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# ON THETA CORRESPONDENCES FOR $(\mathrm{GSp}_4, \mathrm{GSO}_{4,2})$

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ABSTRACT. We consider local and global theta correspondences for $\mathrm{GSp}_4$ and $\mathrm{GSO}_{4,2}$. Because of the accidental isomorphism $\mathrm{PGSO}_{4,2} \simeq \mathrm{PGU}_{2,2}$, these correspondences give rise to those between $\mathrm{GSp}_4$ and $\mathrm{GU}_{2,2}$ for representations with trivial central characters. Also we characterize representations which have Shalika period using theta correspondences. In this note, we give results without a proof, and details will appear in [8].

## 1. GLOBAL THETA CORRESPONDENCE

Let $F$ be a number field, and we denote its ring of adeles by $\mathbb{A}_F$. Let $E$ be a quadratic extension of $F$ and $\mathbb{A}_E$ its ring of adeles. We choose $d \in F^\times \setminus (F^\times)^2$ such that $E = F(\eta)$ with $\eta = \sqrt{d}$ and $\bar{\eta} = -\eta$.

We define the similitude unitary group $\mathrm{GU}_{2,2}$ by

$$\mathrm{GU}_{2,2}(F) = \left\{ g \in \mathrm{GL}_4(E) \mid {}^t\bar{g} \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \lambda(g) \in F^\times \right\}.$$

and the similitude symplectic group $\mathrm{GSp}_4$ by

$$\mathrm{GSp}_4(F) = \mathrm{GU}_{2,2}(F) \cap \mathrm{GL}_4(F).$$

Let $\mathrm{GO}_{4,2}$ be the similitude orthogonal group defined by

$$\mathrm{GO}_{4,2} = \left\{ g \in \mathrm{GL}_6 \mid {}^t g S g = \mu(g) S,\ \mu(g) \in \mathbb{G}_m \right\}$$

where

$$S = \begin{pmatrix} &&&&& 1 \\ &&&& 1 & \\ && 2 &&& \\ &&& -2d && \\ & 1 &&&& \\ 1 &&&&& \end{pmatrix}.$$

Denote

$$\mathrm{GSO}_{4,2} = \left\{ g \in \mathrm{GO}_{4,2} \mid \det(g) = \mu(g)^3 \right\}.$$

Then we note that the group $\mathrm{GSO}_{4,2}$ is closely related to $\mathrm{GU}_{2,2}$. Indeed we have

$$\mathrm{PGSO}_{4,2} \simeq \mathrm{PGU}_{2,2}. \tag{1.0.1}$$

Then we shall study the global theta correspondence for $(\mathrm{GSp}_4^+, \mathrm{GU}_{2,2})$ because of the accidental isomorphism. Here for an algebra $R$ over $F$, we denote

$$\mathrm{GSp}_4(R)^+ = \{ g \in \mathrm{GSp}_4(R) \mid \lambda(g) = \mu(h) \text{ for some } h \in \mathrm{GSO}_{4,2}(R) \}.$$

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Indeed, we give a characterization of automorphic representations which have Shalika period in terms of the global theta correspondence.

Let us define unitary analogue of Shalika period on $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ as follows. Let $\xi$ be an idele class character of $\mathbb{A}_F^\times/F^\times$. Let $(\pi, V_\pi)$ be an irreducible cuspidal unitary automorphic representation of $\mathrm{GU}(2,2)\,(\mathbb{A}_F)$ with the central character $\omega_\pi$ satisfying $\omega_\pi\mid_{\mathbb{A}_F^\times}=\xi^{-2}$. Let $\psi$ be a non-trivial additive character of $\mathbb{A}_F/F$, and we regard $\psi$ as a character of

$$N(\mathbb{A}_F)=\left\{\begin{pmatrix}1_2 & X\\ 0 & 1_2\end{pmatrix}\mid {}^t\overline{X}=X\in \mathrm{Mat}_{2\times 2}(\mathbb{A}_E)\right\}$$

by

$$\psi\begin{pmatrix}1_2 & X\\ 0 & 1_2\end{pmatrix}=\psi\left(\mathrm{tr}\left(X\begin{pmatrix}0 & \eta\\ -\eta & 0\end{pmatrix}\right)\right).$$

Then we define the Shalika period of $\varphi\in V_\pi$ by

$$\int_{\mathbb{A}_F^\times \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)}\varphi\left(n\begin{pmatrix}g & 0\\ 0 & \det g\cdot {}^t g^{-1}\end{pmatrix}\right)\psi(n)\xi(\det g)\,dn\,dg.$$

Further, we can define a period on $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ which corresponds to Shalika period with respect to the trivial character from the isomorphism (1.0.1). We also call this period Shalika period of $\mathrm{GSO}_{4,2}$.

Recall that we have the following characterization of irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ which have Shalika period, which is an analogue of Jacquet-Shalika's theorem [5].

**Theorem 1.1** (Theorem 4.1 in [1]). *With the above notations, the following two conditions are equivalent:*

(1) *The Shalika period with respect to $\xi$ does not vanish on the space of $\pi$.*
(2) *$\pi$ is globally generic and the partial twisted exterior square L-function $L^S(s,\pi,\wedge_t^2\otimes\xi)$ has a simple pole at $s=1$.*

By the standard method (e.g. see [2], [11]), we can show that Whittaker period of the theta lift from $\mathrm{GSp}_4^+(\mathbb{A}_F)$ to $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ is expressed by Whittaker period on $\mathrm{GSp}_4^+(\mathbb{A}_F)$. Similarly, it is shown that Whittaker period of the theta lift from $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ to $\mathrm{GSp}_4^+(\mathbb{A}_F)$ is expressed by Shalika period of $\mathrm{GSO}_{4,2}$. Then as in the well-known case of $(\mathrm{GSp}_4,\mathrm{GSO}_{3,3})$, we obtain the following characterization of irreducible cuspidal automorphic representations of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ which have Shalika period, via the global theta correspondence.

**Theorem 1.2.** *Let $(\sigma, V_\sigma)$ be an irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ with trivial central character. Then $\sigma$ has Shalika period if and only if $\sigma=\theta^*(\Pi)$ for some generic irreducible cuspidal automorphic representation $\Pi$ of $\mathrm{GSp}_4(\mathbb{A}_F)$ with trivial central character. Here we denote $\sigma=\theta^*(\Pi)$ when $\sigma=\theta(\Pi^+)$ for some irreducible constituent $\Pi^+$ of $\Pi|_{\mathrm{GSp}_4(\mathbb{A}_F)^+}$.*

We remark that Takeo Okazaki, motivated by a conjecture of van Geeman and van Straten, also studied independently the global aspect of this theta correspondence and gave a sketch of proof for the relationship between the non-vanishing of theta lift and the existence of Shalika period in [9]. Though he did not discuss any local theory, as far as the author knows.

## 2. Local theta correspondence

Let $F$ be a nonarchimedean local field of characteristic zero. According to the global case, we can define a local analogue of Shalika period of $\mathrm{GSO}_{4,2}$.

We shall consider the local theta correspondence for $(\mathrm{GSp}_4^+(F), \mathrm{GSO}_{4,2}(F))$. More precisely, we consider a local analogue of characterization given in Theorem 1.2. As a first step for the characterization, we study local theta correspondence itself.

**Theorem 2.1.** *For the dual pair $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$, the Howe duality holds over any nonarchimedean local field of characteristic zero. Moreover, we can compute explicitly local theta correspondence from $\mathrm{GSp}_4^+(F)$ to $\mathrm{GSO}_{4,2}(F)$.*

We note that for a proof of this theorem and an explicit computation of the theta correspondence, we need classifications of non-supercuspidal irreducible admissible representations of $\mathrm{GSp}_4^+(F)$ and $\mathrm{GSO}_{4,2}(F)$. The classification for $\mathrm{GSp}_4^+(F)$ is deduced from that of $\mathrm{GSp}_4(F)$ and the study of restrictions of irreducible representations of $\mathrm{GSp}_4(F)$ to $\mathrm{GSp}_4^+(F)$ (cf. [3]). On the other hand, the classification for $\mathrm{GSO}_{4,2}(F)$ is essentially new. We can give the classification using a method in Sally–Tadic [10] and Konno [6]. Then we can compute the local theta correspondence explicitly as in [4].

Computing twisted Jacquet modules of the extended Weil representation (cf. [2], [7]), we obtain a characterization for generic irreducible representations of $\mathrm{GSO}_{4,2}(F)$, which have Shalika period.

**Proposition 2.1.** *Let $\sigma$ be a generic irreducible representation of $\mathrm{GSO}_{4,2}(F)$. Then the following conditions are equivalent:*

(1) *$\sigma$ has Shalika period.*
(2) *the small theta lift $\theta(\sigma)$ of $\sigma$ to $\mathrm{GSp}_4^+(F)$ is non-zero.*
(3) *the small theta lift $\theta(\sigma)$ of $\sigma$ to $\mathrm{GSp}_4^+(F)$ is generic.*

Using explicit computation of local theta lifts from $\mathrm{GSO}_{4,2}(F)$ to $\mathrm{GSp}_4^+(F)$, we get a necessary condition for essentially tempered representations of $\mathrm{GSO}_{4,2}(F)$ to have Shalika period.

**Proposition 2.2.** *Let $\sigma$ be an irreducible representation of $\mathrm{GSO}_{4,2}(F)$. Suppose that $\sigma$ is essentially tempered. If $\sigma$ has Shalika period, then $\sigma$ is generic.*

From this proposition, we obtain the following local analogue of Theorem 1.2 for essentially tempered irreducible representations of $\mathrm{GSO}_{4,2}(F)$.

**Theorem 2.2.** *Let $\sigma$ be as in Proposition 2.2. Then $\sigma$ has Shalika period if and only if $\sigma = \theta(\pi)$ for some generic irreducible representation $\pi$ of $\mathrm{GSp}_4^+(F)$.*

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