpha| y) y^s \frac{dy}{y}.$$

Relation between $\Lambda_f(s,\alpha)$ and $\Lambda_g(s,-1/N\alpha)$? Now, the one piece of information that we haven't yet used is the modularity relation relating f and g at z an $-1/Nz$, respectively. In view of this, one might guess that the additive twist by α is the same as that by $\beta = -1/N\alpha$. That turns out to be a bit too naive, however, because if we map the whole half-line passing through α under the Möbius transformation $z \mapsto -1/Nz$, the result is no longer a straight line, but the semi-circular geodesic which meetings the x-axis at β and 0:



However, as the diagram shows, this is at least *tangent* to the vertical half-line passing through β, so we expect $\Lambda_f(s,\alpha)$ and $\Lambda_g(s,\beta)$ to agree to leading order. Precisely, we have

$$-\frac{1}{N(\alpha+i|\alpha|y)} = \beta + i|\beta|y - \frac{\beta y^2}{1-i\,\mathrm{sgn}(\beta)y}, \quad \beta = -\frac{1}{N\alpha},$$

and substituting this into the modularity relation gives

$$\begin{aligned} f(\alpha+i|\alpha|y) &= \epsilon\left(\frac{i}{\sqrt{N}(\alpha+i|\alpha|y)}\right)^k g\left(-\frac{1}{N(\alpha+i|\alpha|y)}\right), \\ &= \epsilon(-i\sqrt{N}\beta)^k \sum_{n=1}^{\infty} g_n e^{2\pi i\beta n} e^{-2\pi|\beta|ny} \\ &\qquad \cdot \left(1-i\,\mathrm{sgn}(\beta)y\right)^{-k} \exp\left(-\frac{2\pi i\beta n y^2}{1-i\,\mathrm{sgn}(\beta)y}\right) \end{aligned}$$

Note that if not for the correction factor $\left(1-i\,\mathrm{sgn}(\beta)y\right)^{-k}\exp\left(-\frac{2\pi i\beta ny^2}{1-i\,\mathrm{sgn}(\beta)y}\right)$, which tends to 1 as $y \to 0^+$, the Mellin transform of the right-hand side would be, up to a constant factor, $\Lambda_g(s,\beta)$, as expected. Our strategy is to use the Taylor expansion

$$\left(1-i\,\mathrm{sgn}(\beta)y\right)^{-k}\exp\left(-\frac{2\pi i\beta ny^2}{1-i\,\mathrm{sgn}(\beta)y}\right) = \sum_{m=0}^{\infty}\left(i\,\mathrm{sgn}(\beta)y\right)^m \sum_{j=0}^{m}\binom{m+k-1}{m-j}\frac{\left(-2\pi n|\beta|y\right)^j}{j!}$$

and take the Mellin transform of each term separately:

$$f(\alpha + i|\alpha|y) = \epsilon(-i\sqrt{N}\beta)^k \sum_{m=0}^{M-1} \left(i\,\mathrm{sgn}(\beta)\right)^m \sum_{j=0}^{m} \binom{m+k-1}{m-j}$$
$$\cdot \frac{y^{j+m}}{j!} \frac{d^j}{dy^j} \sum_{n=1}^{\infty} g_n e^{2\pi i\beta n} e^{-2\pi|\beta|ny} + O\left(y^{M-\sigma-2}\right)$$
$$= \epsilon(-i\sqrt{N}\beta)^k \sum_{m=0}^{M-1} \left(-i\,\mathrm{sgn}(\beta)\right)^m$$
$$\cdot \frac{1}{2\pi i} \int_{\Re(s)=\sigma+2} \binom{s+m-k}{m} \Lambda_g(s+m,\beta) y^{-s}\, ds + O\left(y^{M-\sigma-2}\right).$$

Thus,

$$\Lambda_f(s,\alpha) - \epsilon(-i\sqrt{N}\beta)^k \sum_{m=0}^{M-1} \left(-i\,\mathrm{sgn}(\beta)\right)^m \binom{s+m-k}{m} \Lambda_g(s+m,\beta)$$

is holomorphic for $\Re(s) > \sigma + 2 - M$.

As an example, suppose we know that $\Lambda_f(s,\alpha)$ and $\Lambda_g(s,\beta)$ are holomorphic outside of the strip $\Re(s) \in (0,1)$. (Note that this and all of the other assumptions that we made above hold for Artin L-functions, so we have not lost all generality.) Then

$$\begin{aligned} &\Lambda_f(s,\alpha) \text{ has a pole at } s = s_0 \\ &\Longrightarrow \Lambda_g(s,\beta) \text{ has a pole at } s = s_0 \\ &\Longrightarrow \Lambda_g(s+1,\beta) \text{ has a pole at } s = s_0 - 1 \\ &\Longrightarrow \Lambda_f(s,\alpha) \text{ has a pole at } s = s_0 - 1. \end{aligned}$$

But this is a contradiction, so $\Lambda_f(s,\alpha)$ has no poles!

4. Saito-Kurokawa lifts

Finally, I mention one application of our classical converse theorem to Siegel modular forms.

Siegel modular forms. Let $\mathcal{H}_g = \{Z \in \mathrm{Mat}_{g\times g}(\mathbb{C}) : Z^T = Z, \Im(Z) \text{ positive definite}\}$. This has an action of

$$\mathrm{Sp}(2g,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A,B,C,D \in \mathrm{Mat}_{g\times g}(\mathbb{R}), \right.$$
$$\left. AB^T = BA^T, CD^T = DC^T, AD^T - BC^T = I_{g\times g} \right\}$$

defined by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right).Z = (AZ+B)(CZ+D)^{-1}$. A (classical) Siegel modular form of weight k and genus g is a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ such that $f(\gamma.Z) = \det(CZ+D)^k f(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2g,\mathbb{Z})$.

Saito-Kurokawa lift. Saito and Kurokawa (independently) conjectured the existence of an injective linear map

$$M_{2\kappa}(\mathrm{SL}(2,\mathbb{Z})) \hookrightarrow M_{\kappa+1}(\mathrm{Sp}(4,\mathbb{Z})),$$

with an explicit relationship between the corresponding L-functions. This was proven by Maass, Kohnen and Zagier; in fact, their proof shows more:

$$M_{2\kappa}(\mathrm{SL}(2,\mathbb{Z})) \xrightarrow[\sim]{\text{Shimura}} M^{+}_{\kappa+\frac{1}{2}}(\Gamma_0(4)) \hookrightarrow M_{\kappa+1}(\mathrm{Sp}(4,\mathbb{Z})),$$

where the first arrow is the Shimura correspondence, and $M^{+}_{\kappa+\frac{1}{2}}(\Gamma_0(4))$ is Kohnen's "plus space" of half-integral weight modular forms.

The Saito-Kurokawa lift can be generalized in various ways, e.g. to congruence subgroups, higher genus, etc. A natural question, for a given generalization, is whether one can characterize its image in terms of local data, i.e. Hecke eigenvalues. D. Lanphier [6] answered this question for a generalization to congruence groups using, among other tools, our version of the classical converse theorem with poles.

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