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UMBRAL MOONSHINE AND LINEAR CODES

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Abstract

We exhibit each of the finite groups of umbral moonshine as a distinguished subgroup of the automorphism group of a distinguished linear code, each code being defined over a different quotient of the ring of integers. These code constructions entail permutation representations which we use to give a description of the multiplier systems of the vector-valued mock modular forms attached to the conjugacy classes of the umbral groups by umbral moonshine.

1 Introduction

In 2010 Eguchi–Ooguri–Tachikawa made a remarkable observation [1] relating the elliptic genus of a $K3$ surface to the largest Mathieu group M_{24} via a decomposition of the former into a linear combination of characters of irreducible representations of the small $N = 4$ superconformal algebra. The elliptic genus is a topological invariant and for any $K3$ surface it is given by the weak Jacobi form

$$Z_{K3}(\tau, z) = 8\left(\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2 + \left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right) \tag{1.1}$$

of weight 0 and index 1. The θ_i here denote Jacobi theta functions (cf. (B.3)). The decomposition into $N = 4$ characters leads to an expression

$$Z_{K3}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\left(24\,\mu(\tau,z) + q^{-1/8}\big(-2 + \sum_{n=1}^{\infty} t_n q^n\big)\right) \tag{1.2}$$

for some $t_n \in \mathbb{Z}$ (cf. [2]) where $\theta_1(\tau, z)$ and $\mu(\tau, z)$ are defined in (B.3-B.4). By inspection, the first five t_n are given by $t_1 = 90$, $t_2 = 462$, $t_3 = 1540$, $t_4 = 4554$, and $t_5 = 11592$; the observation

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of [1] is that each of these t_n is twice the dimension of an irreducible representation of M_{24} (cf. [3]).

Experience with monstrous moonshine [4, 5, 6], for example, leads us to conjecture that every t_n may be interpreted as the dimension of an M_{24}-module $K^{(2)}_{n-1/8}$, and that, assuming such a module structure to be known, we may obtain interesting functions by replacing $t_n = \mathrm{tr}|_{K^{(2)}_{n-1/8}} 1$ with $\mathrm{tr}|_{K^{(2)}_{n-1/8}} g$ for non-identity elements $g \in M_{24}$. If we define $H^{(2)}(\tau)$ by requiring

$$Z_{K3}(\tau, z)\eta(\tau)^3 = \theta_1(\tau, z)^2(a\mu(\tau, z) + H^{(2)}(\tau)), \tag{1.3}$$

then $a = 24$ and

$$H^{(2)}(\tau) = q^{-1/8}\left(-2 + \sum_{n>0} t_n q^n\right) \tag{1.4}$$

is a slight modification of the generating function of the t_n. The inclusion of the term -2 and the factor $q^{-1/8} = e^{-2\pi i\tau/8}$ has the effect of improving the modularity: $H^{(2)}(\tau)$ is a *(weak) mock modular form* for $SL_2(\mathbb{Z})$ with multiplier ϵ^{-3} (cf. (B.2)), weight 1/2, and shadow $24\eta^3$ (cf. (B.1)), meaning that if we define the *completion* $\hat{H}^{(2)}(\tau)$ of the holomorphic function $H^{(2)}(\tau)$ by setting

$$\hat{H}^{(2)}(\tau) = H^{(2)}(\tau) + 24\,(4i)^{-1/2}\int_{-\bar{\tau}}^{\infty}(z+\tau)^{-1/2}\overline{\eta(-\bar{z})^3}\mathrm{d}z, \tag{1.5}$$

then $\hat{H}^{(2)}(\tau)$ transforms as a modular form of weight 1/2 on $SL_2(\mathbb{Z})$ with multiplier system conjugate to that of $\eta(\tau)^3$, so that we have

$$\epsilon(\gamma)^{-3}\hat{H}^{(2)}(\gamma\tau)j(\gamma,\tau)^{1/2} = \hat{H}^{(2)}(\tau)$$

for $\gamma \in SL_2(\mathbb{Z})$ where $j(\gamma, \tau) = (c\tau + d)^{-1}$ when (c, d) is the lower row of γ.

The *McKay–Thompson series* $H^{(2)}_g$ for $g \in M_{24}$ is now defined—assuming knowledge of the M_{24}-module structure on $K^{(2)} = \bigoplus_n K^{(2)}_{n-1/8}$—by setting

$$H^{(2)}_g(\tau) = -2q^{-1/8} + \sum_{n=1}^{\infty}\mathrm{tr}|_{K_{n-1/8}}(g)q^{n-1/8} \tag{1.6}$$

where $q = \mathrm{e}(\tau) = e^{2\pi i\tau}$. Actually the functions $H^{(2)}_g$ are more accessible than M_{24}-module $K^{(2)}$ (for which no concrete construction is yet known) since one only needs to know $\mathrm{tr}|_{K_{n-1/8}}g$ for a few values of n if one assumes the function $H^{(2)}_g$ to have good modular properties. Concrete proposals made in [7, 8, 9, 10] entail the prediction that $H^{(2)}_g$ should be a certain concretely defined (cf. [2]) mock modular form of weight 1/2 for $\Gamma_0(n_g)$ with shadow proportional to ϵ^{-3}

where n_g is the order of $g \in M_{24}$, and the existence of a compatible M_{24}-module $K^{(2)}$ has now been established by Gannon [11].

In [12] it was shown that the observation of Eguchi–Ooguri–Tachikawa belongs to a family of relationships—*umbral moonshine*—between finite groups $G^{(\ell)}$ and vector-valued mock modular forms $H_g^{(\ell)} = (H_{g,1}^{(\ell)}, \dots, H_{g,\ell-1}^{(\ell)})$ for $g \in G^{(\ell)}$, that support the existence of infinite-dimensional bi-graded $G^{(\ell)}$-modules

$$K^{(\ell)} = \bigoplus_{0<r<\ell} \bigoplus_{n} K_{r,n-r^2/4\ell}^{(\ell)}, \tag{1.7}$$

where the $G^{(\ell)}$-module structure on $K^{(\ell)}$ is conjectured to be related to the vector-valued mock modular form $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$ via

$$H_{g,r}^{(\ell)}(\tau) = -2\delta_{r,1} q^{-1/4\ell} + \sum_{\substack{n \in \mathbb{Z} \\ r^2-4n\ell<0}} \operatorname{tr}|_{K_{r,n-r^2/4\ell}^{(\ell)}}(g) q^{n-r^2/4\ell}. \tag{1.8}$$

The cases of umbral moonshine presented in [12] are indexed by the positive integers ℓ such that $\ell - 1$ divides 12. In this note we give constructions of the umbral groups $G^{(\ell)}$ as automorphisms of linear codes over rings $\mathbb{Z}/\ell$, and we show, as an application, how to use the resulting permutation representations to describe the multiplier systems of the umbral McKay–Thompson series $H_g^{(\ell)}$.

Table 1: The groups of umbral moonshine.

ℓ	2	3	4	5	7	13
$G^{(\ell)}$	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$SL_2(3)$	$\mathbb{Z}/4\mathbb{Z}$

It is striking that the codes arising all appear in the article [13] written in connection with the Leech lattice. One consequence is that each of the umbral groups $G^{(\ell)}$ may be regarded as a subgroup of the Conway group Co_0, this being the automorphism group of the Leech lattice. Another consequence is the suggestion that there might be analogous cases of umbral moonshine for the remaining 17 codes (or equivalently, Niemeier root systems) appearing in [13]. Confirmation of this suggestion will be demonstrated in a forthcoming article [14].

2 Codes

In this section we review the notion of linear code over a ring $\mathbb{Z}/m$ and define a distinguished linear code $\mathcal{G}^{(\ell)}$ over $\mathbb{Z}/\ell$ for each ℓ such that $\ell - 1$ divides 12.

A (linear) code over $\mathbb{Z}/m$ of length n is a $\mathbb{Z}/m$-submodule of $(\mathbb{Z}/m)^n$. Let $\{e_i \mid i \in \Omega\}$ denote the standard basis for $(\mathbb{Z}/m)^n$, the index set Ω having cardinality n. We equip $(\mathbb{Z}/m)^n$ with a $\mathbb{Z}/m$-valued $\mathbb{Z}/m$-bilinear form by setting $(C, C') = \sum_{i\in\Omega} c_i c'_i$ in case $C = \sum_{i\in\Omega} c_i e_i$ and $C' = \sum_{i\in\Omega} c'_i e_i$, and given $S \subset (\mathbb{Z}/m)^n$ we define $S^\perp = \{D \in (\mathbb{Z}/m)^n \mid (C, D) = 0, \forall C \in S\}$. We say that a code $\mathcal{C} < (\mathbb{Z}/m)^n$ is *self-orthogonal* in case $\mathcal{C} \subset \mathcal{C}^\perp$ and we say that $\mathcal{C}$ is *self-dual* if it is maximally self-orthogonal, meaning that $\mathcal{C} = \mathcal{C}^\perp$. Given a code $\mathcal{C}$ of length n over $\mathbb{Z}/m$ we define $\mathrm{Aut}(\mathcal{C})$ to be the subgroup of $GL_n(\mathbb{Z}/m)$ that stabilizes the subspace $\mathcal{C} < (\mathbb{Z}/m)^n$, and we define $\mathrm{Aut}^\pm(\mathcal{C})$ to be the subgroup of $\mathrm{Aut}(\mathcal{C})$ consisting of signed coordinate permutations, meaning that

$$\mathrm{Aut}^\pm(\mathcal{C}) = \{\gamma \in GL_n(\mathbb{Z}/m) \mid \gamma(\mathcal{C}) \subset \mathcal{C} \,\mathrm{and}\, \gamma(B) \subset B\} \tag{2.1}$$

where B denotes the set $\{\pm e_i \mid i \in \Omega\}$. Observe that $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Aut}^\pm(\mathcal{C})$ coincide when $m \in \{2, 3, 4\}$, but are generally different otherwise.

We now identify a distinguished linear code $\mathcal{G}^{(\ell)}$ over $\mathbb{Z}/\ell$ for each ℓ such that $\ell - 1$ divides 12. The code $\mathcal{G}^{(\ell)}$ will have length $24/(\ell - 1)$ and the construction we give will be either a rephrasing or direct reproduction of a construction given (much earlier) in [13]. In particular, it will develop that $\mathcal{G}^{(2)}$ is the extended binary Golay code and $\mathcal{G}^{(3)}$ is the extended ternary Golay code. The remaining $\mathcal{G}^{(\ell)}$ are visible, in a certain sense, inside the Leech lattice (cf. [13]) and may be regarded as natural analogues of the extended binary and ternary Golay codes defined over larger quotients of the ring of integers.

To define $\mathcal{G}^{(2)}$ equip $(\mathbb{Z}/2)^{24}$ with the standard basis $\{e_i\}$ and index this basis with the set $\Omega^{(2)} = \{\infty\} \cup \mathbb{Z}/23$. Let N be the subset of $\Omega^{(2)}$ consisting of the elements of $\mathbb{Z}/23$ that are not squares in $\mathbb{Z}/23$, so that $N = \{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22\}$, and define

$$C_i = e_\infty + \sum_{n\in N} e_{n+i} \in (\mathbb{Z}/2)^{24} \tag{2.2}$$

for $i \in \mathbb{Z}/23$. Then the subspace of $(\mathbb{Z}/2)^{24}$ generated by the set $\{C_i \mid i \in \mathbb{Z}/23\}$ is a self-dual linear code over $\mathbb{Z}/2$ of length 24 which we denote $\mathcal{G}^{(2)}$. In fact, $\mathcal{G}^{(2)}$ is a copy of the extended binary Golay code (see various chapters in [15] for more details) and the automorphism group of $\mathcal{G}^{(2)}$ is isomorphic to the largest Mathieu group, M_{24}. The group $G^{(2)}$ defined in [12] is also

isomorphic to M_{24} so we have $\mathrm{Aut}^{\pm}(\mathcal{G}^{(2)}) = \mathrm{Aut}(\mathcal{G}^{(2)}) \simeq G^{(2)}$.

To define $\mathcal{G}^{(3)}$ equip $(\mathbb{Z}/3)^{12}$ with the standard basis $\{e_i\}$ and take the index set to be $\Omega^{(3)} = \{\infty\} \cup \mathbb{Z}/11$. The set of non-squares in $\mathbb{Z}/11$ is $N = \{2,6,7,8,10\}$. Let Q be the complement of N in $\mathbb{Z}/11$, so that $Q = \{0,1,3,4,5,9\}$, and define $C_i \in (\mathbb{Z}/3)^{12}$ for $i \in \mathbb{Z}/11$ by setting

$$C_i = 2e_\infty + \sum_{n\in N} 2e_{n+i} + \sum_{n\in Q} e_{n+i} \in (\mathbb{Z}/3)^{12}. \tag{2.3}$$

Then the code $\mathcal{G}^{(3)}$ generated by the C_i is a copy of the extended ternary Golay code (cf. [15]) and the automorphism group of $\mathcal{G}^{(3)}$ is isomorphic to a group $2.M_{12}$, being the unique (up to isomorphism) non-trivial double cover of the Mathieu group M_{12} (cf. [3]). Again we have $\mathrm{Aut}^{\pm}(\mathcal{G}^{(3)}) = \mathrm{Aut}(\mathcal{G}^{(3)})$ and the group $G^{(3)}$ defined in [12] is also isomorphic to $2.M_{12}$, so $\mathrm{Aut}^{\pm}(\mathcal{G}^{(3)}) \simeq G^{(3)}$.

For $\ell = 4$ equip $(\mathbb{Z}/4)^8$ with the standard basis, indexed by $\Omega_{(4)} = \{\infty\} \cup \mathbb{Z}/7$, let N denote the set $\{3,5,6\}$ of non-squares in $\mathbb{Z}/7$, and define $C_i \in (\mathbb{Z}/4)^8$ for $i \in \mathbb{Z}/7$ by setting

$$C_i = 3e_\infty + 2e_i + \sum_{n\in N} e_{n+i} \in (\mathbb{Z}/4)^8. \tag{2.4}$$

Define $\mathcal{G}^{(4)}$ to be the $\mathbb{Z}/4$-submodule of $(\mathbb{Z}/4)^8$ generated by the set $\{C_i \mid i \in \mathbb{Z}/7\}$. Then $\mathcal{G}^{(4)}$ is a copy of the octacode [16] (see also [17, §3.2]). The automorphism group of $\mathcal{G}^{(4)}$ has a central subgroup of order 2, generated by the symmetry $C = (c_i) \mapsto (-c_i)$, and modulo this central subgroup we obtain the affine general linear group of degree 3 over a field with 2 elements, which is the same as the stabilizer in $GL_4(2)$ of a line in $(\mathbb{Z}/2)^4$. Comparing with the definition of $G^{(4)}$ given in [12] we find that $\mathrm{Aut}^{\pm}(\mathcal{G}^{(4)}) = \mathrm{Aut}(\mathcal{G}^{(4)}) \simeq G^{(4)}$.

Now consider the case that $\ell = 5$. Index the standard basis of $(\mathbb{Z}/5)^6$ with the set $\Omega^{(5)} = \{\infty\} \cup \{0,1,2,3,4\}$ and define

$$C_i = e_\infty + e_{1+i} + 4e_{2+i} + 4e_{3+i} + e_{4+i} \in (\mathbb{Z}/5)^6 \tag{2.5}$$

for $i \in \mathbb{Z}/5$. Then the C_i generate a self-dual code $\mathcal{G}^{(5)} < (\mathbb{Z}/5)^6$. We see that $\mathrm{Aut}^{\pm}(\mathcal{G}^{(5)})$ is a proper subgroup of $\mathrm{Aut}(\mathcal{G}^{(5)})$ since the latter contains the central element $e_i \mapsto 2e_i$, for example, which does not preserve $B = \{\pm e_i\}$. The group $\mathrm{Aut}^{\pm}(\mathcal{G}^{(5)})$ is a double cover of S_5, regarded as a permutation group on 6 points via the isomorphism $S_5 \simeq PGL_2(5)$. The particular double cover arising is perhaps a little unfamiliar in that it does not contain the Schur double cover of A_5 as a subgroup; it can be realized explicitly as the quotient of $GL_2(5)$ by its unique central

subgroup of order 2. (Note that the centre of $GL_2(5)$ is cyclic of order 4.) Upon comparison with [12] we conclude that $\mathrm{Aut}^{\pm}(\mathcal{G}^{(5)}) \simeq G^{(5)}$.

For $\ell = 7$ we take $\Omega^{(7)} = \{\infty\} \cup \mathbb{Z}/3$ as an index set for the standard basis of $(\mathbb{Z}/7)^4$ and we define $C_i \in (\mathbb{Z}/7)^4$ by setting

$$C_i = e_\infty + 2e_i + e_{1+i} + 6e_{2+i} \in (\mathbb{Z}/7)^4 \tag{2.6}$$

for $i \in \mathbb{Z}/3$. We define $\mathcal{G}^{(7)}$ to be the $\mathbb{Z}/7$-submodule generated by the C_i for $i \in \mathbb{Z}/3$ and observe that $\mathcal{G}^{(7)}$ is a self-dual code over $\mathbb{Z}/7$ with $\mathrm{Aut}^{\pm}(\mathcal{G}^{(7)})$ a double cover of $PSL_2(3) \simeq A_4$. In fact the double cover arising is $SL_2(3)$ and we have $\mathrm{Aut}^{\pm}(\mathcal{G}^{(7)}) \simeq G^{(7)}$.

The remaining code is $\mathcal{G}^{(13)}$ which has length $2 = 24/(13-1)$ and which we may take to be generated by $e_\infty + 5e_0 \in (\mathbb{Z}/13)^2$. (In this case we set $\Omega^{(13)} = \{\infty\} \cup \mathbb{Z}/1$.) The code $\mathcal{G}^{(13)}$ is self-dual (since $1^2 + 5^2 \equiv 0 \pmod{13}$) and $\mathrm{Aut}^{\pm}(\mathcal{G}^{(13)})$ is cyclic of order 4, generated explicitly by $(c_\infty, c_0) \mapsto (c_0, -c_\infty)$. Once again we find $\mathrm{Aut}^{\pm}(\mathcal{G}^{(13)}) \simeq G^{(13)}$ and we conclude that $\mathrm{Aut}^{\pm}(\mathcal{G}^{(\ell)}) \simeq G^{(\ell)}$ for all ℓ (such that $\ell - 1$ divides 12).

3 Automorphy

We now take $G^{(\ell)} = \mathrm{Aut}^{\pm}(\mathcal{G}^{(\ell)})$ for $\ell \in \{2, 3, 4, 5, 7, 13\}$ and give an explanation of how these constructions may be used to describe the automorphy of the vector-valued mock modular forms $H_g^{(\ell)}$ attached (in [12]) to the elements of $G^{(\ell)}$ via umbral moonshine.

Observe that $\mathcal{G}^{(\ell)}$ is a code of length $24/(\ell-1)$ over $\mathbb{Z}/\ell$ for each ℓ. Thus we obtain a permutation representation of degree 24 for $G^{(\ell)}$ by considering its action on the set

$$\{ce_i \mid i \in \Omega^{(\ell)},\ c \in \mathbb{Z}/\ell,\ c \neq 0\} \tag{3.1}$$

of non-zero multiplies of the basis vectors e_i. Write $\tilde{\Pi}_g$ for the cycle shape attached to $g \in G^{(\ell)}$ arising from this permutation representation and write $g \mapsto \tilde{\chi}_g$ for the corresponding character of $G^{(\ell)}$. Then, for example, $\tilde{\Pi}_g = 2^{12}$ if g is the central involution in $G^{(\ell)}$ and $\ell \neq 4$. (For each ℓ the group $G^{(\ell)}$ contains the transformation $e_i \mapsto -e_i$, which is the unique central involution of $G^{(\ell)}$ if $\ell > 2$.) In the case that $\ell = 4$ and g is the central involution of $G^{(4)}$ we have $\tilde{\Pi}_g = 1^8 2^8$.

Define a second permutation representation of degree $24/(\ell-1)$ for $G^{(\ell)}$ by considering the action of $G^{(\ell)}$ on the sets $E_i = \{ce_i \mid c \in \mathbb{Z}/\ell,\ c \neq 0\}$ for $i \in \Omega^{(\ell)}$, which constitute a system of imprimitivity for the degree 24 permutation representation of $G^{(\ell)}$ just defined. Write $\bar{\Pi}_g$ for the corresponding cycle shapes and $g \mapsto \bar{\chi}_g$ for the character, and observe that this permutation

representation is generally not faithful, for the central involution $e_i \mapsto -e_i$ acts trivially. We write $g \mapsto \bar{g}$ for the natural map from $G^{(\ell)}$ to its quotient $\bar{G}^{(\ell)}$ by the central subgroup generated by $e_i \mapsto -e_i$. (We have $G^{(\ell)} \simeq \bar{G}^{(\ell)}$ when $\ell = 2$ since $e_i = -e_i$ for $i \in \Omega^{(2)}$.)

Observe that the smaller permutation representation $\bar{\chi}$ is an irreducible constituent of the larger one $\tilde{\chi}$. Indeed, the latter contains $\lfloor \ell/2 \rfloor$ copies of the former, and $\lfloor (\ell-1)/2 \rfloor$ copies of a faithful representation of degree $24/(\ell-1)$, whose character we denote $g \mapsto \chi_g$, which is just that which you obtain by taking the matrices representing the action of $G^{(\ell)} = \mathrm{Aut}^{\pm}(\mathcal{G}^{(\ell)})$ as elements of $GL_n(\mathbb{Z}/\ell)$—these matrices having exactly one non zero entry ± 1 in each row and column—and regarding them as elements of $GL_n(\mathbb{C})$. (Here $n = 24/(\ell-1)$.)

$$\tilde{\chi}_g = \lfloor \ell/2 \rfloor \bar{\chi}_g + \lfloor (\ell-1)/2 \rfloor \chi_g \tag{3.2}$$

It is now easy to describe the shadow of the vector-valued mock modular form $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$, for it is given by $S_g^{(\ell)} = (S_{g,r}^{(\ell)})$ where $S_{g,r}^{(\ell)} = \bar{\chi}_g S_{\ell,r}$ for r odd, and $S_{g,r}^{(\ell)} = \chi_g S_{\ell,r}$ for r even, where $S_{m,r}$ denotes the unary theta series

$$S_{m,r}(\tau) = \sum_{k \in \mathbb{Z}} (2km + r) q^{(2km+r)^2/4m}. \tag{3.3}$$

Note that $S_m = (S_{m,r})$ is a vector-valued cusp form of weight $3/2$ for the modular group $SL_2(\mathbb{Z})$.

Given a cycle shape $\Pi = m_1^{n_1} \cdots m_k^{n_k}$ with $n_i > 0$ for $1 \leq i \leq k$ and $m_1 < m_2 < \cdots < m_k$ call m_k the *largest factor* of Π and call m_1 the *smallest factor*. For each $g \in G^{(\ell)}$ define n_g to be the largest factor of $\bar{\Pi}_g$ (this turns out to be the same as the order of $\bar{g}$) and define N_g to be the product of the smallest and largest factors of $\tilde{\Pi}_g$. The significance of these values for the automorphy of $H_g^{(\ell)}$ is that n_g is the level of $H_g^{(\ell)}$—i.e., the smallest positive integer such that the vector-valued mock modular form $H_g^{(\ell)}$ is a mock modular form for $\Gamma_0(n_g)$—and N_g is the smallest positive integer such that the multiplier system for $H_g^{(\ell)}$ coincides with that of $S_\ell = (S_{\ell,r})$ when restricted to $\Gamma_0(N_g)$.

The expression for $S_g^{(\ell)}$ just given determines the multiplier system of $H_g^{(\ell)}$—since the multiplier system of a mock modular form is the inverse of the multiplier system of its shadow (cf. [18])—in the case that $\bar{\chi}_g$ and χ_g are both non-zero. The multiplier system of $H_g^{(\ell)}$ may be described as follows in the case that $\bar{\chi}_g \chi_g = 0$.

Define $v^{(\ell)} = \ell + 2$ in case ℓ is odd (i.e. $\ell \in \{3, 5, 7, 13\}$), and set $v^{(\ell)} = \ell - 1$ when ℓ is even (i.e. $\ell \in \{2, 4\}$). Let $\sigma^{(\ell)}$ denote the inverse of the multiplier system of the cusp form $S_\ell = (S_{\ell,r})$

and define $\psi^{(\ell)}_{n|h} : \Gamma_0(n) \to GL_{\ell-1}(\mathbb{C})$ for positive integers n and h by setting

$$\psi^{(\ell)}_{n|h}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{e}\left(-v^{(\ell)}\frac{cd}{nh}\right)\sigma^{(\ell)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.4}$$

in case h divides n, and otherwise

$$\psi^{(\ell)}_{n|h}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{e}\left(-v^{(\ell)}\frac{cd}{nh}\frac{(n,h)}{n}\right)\sigma^{(\ell)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} J^{c(d+1)/n}K^{c/n} \tag{3.5}$$

where J is the diagonal matrix $J = \mathrm{diag}(1, -1, 1, \cdots)$ with alternating ± 1 along the diagonal, and K is the "reverse shuffle" permutation matrix corresponding to the permutation

$$(1, \ell-1)(2, \ell-2)(3, \ell-3)\cdots \tag{3.6}$$

of the standard basis $\{e_1, \ldots, e_{\ell-1}\}$ of $\mathbb{C}^{\ell-1}$. Now the multiplier system of $H^{(\ell)}_g$ is given by $\psi^{(\ell)}_{n|h}$ for $n = n_g$ and $h = h_g = N_g/n_g$ when $\bar{\chi}_g\chi_g = 0$.

Note that the factor $(n,h)/n$ can usually be ignored in practice, for there is just one case in which $\bar{\chi}_g\chi_g = 0$ and $h = h_g$ does not divide $n = n_g$ and $(n,h) \neq n$; viz., the case that $\ell = 3$ and $g \in G^{(3)}$ satisfies $\bar{\Pi}_g = 2^1 10^1$ and $\tilde{\Pi}_g = 4^1 20^1$.

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A Characters

The tables in this section describe the character values $\bar{\chi}_g$, and χ_g, as well as the cycle shapes $\tilde{\Pi}_g$ and $\bar{\Pi}_g$, and also the symbols $n_g|h_g$ for each $g \in G^{(\ell)}$, for each $\ell \in \{2,3,4,5,7,13\}$. In case $\ell = 2$ we have $\tilde{\chi} = \bar{\chi} = \chi$, so we just give the $\bar{\chi}_g$ and $\bar{\Pi}_g$. The conjugacy class names are chosen to coincide with those of [12].

Table 2: Characters and cycle shapes at $\ell = 2$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
$n_g\|h_g$	1|1	2|1	2|2	3|1	3|3	4|2	4|1	4|4	5|1	6|1	6|6
$\bar{\chi}_g$	24	8	0	6	0	0	4	0	4	2	0
$\bar{\Pi}_g$	1^{24}	$1^8 2^8$	2^{12}	$1^6 3^6$	3^8	$2^4 4^4$	$1^4 2^2 4^4$	4^6	$1^4 5^4$	$1^2 2^2 3^2 6^2$	6^4

$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
$n_g\|h_g$	7|1	8|1	10|2	11|1	12|2	12|12	14|1	15|1	21|3	23|1
$\bar{\chi}_g$	3	2	0	2	0	0	1	1	0	1
$\bar{\Pi}_g$	$1^3 7^3$	$1^2 2^1 4^1 8^2$	$2^2 10^2$	$1^2 11^2$	$2^1 4^1 6^1 12^1$	12^2	$1^1 2^1 7^1 14^1$	$1^1 3^1 5^1 15^1$	$3^1 21^1$	$1^1 23^1$

Table 3: Characters and cycle shapes at $\ell = 3$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
$n_g\|h_g$	1	1|4	2|8	2	2|2	3	3|4	3|3	3|12	4|2	4	5	5|4	6|24	6	6|2	8|4	8	10|8	11	11|4
$\bar{\chi}_g$	12	12	0	4	4	3	3	0	0	0	4	2	2	0	1	1	0	2	0	1	1
χ_g	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	1	−1
$\bar{\Pi}_g$	1^{12}	1^{12}	2^6	$1^4 2^4$	$1^4 2^4$	$1^3 3^3$	$1^3 3^3$	3^4	3^4	$2^2 4^2$	$1^4 4^2$	$1^2 5^2$	$1^2 5^2$	6^2	$1^1 2^1 3^1 6^1$	$1^1 2^1 3^1 6^1$	$4^1 8^1$	$1^2 2^1 8^1$	$2^1 10^1$	$1^1 11^1$	$1^1 11^1$
$\tilde{\Pi}_g$	1^{24}	2^{12}	4^6	$1^8 2^8$	2^{12}	$1^6 3^6$	$2^3 6^3$	3^8	6^4	$2^4 4^4$	$1^4 2^2 4^4$	$1^4 5^4$	$2^2 10^2$	12^2	$1^2 2^2 3^2 6^2$	$2^2 6^2$	$4^2 8^2$	$1^2 2^1 4^1 8^2$	$4^1 20^1$	$1^2 11^2$	$2^1 22^1$

Table 4: Characters and cycle shapes at $\ell = 4$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$n_g\|h_g$	1|1	1|2	2|2	2|4	4|4	2|1	3|1	3|2	6|2	4|8	4|1	7|1	7|2
$\bar{\chi}_g$	8	8	0	0	0	4	2	2	0	0	2	1	1
χ_g	8	−8	0	0	0	0	2	−2	0	0	0	1	−1
$\bar{\Pi}_g$	1^8	1^8	2^4	2^4	4^2	$1^4 2^2$	$1^2 3^2$	$1^2 3^2$	$2^1 6^1$	4^2	$1^2 2^1 4^1$	$1^1 7^1$	$1^1 7^1$
$\tilde{\Pi}_g$	1^{24}	$1^8 2^8$	2^{12}	$2^4 4^4$	4^6	$1^8 2^8$	$1^6 3^6$	$1^2 2^2 3^2 6^2$	$2^3 6^3$	$4^2 8^2$	$1^4 2^2 4^4$	$1^3 7^3$	$1^1 2^1 7^1 14^1$

Table 5: Characters and cycle shapes at $\ell = 5$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$n_g\|h_g$	1|1	1|4	2|2	2|1	3|3	3|12	5|1	5|4	2|8	4|1	6|24
$\bar{\chi}_g$	6	6	2	2	0	0	1	1	0	2	0
χ_g	6	−6	−2	2	0	0	1	−1	0	0	0
$\bar{\Pi}_g$	1^6	1^6	$1^2 2^2$	$1^2 2^2$	3^2	3^2	$1^1 5^1$	$1^1 5^1$	2^3	$1^2 4^1$	6^1
$\tilde{\Pi}_g$	1^{24}	2^{12}	2^{12}	$1^8 2^8$	3^2	6^4	$1^4 5^4$	$2^2 10^2$	4^6	$1^4 2^2 4^4$	12^2

Table 6: Characters and cycle shapes at $\ell = 7$

$[g]$	1A	2A	4A	3AB	6AB
$n_g\|h_g$	1	1|4	2|8	3	3|4
$\bar{\chi}_g$	4	4	0	1	1
χ_g	4	-4	0	1	-1
$\bar{\Pi}_g$	1^4	1^4	2^2	$1^1 3^1$	$1^1 3^1$
$\tilde{\Pi}_g$	1^{24}	2^{12}	4^6	$1^6 3^6$	$2^3 6^3$

Table 7: Characters and cycle shapes at $\ell = 13$

$[g]$	1A	2A	4AB
$n_g\|h_g$	1	1|4	2|8
$\bar{\chi}_g$	2	2	0
χ_g	2	-2	0
$\bar{\Pi}_g$	1^2	1^2	2^1
$\tilde{\Pi}_g$	1^{24}	2^{12}	4^6

B Special Functions

The *Dedekind eta function*, denoted $\eta(\tau)$, is a holomorphic function on the upper half-plane defined by the infinite product

$$\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n) \tag{B.1}$$

where $q = \mathrm{e}(\tau) = e^{2\pi i \tau}$. It is a modular form of weight $1/2$ for the modular group $SL_2(\mathbb{Z})$ with multiplier $\epsilon : SL_2(\mathbb{Z}) \to \mathbb{C}$ so that

$$\eta(\gamma\tau)\epsilon(\gamma)\,\mathrm{j}(\gamma,\tau)^{1/2} = \eta(\tau) \tag{B.2}$$

for all $\gamma \in SL_2(\mathbb{Z})$, where $\mathrm{j}(\gamma,\tau) = (c\tau + d)^{-1}$ in case (c, d) is the lower row of γ.

Setting $q = \mathrm{e}(\tau)$ and $y = \mathrm{e}(z)$ we use the following conventions for the four standard *Jacobi theta functions.*

$$\begin{aligned}
\theta_1(\tau,z) &= -iq^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}) \\
\theta_2(\tau,z) &= q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^{n-1}) \\
\theta_3(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1+y\,q^{n-1/2})(1+y^{-1}q^{n-1/2}) \\
\theta_4(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1-y\,q^{n-1/2})(1-y^{-1}q^{n-1/2})
\end{aligned} \tag{B.3}$$

We write $\mu(\tau, z)$ for the *Appell-Lerch sum* defined by setting

$$\mu(\tau,z) = \frac{-iy^{1/2}}{\theta_1(\tau,z)} \sum_{\ell\in\mathbb{Z}} \frac{(-1)^\ell y^n q^{\ell(\ell+1)/2}}{1-yq^\ell}. \tag{B.4}$$

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