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m-WALK-REGULAR GRAPHS, A GENERALIZATION OF DISTANCE-REGULAR GRAPHS

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1. INTRODUCTION

This paper is based on joint work with Marc Cámara, Edwin van Dam and Jongyook Park and it is a preliminary version of [7] which we are still working on. In [7], you can find all the details. Walk-regular graphs were introduced by Godsil and McKay [26] in their study of cospectral graphs. They showed that the property that the vertex-deleted subgraphs of a graph Γ are all cospectral is equivalent to the property that the number of closed walks of a given length ℓ in Γ is independent of the starting vertex, for every ℓ. They also observed that walk-regular graphs generalize both vertex-transitive graphs and distance-regular graphs. Distance-regular graphs [5, 16] play a crucial role in the area of algebraic combinatorics, and it was shown by Rowlinson [33] that such graphs can be characterized in terms of the numbers of walks between two vertices; in particular that this number only depends on their length and the distance between the two vertices. Motivated by this characterization, Dalfó, Fiol, and Garriga [22, 11] introduced t-walk-regular graphs; such graphs have the property of Rowlinson's characterization at least for those vertices that are at distance at most t. These t-walk-regular graphs were further studied by Dalfó, Fiol, and coauthors [12, 13, 14, 15]. Dalfó, Van Dam, and Fiol [14] characterized t-walk-regular graphs in terms of the cospectrality of certain perturbations, thus going back to the roots of walk-regular graphs. Dalfó, Van Dam, Fiol, Garriga, and Gorissen [15] among others raised the question of when t-walk-regularity implies distance-regularity.

Our motivation for studying t-walk-regular graphs lies in the generalization of distance-regular graphs. In order to better understand the latter, we would like to know which results for these graphs can be generalized to t-walk-regular graphs. In this way, we aim to have a better understanding of which properties of distance-regular graphs are most relevant.

Here we will focus on 1- and in particular 2-walk-regular graphs. We will for example generalize Delsarte's clique bound [17] to 1-walk-regular graphs. It seems however that 1-walk-regularity is still far away from distance-regularity, but going to 2-walk-regularity is an important step (or jump) forward. Indeed, we will see that several important results on distance-regular graphs have interesting generalizations to 2-walk-regular graphs (but not to 1-walk-regular graphs), such as Godsil's multiplicity bound [23] and Terwilliger's analysis of the local structure

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[36]. On the other hand, there are very basic construction methods for 1-walk-regular graphs that cannot be generalized to 2-walk-regular graphs; indeed, most know examples of the latter come from groups as graphs that are obtained in an elementary way (such as the line graph and halved graph) from s-arc-transitive graphs. We will indeed show that 2-walk-regular graphs have a much richer combinatorial structure than 1-walk-regular graphs. We will show that there are finitely many non-geometric 2-walk-regular graphs with given smallest eigenvalue and given diameter (a geometric graph is the point graph of a special partial linear space); a result that is analogous to a result on distance-regular graphs. In fact, this result shows that the class of 2-walk-regular graphs is quite limited. Again, such a result does not hold for 1-walk-regular graphs, as our construction methods (Proposition 3.5, in particular) will show.

This paper is organized as follows: in the next section, we give some technical background. In Section 3, we give elementary construction methods for t-walk-regular graphs that we will use in the remaining sections. In Section 4, Godsil's multiplicity bound for distance-regular graphs is generalized to 2-walk-regular graphs. Similarly we generalize in Section 5 Terwilliger's analysis of local graphs. In Section 6, we study t-walk-regular graphs with an eigenvalue with small multiplicity. Finally, in Section 7, we generalize Delsarte's clique bound and study geometric 2-walk-regular graphs.

2. Preliminaries

Let Γ be a connected graph with vertex set $V = V(\Gamma)$ and denote $x \sim y$ if the vertices $x, y \in V$ are adjacent. The *distance* $\mathrm{dist}_\Gamma(x, y)$ between two vertices $x, y \in V$ is the length of a shortest path connecting x and y (we omit the index Γ when this is clear from the context). The maximum distance between two vertices in Γ is the *diameter* $D = D(\Gamma)$. We use $\Gamma_i(x)$ for the set of vertices at distance i from x and write, for the sake of simplicity, $\Gamma(x) := \Gamma_1(x)$. The *degree* of x is the number $|\Gamma(x)|$ of vertices adjacent to it. A graph is *regular* with *valency* k if the degree of each of its vertices is k.

For a connected graph Γ with diameter D, the *distance-i graph* Γ_i of Γ $(1 \le i \le D)$ is the graph whose vertices are those of Γ and whose edges are the pairs of vertices at mutual distance i in Γ. In particular, $\Gamma_1 = \Gamma$. The *distance-i matrix* $A_i = A_i(\Gamma)$ is the matrix whose rows and the columns are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever $\mathrm{dist}(x, y) = i$ and 0 otherwise. The *adjacency matrix* A of Γ equals A_1.

The *eigenvalues* of the graph Γ are the eigenvalues of A. We use $\{\theta_0 > \ldots > \theta_d\}$ for the set of distinct eigenvalues of Γ. The *multiplicity* of an eigenvalue θ is denoted by $m(\theta)$. Note that if Γ is connected and regular with valency k, then $\theta_0 = k$ and $m(\theta_0) = 1$. Let $\{v_1, \ldots, v_{m(\theta)}\}$ be an orthonormal basis of eigenvectors with eigenvalue θ, and let U be a matrix whose columns are these vectors. Then the matrix $E_\theta = UU^\top$ is called a *minimal idempotent* associated to θ. We abbreviate E_{θ_i} by E_i $(i = 0, 1, \ldots, d)$.

Fiol and Garriga [22] introduced t-walk-regular graphs as a generalization of both distance-regular and walk-regular graphs. A graph is *t-walk-regular* if the number of walks of every given length ℓ between two vertices $x, y \in V$ only depends on the distance between them, provided that $\mathrm{dist}(x, y) \le t$ (where it is implicitly assumed that the diameter of the graph is at least t). The 'Spectral Decomposition Theorem' leads immediately to

$$A^\ell = \sum_{i=0}^{d} \theta_i^\ell E_i.$$

From that, we obtain that a graph is t-walk-regular if and only if for every minimal idempotent the (x, y)-entry only depends on $\operatorname{dist}(x, y)$, provided that the latter is at most t (see Dalfó, Fiol, and Garriga [11]). In other words, for a fixed eigenvalue θ with minimal idempotent E, there exist constants $\alpha_j := \alpha_j(\theta)$ $(0 \le j \le t)$, such that $A_j \circ E = \alpha_j A_j$, where $\circ$ is the entrywise product.

Given a vertex x in a graph Γ and a vertex y at distance j from x, we consider the numbers $a_j(x, y) = |\Gamma(y) \cap \Gamma_j(x)|$, $b_j(x, y) = |\Gamma(y) \cap \Gamma_{j+1}(x)|$, and $c_j(x, y) = |\Gamma(y) \cap \Gamma_{j-1}(x)|$. A graph Γ with diameter D is *distance-regular* if these parameters do not depend on x and y, but only on j, for $0 \le j \le D$. If this is the case then these numbers are denoted simply by a_j, b_j, and c_j, for $0 \le j \le D$, and they are called the *intersection numbers* of Γ. Also, if a graph Γ is t-walk-regular, then the intersection numbers are well-defined for $0 \le j \le t$, as they do not depend neither on x nor on the chosen $y \in \Gamma_j(x)$ (see Dalfó et al. [15, Prop. 3.15]). More generally, let x and y be two vertices at distance h in a t-walk-regular graph. Then the numbers $p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$ exist (i.e., they only depend on h, i and j) for nonnegative integers $h, i, j \le t$. This follows from working out the product $A_i A_j \circ A_h$, for example; see also Dalfó, Fiol, and Garriga [13, Prop. 1]. Moreover, if $k_h = |\Gamma_h(x)|$, then relations such as $k_h p^h_{ij} = k_i p^i_{hj}$ hold. From the above it is clear that a D-walk-regular graph is distance-regular.

Let E denote the idempotent associated to an eigenvalue θ of a t-walk-regular graph with adjacency matrix A. By looking at an (x, y)-entry with $\operatorname{dist}(x, y) = j$ in the equation $AE = \theta E$ we obtain the following relations:

$$k\alpha_1 = \theta\alpha_0 \tag{1}$$

$$c_j\alpha_{j-1} + a_j\alpha_j + b_j\alpha_{j+1} = \theta\alpha_j \qquad (1 \le j \le t-1) \tag{2}$$

Let θ be an eigenvalue of Γ and recall the matrix U whose columns form an orthonormal basis of the eigenspace of θ. For every vertex $x \in V$ we denote by $\hat{x}$ the x-th row of U. From $AU = \theta U$, it follows that

$$\theta\hat{x} = \sum_{y \sim x} \hat{y}. \tag{3}$$

The map $x \mapsto \hat{x}$ is called a representation (associated to θ) of Γ. Note that if we assume t-walk-regularity with $t \ge 0$, then the vectors $\hat{x}$ $(x \in V)$ all have the same length (the square of which is $E_{xx} = \alpha_0$); in this case we call the representation spherical. Given two vertices $x, y \in V$, we will often refer to $u_{xy} := E_{xy}/\alpha_0$ as the xy-cosine of the eigenvalue θ, as it can be interpreted as the cosine of the angle between the vectors $\hat{x}$ and $\hat{y}$. We remark that if Γ is t-walk-regular and $\operatorname{dist}(x, y) = s \le t$, then $u_{xy} = \alpha_s/\alpha_0$ does not depend on x and y, but only on s. In this case, we write $u_s := \alpha_s/\alpha_0$. For a distance-regular graph Γ, the sequence $(u_0, u_1, \ldots, u_D)$ is known as the *standard sequence* of Γ with respect to θ.

A graph Γ is called *bipartite* if it has no odd cycle. For a connected graph Γ, the *bipartite double* $\widetilde{\Gamma}$ of Γ is the graph whose vertices are the symbols x^+, x^- $(x \in V)$ and where x^+ is adjacent to y^- if and only of x is adjacent to y in Γ.

Let $\overline{\Gamma}$ be a graph with vertex set $V(\overline{\Gamma})$. Let Γ be a graph whose vertices are partitioned in $|V(\overline{\Gamma})|$ classes of the same size. We say that Γ is a *cover* of $\overline{\Gamma}$ if the following three properties hold: The vertices of each class induce an empty graph in Γ; the classes give an equitable partition in Γ (that is, for every two classes, every vertex in one of these classes has the same number of neighbors in the other class); and the quotient graph provided by the classes (that is, the graph on the classes, where two classes are adjacent if there are edges (of Γ) between them) is isomorphic to $\overline{\Gamma}$. This quotient graph is also called the folded graph of Γ.

Given a graph Γ and $x \in V$, the *local graph* $\Delta(x)$ *at vertex* x is the subgraph of Γ induced on the vertices that are adjacent to x. When all the local graphs are isomorphic, we simply write Δ, and say that Γ is locally Δ.

3. Construction methods

Highly symmetric examples of t-walk-regular graphs exist for $t \leq 7$ in the form of t-arc-transitive graphs. For example, the infinite family of 3-arc-transitive graphs constructed by Devillers, Giudici, Li, and Praeger [19] is also an infinite family of 3-walk-regular graphs. Indeed, every t-arc-transitive graph with diameter at least t is t-walk-regular. By a covering construction due to Conway (see [2, Ch. 19]) and independently Djoković [21], infinite families of 5-arc-transitive graphs with valency 3 and 7-arc-transitive graphs with valency 4 were constructed. Conder and Walker [9] also constructed infinitely many 7-arc-transitive graphs with valency 4. In turn, these give rise to infinite families of cubic 5-walk-regular graphs and 7-walk-regular graphs with valency 4. The validity of the Bannai-Ito conjecture [1] (in particular the fact that there are finitely many distance-regular graphs with valency four [6]) for example implies that there are infinitely many 7-walk-regular graphs that are not distance-regular.

It is worth mentioning that less-known (and less restrictive) concepts such as t-geodesic-transitivity and t-distance-transitivity have been introduced by Devillers, Jin, Li, and Praeger [20], and both concepts are stronger than t-walk-regularity.

It is rather straightforward to show that the bipartite double of a t-arc-transitive graph is again t-arc-transitive. This could for example be applied to the infinite family of non-bipartite 2-arc-transitive graphs constructed by Nochefranca [32], to obtain also an infinite family of bipartite such graphs. For t-walk-regular graphs, a similar result holds, but we have to take into account the odd-girth (note that for t-arc-transitive graphs with diameter at least t, the odd-girth is at least $2t + 1$).

Proposition 3.1. *Let* Γ *be a* t*-walk-regular graph with odd-girth* $2s + 1$*. Then the bipartite double of* Γ *is* $\min\{s, t\}$*-walk-regular.*

The graph on the flags of the 11-point biplane as described by Dalfó, Van Dam, Fiol, Garriga, and Gorissen [15] and characterized by Blokhuis and Brouwer [3] is 3-walk-regular with odd-girth 5, so its bipartite double is 2-walk-regular (and it is not 3-walk-regular). Proposition 3.1 also shows that the bipartite double of the dodecahedral graph is 2-walk-regular because the dodecahedral graph is 5-walk-regular with odd-girth 5. This bipartite double is even 3-walk-regular, however.

Proposition 3.2. *Let* $t \geq 2$*, and let* Γ *be a* t*-walk-regular graph with valency* k *and odd-girth* $2s + 1$*. If* Γ *is not complete multipartite, then the distance-*2 *graph* Γ_2 *of* Γ *is* $\min\{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor\}$*-walk-regular.*

For example, the distance-2 graph of the dodecahedral graph is 1-walk-regular but not 2-walk-regular. Another example comes from the Biggs-Smith graph, whose distance-2 graph is 2-walk-regular.

We remark that the halved graphs of a bipartite graph are degenerate cases of the distance-2 graph. We thus obtain the following.

Corollary 3.3. *Let $t \geq 2$, and let Γ be a t-walk-regular bipartite graph. Then the halved graphs of Γ are $\lfloor t/2 \rfloor$-walk-regular.*

Using that the minimal idempotents of the line graph of a regular graph are easily deduced from the minimal idempotents of the graph, we obtain the following.

Proposition 3.4. *Let $t \geq 0$. Let Γ be a $(t+1)$-walk-regular graph with valency k and girth larger than $2t+1$. Then the line graph of Γ is t-walk-regular.*

An example is the already mentioned graph on the flags of the 11-point biplane. Since this graph has girth 5 and it is 3-walk-regular (and therefore 2-walk-regular), its line graph is 1-walk-regular (and not 2-walk-regular). This shows that the condition on the girth is necessary. Also the line graphs of s-arc-transitive graphs (with large girth) provide new examples of t-walk-regular graphs. Note by the way that the line graph of a $(t+1)$-arc-transitive graph with valency at least 3 is not t-arc-transitive (for $t \geq 2$), since it has triangles.

We will finish this section with a straightforward construction method for 1-walk-regular graphs. Let us first recall the coclique extension of a graph Γ, that is, the graph with adjacency matrix $A \otimes J$, where A is the adjacency matrix of Γ, J is a square all-ones matrix and $\otimes$ stands for the Kronecker product. It is fairly easy to see (combinatorially) that if Γ is a 1-walk-regular graph, then also every coclique extension of Γ is 1-walk-regular. A variation on the coclique extension is the Kronecker product $\Gamma \otimes \Gamma'$ of two graphs Γ and Γ', that is, the graph with adjacency matrix $A \otimes B$, where A and B are the adjacency matrices of Γ and Γ'.

Proposition 3.5. *Let Γ and Γ' be two 1-walk-regular graphs. Then the Kronecker product $\Gamma \otimes \Gamma'$ is 1-walk-regular.*

Finally, we remark that the sum [10] $\Gamma \oplus \Gamma'$ — also called Cartesian product — of two 1-walk-regular graphs Γ and Γ', that is, the graph with adjacency matrix $A \otimes I + I \otimes B$, is in general not 1-walk-regular. However, the particular case $\Gamma \oplus \Gamma$ is again 1-walk-regular, as one can easily show (the idempotents are $E_i \otimes E_j + E_j \otimes E_i$ $(i \neq j)$ and $E_i \otimes E_i$).

4. Godsil's multiplicity bound

Let $m \geq 2$ and let Γ be a connected regular graph with an eigenvalue $\theta \neq \pm k$ with multiplicity m. Godsil [23] proved that if such a graph is distance-regular and not complete multipartite, then both its diameter and its valency are bounded by a function of m. In particular, this assures that there are finitely many such distance-regular graphs. In this section we extend some of Godsil's results and reasonings to the class of 2-walk-regular graphs. The main difference with distance-regular graphs is that we are not able to bound the diameter.

We start by pointing out that, as it happens with distance-regular graphs, the images of two vertices at distance at most 2 under a representation associated to $\theta \neq \pm k$ cannot be collinear. The following lemma can indeed be read between the lines in a proof by Godsil [23].

Lemma 4.1. *Let Γ be a 2-walk-regular graph different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq \pm k$. Let x and y be vertices of Γ and consider a representation associated to θ. If $\hat{x} = \pm \hat{y}$, then* $\mathrm{dist}(x, y) > 2$.

An immediate corollary is the following.

Corollary 4.2. *Let Γ be a 2-walk-regular graph different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq k$, and consider the associated representation. If $u_2 = \pm 1$, then $\theta = -k$ and Γ is bipartite.*

Let $\theta \neq \pm k$ be an eigenvalue with multiplicity m of a 2-walk-regular graph Γ (with valency k), and consider the associated (spherical) representation. Let x be a vertex of Γ and consider the set of vectors $\{\hat{y} \mid y \in \Gamma(x)\}$. These vectors lie in the hyperplane of all vectors having inner product α_1 with $\hat{x}$, so they lie in an $(m-1)$-dimensional sphere (in $\mathbb{R}^m$). Lemma 4.1 ensures that the cardinality of the set is k. Also, the inner product between two of its elements is either α_1 or α_2, so it is a (spherical) 2-distance set. As pointed out by Godsil [23, Lemma 4.1], Delsarte, Goethals, and Seidel [18, Ex. 4.10] provide a bound for the size of such a set, and we have the following:

Theorem 4.3. *(cf. [23, Thm. 1.1]) Let Γ be a 2-walk-regular graph, not complete multipartite, with valency $k \geq 3$. Assume that Γ has an eigenvalue $\theta \neq \pm k$ with multiplicity m. Then $k \leq \frac{(m+2)(m-1)}{2}$.*

This bound will be key in Section 7, as well as for the study of 2-walk-regular graphs with an eigenvalue with multiplicity 3 in Section 6.3. In both cases we will also use properties of the local graph of 2-walk-regular graphs; we will study these in the next section. Note that also some of the results in Terwilliger's 'tree bound' paper [35] on t-arc-transitive graphs and in Hiraki and Koolen's paper [27] with improvements of Godsil's bound can be generalized to t-walk-regular graphs with large enough girth.

5. The local structure of 2-walk-regular graphs

In [36] Terwilliger gave bounds for the eigenvalues of the local graphs of a distance-regular graph. We start this section showing that these bounds also hold for 2-walk-regular graphs. We follow the proof as given by Godsil [24, Ch. 13].

Proposition 5.1. *(cf. [5, Thm. 4.4.3] and [24, Cor. 4.3, p. 269]) Let Γ be a 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_d$. Let x be a vertex of Γ and let Δ be the subgraph of Γ induced on the neighbors of x. Let $a_1 = \eta_0 \geq \eta_1 \geq \ldots \geq \eta_{k-1}$ be the eigenvalues of Δ. Then*

$$\eta_{k-1} \geq -1 - \frac{b_1}{\theta_1 + 1},$$
$$\eta_1 \leq -1 - \frac{b_1}{\theta_d + 1}.$$

We remark that the 2-coclique extensions of the lattice graphs $L_2(n)$ provide examples of 1-walk-regular graphs for which the upper bound for the eigenvalues of the local graphs in the above proposition is not valid. In this case $\eta_1 = a_1 = 2n-4$ (the local graph consists of 2 cocktailparty graphs), $b_1 = 2n-1$, and $\theta_d = -4$.

In what follows the symbol $\delta_{x,y}$ stands for the Kronecker delta, that is, $\delta_{x,y} = 1$ if $x = y$ and 0 otherwise.

Proposition 5.2. *Let Γ be a 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_d$ and local graph Δ. Let $\theta \neq k$ be an eigenvalue of Γ with* *multiplicity m. If $m < k$, then $\theta \in \{\theta_1, \theta_d\}$ and $b := -1 - \frac{b_1}{\theta+1}$ is an eigenvalue of Δ with multiplicity at least $k - m + \delta_{b,a_1}$.*

In the next part we are going to derive the 'fundamental bound' for 2-walk-regular graphs. This bound was obtained for distance-regular graphs by Jurišić, Koolen, and Terwilliger [29]. We start with the following lemma.

Lemma 5.3. [28, Thm. 2.1] *Let Δ be a regular graph with valency k and n vertices. Let $k = \eta_0 \geq \eta_1 \geq \ldots \geq \eta_{n-1}$ be the eigenvalues of Δ. Let σ and τ be numbers such that $\sigma \geq \eta_1 \geq \eta_{n-1} \geq \tau$. Then $n(k + \sigma\tau) \leq (k - \sigma)(k - \tau)$, with equality if and only if $\eta_i \in \{\sigma, \tau\}$ $(1 \leq i \leq n-1)$. In particular, if equality holds then Δ is empty, complete, or strongly regular.*

As a consequence of Proposition 5.1 and Lemma 5.3 we obtain the following 'fundamental bound'.

Theorem 5.4. *(cf. [29, Thm. 6.2] and [28, Thm. 2.1]) Let Γ be a 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_d$. Then*

$$(\theta_1 + \frac{k}{a_1+1})(\theta_d + \frac{k}{a_1+1}) \geq -\frac{k a_1 b_1}{(a_1+1)^2}.$$

If $a_1 \neq 0$, then equality holds if and only if every local graph Δ is strongly regular with eigenvalues a_1, $-1 - \frac{b_1}{\theta_d+1}$, and $-1 - \frac{b_1}{\theta_1+1}$. If $a_1 = 0$, then equality holds if and only if Γ is bipartite.

6. Small multiplicity

This section is devoted to study t-walk-regular graphs having eigenvalues with small multiplicity. We start by answering the following question: How small can the multiplicity of an eigenvalue be of a t-walk-regular graph that is not distance-regular? Afterwards, in Sections 6.2 and 6.3, we will use this answer and the results in the previous sections to describe 1- and 2-walk-regular graphs having an eigenvalue (with absolute value smaller than the spectral radius) with small multiplicity.

6.1. Distance-regularity from a small multiplicity.

Dalfó, Van Dam, Fiol, Garriga and Gorissen [15] posed the following problem: What is the smallest t such that every t-walk-regular graph is distance-regular? More precisely, they considered t as a function of either the diameter D of Γ or the number $d + 1$ of distinct eigenvalues. We will give an answer to this question, but in terms of the minimum multiplicity of an eigenvalue $\theta \neq \pm k$ of Γ (where k is the valency). Notice that this minimum multiplicity is related to d and the number of vertices. The following result follows from revisiting the proof of a result by Godsil [23, Thm. 3.2].

Proposition 6.1. *Let $t \geq 2$ and let Γ be a t-walk-regular graph with valency $k \geq 3$ and diameter $D > t$. If Γ has an eigenvalue $\theta \neq \pm k$ with multiplicity at most t, then $b_t = 1$.*

Proposition 6.2. *Let Γ be a t-walk-regular graph. If $b_t = 1$, then Γ is distance-regular.*

The following result now follows immediately.

Theorem 6.3. *Let Γ be a t-walk-regular graph with an eigenvalue $\theta \neq \pm k$ with multiplicity at most t. If $t \geq 2$, then Γ is distance-regular.*

Note that we can extend this result with $t = 1$, as we will show next that 1-walk-regular graphs with an eigenvalue $\theta \neq \pm k$ with multiplicity 1 do not exist.

6.2. **1-Walk-regular graphs with a small multiplicity.** Let Γ be a 1-walk-regular graph, and suppose that it has an eigenvalue θ with multiplicity 1. Let x and y be two adjacent vertices. Since the matrix E_θ has rank 1, by considering the determinant of the 2×2 principal submatrix of E_θ on x and y, it follows that $\alpha_1 = \pm\alpha_0$, and hence by (1) we obtain that $\theta = \pm k$. In other words, a 1-walk-regular graph has no eigenvalues different from $\pm k$ with multiplicity 1. In the following proposition we consider 1-walk-regular graphs with an eigenvalue with multiplicity 2.

Proposition 6.4. *Let Γ be a 1-walk-regular graph with an eigenvalue with multiplicity 2. Then Γ is a cover of a cycle.*

Every coclique extension of a cycle is 1-walk-regular (see Section 3), and it has eigenvalues with multiplicity 2, except for coclique extensions of the 4-cycle (which are complete bipartite graphs). But this certainly does not cover all the possibilities.

Indeed, let Γ be any 1-walk-regular graph (for example, a strongly regular graph) and let Γ' be any cycle, except the 4-cycle. Then by applying Proposition 3.5 one obtains a 1-walk-regular graph, which typically has an eigenvalue with multiplicity 2. Indeed, if k is the valency of Γ and $\theta \neq 0$ is an eigenvalue of Γ' with multiplicity 2, then the product $k\theta$ is a good candidate eigenvalue with multiplicity 2 of $\Gamma \otimes \Gamma'$; sometimes however this eigenvalue coincides with other (product) eigenvalues. The latter clearly happens when Γ' is the 4-cycle, because its only eigenvalue with multiplicity 2 is $\theta = 0$.

We end this section by observing that the smallest multiplicity of an eigenvalue different from $\pm k$ in a 1-walk-regular graph provides a bound for its clique number.

Proposition 6.5. *Let Γ be a 1-walk-regular graph with valency k. Let $\theta \neq k$ be an eigenvalue of Γ with multiplicity m. Then every clique of Γ has at most $m + 1$ vertices.*

The coclique extensions of the triangle satisfy the bound with equality (with $m = 2$), for example.

6.3. **2-Walk-regular graphs with a small multiplicity.** Let $\theta \neq k$ be an eigenvalue of a 2-walk-regular graph Γ with valency k. Recall that θ, as proven in Section 6.2, cannot have multiplicity one. If θ has multiplicity 2, then by Theorem 6.3 we know that Γ is distance-regular, and the only distance-regular graphs with an eigenvalue with multiplicity 2 are the polygons and the regular complete tripartite graphs (cf. [5, Prop. 4.4.8]). In Theorem 6.7, we will discuss the case of multiplicity 3. For that we use the following lemma, which is interesting on its own.

Lemma 6.6. *(cf.* [35, Thm. 1]*) Let Γ be a 2-walk-regular graph with valency k. Let $\theta \neq \pm k$ be an eigenvalue of Γ with multiplicity m. If $m < k$, then the intersection number a_1 is positive.*

Theorem 6.7. *Let Γ be a 2-walk-regular graph different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq \pm k$ with multiplicity 3. Then Γ is a cubic graph with $a_1 = a_2 = 0$ or a distance-regular graph. Moreover, if Γ is distance-regular, then Γ is the cube, the dodecahedron, or the icosahedron.*

Notice that the complete multipartite graph $K_{(m+1)\times\omega}$ has eigenvalue $-\omega$ with multiplicity m, and hence the complete multipartite graph $K_{4\times\omega}$ has an eigenvalue with multiplicity 3. The only complete multipartite graph having eigenvalue 0 with multiplicity 3 is the earlier mentioned $K_{3\times 2}$. Examples of other 2-walk-regular graphs (not being distance-regular) with an eigenvalue with multiplicity 3 can be found in the Foster census of symmetric cubic graphs [34], such as the graphs $F056A$, $F104A$, $F112C$, as well as the generalized Petersen graphs $G(12,5)$ and $G(24,5)$, which correspond to graphs $F24A$ and $F48A$, respectively. It would be interesting to know whether there are infinitely many 2-walk-regular graphs with a multiplicity 3.

7. The Delsarte bound and geometric graphs

In this section we start by observing that the Delsarte bound [17] for the size of a clique also holds for 1-walk-regular graphs. We will in fact prove a somewhat stronger statement and study the cases when equality is attained. After that, we will focus our attention on the highly related notion of geometric graphs. We will show that there are finitely many non-geometric 2-walk-regular graphs with bounded smallest eigenvalue and fixed diameter.

7.1. The Delsarte bound.

Proposition 7.1. *Let Γ be a connected k-regular graph with an eigenvalue $\theta < 0$ and corresponding minimal idempotent E satisfying $E \circ I = \alpha_0 I$ and $E \circ A = \alpha_1 A$. If C is a clique in Γ with characteristic vector χ, then $|C| \leq 1 - \frac{k}{\theta}$, with equality if and only if $E\chi = 0$.*

We call a clique with size attaining this bound a *Delsarte clique.* Note that if the multiplicity of θ equals $|C| - 1$, that is, the bound of Proposition 6.5 is tight, then C is a Delsarte clique. Clearly, Proposition 7.1 applies to 1-walk-regular graphs, so that we obtain the following Delsarte bound.

Theorem 7.2. *Let Γ be a 1-walk-regular graph with valency k and smallest eigenvalue θ_d. Then every clique of Γ has at most $1 - \frac{k}{\theta_d}$ vertices.*

We remark that if the graph is 1-walk-regular, then equality in Proposition 7.1 can only occur for $\theta = \theta_d$. Line graphs of regular graphs with valency at least 3 constitute a class of graphs for which the bound is satisfied with equality. However, the minimal idempotent corresponding to its smallest eigenvalue does not necessarily satisfy the conditions of Proposition 7.1. On the other hand, the Cartesian product $K_m \oplus K_n \oplus K_p$ of three complete graphs (a generalized Hamming graph) is 0-walk-regular with maximal cliques of size m, n, and p, while the Delsarte 'bound' equals $(m+n+p)/3$, so for particular values of m, n, and p, it has maximal cliques of size attaining the Delsarte bound, but also larger cliques. A final remark is that the same approach works for bounding the maximum number of vertices mutually at distance t in a t-walk-regular graph.

In a distance-regular graph with diameter D, a Delsarte clique C has covering radius (that is, the maximum distance of a vertex to the clique) equal to $D-1$ (note that in every connected graph with diameter D, the covering radius of a clique is either $D-1$ or D). Moreover, C is *completely regular* in the sense that every vertex at distance i from C is at distance i from the same number of vertices ϕ_i of C (and hence it is at distance j from the same number of vertices $\phi_{i,j}$ of C for every j), for $i = 0, 1, \ldots, D-1$. We can generalize this as follows.

Proposition 7.3. *Let Γ be a connected k-regular graph with $d+1$ distinct eigenvalues, and let C be a Delsarte clique. Then the covering radius of C is at most $d-1$. Moreover, if Γ is t-walk-regular, then every vertex at distance i from C is at distance i from the same number of vertices ϕ_i of C, for $i = 0, 1, \ldots, t-1$.*

7.2. **Geometric graphs.** A graph is *geometric* if there exist a set of Delsarte cliques such that every edge lies on exactly one of them. The notion of geometric graph in this sense was introduced by Godsil [25] for distance-regular graphs. Examples of geometric graphs are bipartite graphs (trivially) and line graphs of a regular graphs with valency at least 3.

Koolen and Bang [30] proved that there are only finitely many non-geometric distance-regular graphs with smallest eigenvalue at least $-\omega$ and diameter at least 3. It is also possible to state a similar result for 2-walk-regular graphs. More precisely, Koolen and Bang [30, Thm. 3.3] showed that there are finitely many distance-regular graphs with smallest eigenvalue $-\omega$, diameter $D \geq 3$, and small c_2 (compared with a_1). In order to prove this, they bound the valency k using Godsil's multiplicity bound (the analogue of Theorem 4.3), using the multiplicity of the second largest eigenvalue θ_1. In turn, a bound on $m(\theta_1)$ is derived from the analogue of Proposition 5.2, after showing that $m(\theta_1) < k$. One of the key points for the latter inequality is to give an upper bound for the number of vertices in Γ. Their argument, however, does not apply to 2-walk-regular graphs. The following lemma intents to solve this problem.

Lemma 7.4. *Let $\omega \geq 2$ be an integer. Let Γ be a 2-walk-regular graph with valency k, diameter D, and smallest eigenvalue at least $-\omega$. If ϵ is such that $0 < \epsilon < 1$ and $c_2 \geq a_1\epsilon$, then $|V| < \left(\frac{2\omega^2}{\epsilon}\right)^D Dk$.*

As a consequence of this lemma, the proof by Koolen and Bang [30] also applies to 2-walk-regular graphs, so we have the following result.

Theorem 7.5. *(cf.* [30, Thm. 3.3]*) Let $0 < \epsilon < 1$, and let $\omega \geq 2$ and $D \geq 3$ be integers. Let Γ be a 2-walk-regular graph with valency k, diameter D, smallest eigenvalue at least $-\omega$, and with $c_2 \geq a_1\epsilon$. Then $k < D^2 \left(\frac{2\omega^2}{\epsilon}\right)^{2D+4}$. In particular, there are finitely many such graphs.*

Next is to show, as it happens with distance-regular graphs (see Koolen and Bang [30, Thm. 5.3]), that if a_1 is large enough (compared to c_2), then a 2-walk-regular graph with smallest eigenvalue at least $-\omega$ is geometric. The next result by Metsch [31] is a key point for that purpose.

Proposition 7.6. [31, Result 2.1] *Let $k \geq 2$, $\mu \geq 1$, $\lambda \geq 0$, and $s \geq 1$. Suppose that Γ is a regular graph with valency k such that every two non-adjacent vertices have at most μ common neighbors, and every two adjacent vertices have exactly λ common neighbors. Define a line as a maximal clique in Γ with at least $\lambda + 2 - (s-1)(\mu-1)$ vertices. If $\lambda > (2s-1)(\mu-1) - 1$ and $k < (s+1)(\lambda+1) - s(s+1)(\mu-1)/2$, then every vertex is in at most s lines, and each edge lies in a unique line.*

Proposition 7.7. *Let $\omega \geq 2$ be an integer and let Γ be a 2-walk-regular graph with valency k, diameter $D \geq 2$, and smallest eigenvalue in the interval $[-\omega, 1-\omega)$. If $a_1 > \omega^4 c_2$, then Γ is geometric.*

As a consequence of Theorem 7.5 and Proposition 7.7, we have the following result.

Theorem 7.8. *Let $\omega \geq 2$ and $D \geq 3$. There are finitely many non-geometric 2-walk-regular graphs with diameter D and smallest eigenvalue at least $-\omega$.*

Let us remark that we need to fix both ω and D for the finiteness. Conder and Nedela [8, Prop. 2.5] constructed infinitely many 3-arc-transitive cubic graphs with girth 11. Because a geometric graph without triangles must be bipartite, this shows that there are infinitely many non-geometric 3-walk-regular graphs with smallest eigenvalue larger than -3. To show that we need to fix ω, we consider the symmetric bilinear forms graph. This graph has as vertices the symmetric $n \times n$ matrices over $\mathbb{F}_q$, where two vertices are adjacent if their difference has rank 1; see [5, Sec. 9.5.D]. For q even and $n \geq 4$, this graph is not distance-regular, but it is 2-walk-regular. For $n = 4$, these graphs have diameter 5, and one can show using the distance-distribution diagram (see [4, p. 22]) that the smallest eigenvalue equals $-1 - q^3$. Because the valency equals $q^4 - 1$, this graph cannot be geometric, even though there are 'lines' of size q, but these are not Delsarte cliques.

On the other hand, we need 2-walk-regularity, because the earlier mentioned 2-coclique extensions of the lattice graphs provide an infinite family of non-geometric 1-walk-regular graphs with diameter 2 and smallest eigenvalue -4. Theorem 7.8 thus illustrates once more the important structural gap between 1- and 2-walk-regular graphs.

Note finally that a geometric graph Γ is the point graph of the partial linear space of vertices and (some) Delsarte cliques, and that one can consider also the dual graph on the cliques, that is, the point graph of the dual of this partial linear space. In particular when Γ is locally a disjoint union of cliques (i.e., when $k = -\theta_d(a_1+1)$), this can be used to obtain new examples of t-walk-regular graphs, in the same spirit as in Proposition 3.4, although now one has to consider the so-called geometric girth instead of the usual girth. For example, the Hamming graphs have geometric girth 4 (as $c_2 > 1$), and the dual graphs of the Hamming graph (with diameter at least three) are only 1-walk-regular. The distance-regular near octagon coming from the Hall-Janko group (see [5, Sec. 13.6]) has geometric girth 6 and its dual is 2-walk-regular.

We finish by observing that besides distance-regular graphs and the above mentioned symmetric bilinear forms graphs, we do not know of many examples of 2-walk-regular graphs with $c_2 \geq 2$. We challenge the reader to construct more such examples.

References

[1] S. Bang, A. Dubickas, J. H. Koolen, and V. Moulton. There are only finitely many distance-regular graphs of fixed valency greater than two. *ArXiv e-prints*, Sept. 2009.

[2] N. Biggs. *Algebraic graph theory.* Cambridge University Press, 1974.

[3] A. Blokhuis and A. E. Brouwer. Spectral characterization of a graph on the flags of the eleven point biplane. *Des. Codes Cryptography*, 65(1-2):65–69, 2012.

[4] A. E. Brouwer. Corrections and additions to the book 'Distance-regular Graphs'. `http://www.win.tue.nl/~aeb/drg/BCN-ac.ps.gz`. March 2013.

[5] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-regular graphs.* Berlin etc.: Springer-Verlag, 1989.

[6] A. E. Brouwer and J. H. Koolen. The distance-regular graphs of valency four. *J. Algebr. Comb.*, 10(1):5–24, 1999.

[7] M. Cámara, E. R. van Dam, J. H. Koolen, and J. Park. Geometric aspects of 2-walk-regular graphs. Preprint, March 2013.
[8] M. Conder and R. Nedela. Symmetric cubic graphs of small girth. *J. Combin. Theory, Ser. B*, 97(5):757–768, 2007.
[9] M. D. E. Conder and C. G. Walker. The infinitude of 7-arc-transitive graphs. *J. Algebra*, 208(2):619–629, 1998.
[10] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of graphs. Theory and applications. 3rd edition.* Leipzig: J. A. Barth Verlag, 1995.
[11] C. Dalfó, M. A. Fiol, and E. Garriga. On k-walk-regular graphs. *Electron. J. Combin*, 16(1):R47, 2009.
[12] C. Dalfó, M. A. Fiol, and E. Garriga. On t-cliques in k-walk-regular graphs. *Electron. Notes Discrete Math.*, 34:579–584, 2009.
[13] C. Dalfó, M. A. Fiol, and E. Garriga. Characterizing (ℓ, m)-walk-regular graphs. *Linear Algebra Appl.*, 433(11-12):1821–1826, 2010.
[14] C. Dalfó, E. R. van Dam, and M. A. Fiol. On perturbations of almost distance-regular graphs. *Linear Algebra Appl.*, 435(10):2626–2638, 2011.
[15] C. Dalfó, E. R. van Dam, M. A. Fiol, E. Garriga, and B. L. Gorissen. On almost distance-regular graphs. *J. Combin. Theory, Ser. A*, 118(3):1094–1113, 2011.
[16] E. R. van Dam, J. H. Koolen, and H. Tanaka. Distance-regular graphs. *manuscript*, 2013.
[17] P. Delsarte. *An algebraic approach to the association schemes of coding theory.* Philips Research Reports. Supplements 10, 1973.
[18] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. *Geom. Dedicata*, 6:363–388, 1977.
[19] A. Devillers, M. Giudici, C. H. Li, and C. E. Praeger. An infinite family of biquasiprimitive 2-arc transitive cubic graphs. *J. Algebr. Comb.*, 35(2):173–192, 2012.
[20] A. Devillers, W. Jin, C. H. Li, and C. E. Praeger. On distance, geodesic and arc transitivity of graphs. *ArXiv e-prints*, Oct. 2011.
[21] D. Ž. Djoković. Automorphisms of graphs and coverings. *J. Combin. Theory Ser. B*, 16:243–247, 1974.
[22] M. A. Fiol and E. Garriga. Spectral and geometric properties of k-walk-regular graphs. *Electron. Notes Discrete Math.*, 29:333–337, 2007.
[23] C. D. Godsil. Bounding the diameter of distance-regular graphs. *Combinatorica*, 8(4):333–343, 1988.
[24] C. D. Godsil. *Algebraic combinatorics.* Chapman and Hall, 1993.
[25] C. D. Godsil. Geometric distance-regular covers. *N. Z. J. Math.*, 22(2):31–38, 1993.
[26] C. D. Godsil and B. D. McKay. Feasibility conditions for the existence of walk-regular graphs. *Linear Algebra Appl.*, 30:51–61, 1980.
[27] A. Hiraki and J. Koolen. An improvement of the Godsil bound. *Annals of Combinatorics*, 6:33–44, 2002.
[28] A. Jurišić and J. Koolen. Nonexistence of some antipodal distance-regular graphs of diameter four. *European J. Combin.*, 21(8):1039–1046, 2000.
[29] A. Jurišić, J. Koolen, and P. Terwilliger. Tight distance-regular graphs. *Journal of Algebraic Combinatorics*, 12:163–197, 2000.
[30] J. H. Koolen and S. Bang. On distance-regular graphs with smallest eigenvalue at least $-m$. *J. Combin. Theory, Ser. B*, 100(6):573–584, 2010.
[31] K. Metsch. On a characterization of bilinear forms graphs. *Europ. J. Combinatorics*, 20(4):293–306, 1999.
[32] L. R. Nochefranca. On an infinite class of non-bipartite and non-Cayley graphs having 2-arc transitive automorphism groups. *Graphs and Combinatorics*, 7:271–275, 1991.
[33] P. Rowlinson. Linear algebra. in: L.W. Beineke, R.J. Wilson (Eds.), Graph Connections, Oxford Lecture Ser. Math. Appl., vol. 5, Oxford Univ. Press, New York (1997), pp. 86–99.
[34] G. Royle, M. Conder, B. McKay, and P. Dobscanyi. Cubic symmetric graphs (the Foster census). `http://units.maths.uwa.edu.au/~gordon/remote/foster/index.html`. March 2013.
[35] P. Terwilliger. Eigenvalue multiplicities of highly symmetric graphs. *Discrete Math.*, 41(3):295–302, 1982.
[36] P. Terwilliger. A new feasibility condition for distance-regular graphs. *Discrete Math.*, 61:311–315, 1986.

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