On quotients of Hom-functors

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1. Introduction

A hom-functor on a category $C$ is the functor $\text{Hom}(-, X)$ for an object $X$ of $C$. We consider the quotient functor $\text{Hom}(-, X)/G$ by a subgroup $G$ of $\text{Aut} X$. We are interested in replacing hom-functors in the definitions of limit and adjoint by quotients of hom-functors.

2. limit

We recall the definition of limit in terms of hom-functor. $\text{Set}$ denotes the category of sets. For a small category $C$, $[C^{\text{op}}, \text{Set}]$ denotes the category of contravariant functors $C \rightarrow \text{Set}$. $[C^{\text{op}}, \text{Set}]$ has limits. For instance, the product $F \times G$ of $F$ and $G$ in $[C^{\text{op}}, \text{Set}]$ is given by

$$(F \times G)(A) = F(A) \times G(A) \quad \text{for } A \in C.$$ 

And the final object $1$ of $[C^{\text{op}}, \text{Set}]$ is given by

$$1(A) = \{1\} \quad \text{for } A \in C.$$ 

For $X \in C$, the hom-functor $h_X$ is defined by

$$h_X(A) = \text{Hom}(A, X).$$ 

A functor $F: C^{\text{op}} \rightarrow \text{Set}$ is said to be representable if $F \cong h_X$ for some $X$.

For $X_1, X_2, Z \in C$ we have

$$Z \text{ is a product of } X_1 \text{ and } X_2 \iff h_Z \cong h_{X_1} \times h_{X_2}.$$ 

Therefore

$$\text{product of two objects exists in } C \iff \text{product of two representable functors is representable.}$$

And similarly

$$\text{a final object exists in } C \iff 1 \text{ is representable.}$$

The existence of a limit in $C$ is thus expressed as the representability of a limit of hom-functors. We first aim to replace representability by familial representability.
3. Sum of hom-functors

A functor $F: C^{\text{op}} \to \text{Set}$ is said to be familially representable if

$$F \cong \bigsqcup h_{X_i}$$

for some family $X_i$ of objects in $C$ ([Carboni and Johnstone]).

**Theorem 1.** Let $C$ be a finite category. The following conditions are equivalent to each other.

(i) $h_X \times h_Y$ and 1 are familially representable ($\forall X, Y \in C$).

(ii) Finite limits of hom-functors are familially representable.

(iii) Pushouts and coequalizers exist in $C$.

(iv) Finite connected limits exist in $C$.

Moreover these conditions imply that all morphisms of $C$ are epimorphisms.

**Remark.** “(iii) $\Rightarrow$ (iv)” is generally true.

For the proof of the theorem we may follow the proof of the general representability theorem in [Freyd and Scedrov]. It simplifies owing to our finiteness assumption. We may also use the characterization of familially representable functors ([Leinster]).

An interest with such categories comes from an attempt to define general Burnside rings. Suppose that $C$ satisfies (i) of Theorem 1. For any $X, Y \in C$ we take isomorphisms

$$h_X \times h_Y \cong \bigsqcup h_{Z_i}$$

and

$$1 \cong \bigsqcup h_{W_j}.$$ 

Then the free abelian group based on the isomorphism classes of objects of $C$ becomes a ring by setting

$$[X][Y] = \sum [Z_i],$$

$$1 = \sum [W_j].$$

Here $[X]$ stands for the isomorphism class of an object $X$. This ring may be called the Burnside ring of $C$.

4. The Burnside ring of a finite category

Let $C$ be a finite category. Assume that $C$ satisfies the following conditions.

(B1) For every $X, Y \in C$ there exists a unique family of integers $c_{Z}^{XY}$ such that

$$|\text{Hom}(A, X)||\text{Hom}(A, Y)| = \sum_Z c_{Z}^{XY} |\text{Hom}(A, Z)| \quad (\forall A \in C).$$

(Here $|S|$ stands for the cardinality of a set $S$.)
(B2) There exists a unique family of integers $d_Z$ such that

$$1 = \sum_Z d_Z |\text{Hom}(A, Z)| \ (\forall A \in C).$$

Then the free abelian group based on the isomorphism classes of objects of $C$ becomes a ring:

$$[X][Y] = \sum_Z c_Z^{XY}[Z],$$

$$1 = \sum_Z d_Z[Z].$$

**Theorem.** ([Yoshida]) Assume that a finite category $C$ satisfies the following conditions.

(Y1) $C$ has the unique epi-mono factorization property.

(Y2) $C$ has the coequalizer

$$\text{Coeq}(X \xrightarrow{\alpha} X)$$

for any $\alpha \in \text{Aut} X$.

Then $C$ satisfies (B1) and (B2).

The following diagram shows the relationship between Theorem 1 and Yoshida's theorem:

\[
\begin{array}{ccc}
[X][Y] &=& \sum_Z c_Z^{XY}[Z], \\
\text{pushout, coequalizer exist} &\implies& 1 = \sum_Z d_Z[Z], \\
&\Downarrow& \\
\text{epi-mono factorization,} &\implies& 1 = \sum_Z d_Z[Z], \\
\text{Coeq}(X \xrightarrow{\alpha} X) \text{ exist} &\Downarrow& \\
[X][Y] &=& \sum_Z c_Z^{XY}[Z],
\end{array}
\]

A problem will be to characterize categories satisfying (B1) and (B2).

Here are examples of generalized Burnside rings. Let $G$ be a finite group.

(1) Let $C$ be the category whose objects are $G$-sets $G/H$ for all subgroups $H$, and whose morphisms are $G$-maps. Then $C$ satisfies the condition of Theorem 1. The resulting ring is the ordinary Burnside ring of $G$.

(2) Let $\mathcal{F}$ be a family of subgroups of $G$ which is closed under conjugation and intersection. Let $C$ be the category whose objects are $G$-sets $G/H$ for $H \in \mathcal{F}$. Then $C$ satisfies the condition of Theorem 1.
(3) Let $\mathcal{F}$ be the set of all $p$-centric subgroups of $G$. Let $C$ be the category whose objects are $G$-sets $G/H$ for $H \in \mathcal{F}$. Then $C$ satisfies the condition that $h_X \times h_Y$ are familially representable ([Diaz and Libman], [Oda]). Further examples of $\mathcal{F}$ are found in [Oda and Sawabe].

(4) For a fusion system $\mathcal{F}$ a certain category $O(\mathcal{F}^c)$ is defined. Then $C = O(\mathcal{F}^c)$ satisfies the condition that $h_X \times h_Y$ are familially representable ([Puig], [Diaz and Libman]).

5. Finiteness of connected components of powers of a functor

$\text{FinSet}$ denotes the category of finite sets. Let $K$ be a finite category. We say $G \in [K, \text{FinSet}]$ is connected if $G$ is nonempty and never expressed as a sum of nonempty objects. Every $F \in [K, \text{FinSet}]$ is a sum of connected objects, each of which we call a connected component of $F$. For $F \in [K, \text{FinSet}]$ and $n \geq 0$ we have

$$F^n = F \times \cdots \times F$$

in $[K, \text{FinSet}]$.

**Theorem 2.** For $F \in [K, \text{FinSet}]$, the following are equivalent.

(i) Connected components of $F^n$ for all $n$ have only finitely many isomorphism classes.

(ii) $F(\alpha)$ is injective for every morphism $\alpha$ of $K$.

This theorem relates to Theorem 1 as follows: Let $F : K \to \text{FinSet}$ satisfy (ii) of Theorem 2. Let $C$ be a representative system of isomorphism classes of connected components of $F^n$ for all $n$. Then $C$ is finite. View $C$ as a category (a full subcategory of $[K, \text{FinSet}]$). For $X, Y \in C$, $X \times Y$ is a sum of objects of $C$ and $1$ is a sum of objects of $C$. So $C$ satisfies condition (i) of Theorem 1.

Conversely every finite category satisfying condition (i) of Theorem 1 arises this way.

6. Quotient of hom-functor

Let $C$ be a category. Let $X$ be an object of $C$ and $G$ a subgroup of Aut $X$. We define the functor $h_X/G : C^{op} \to \text{Set}$ by

$$(h_X/G)(A) = \text{Hom}(A, X)/G.$$ 

Here $\text{Hom}(A, X)/G$ is the quotient set relative to the natural action of $G$ on $\text{Hom}(A, X)$.

**Theorem 3.** Let $C$ be a finite category. The following conditions are equivalent to each other.

(i) $h_X \times h_Y$ and $1$ are isomorphic to sums of quotients of hom-functors $(\forall X, Y)$.

(ii) Finite limits of hom-functors are isomorphic to sums of quotients of hom-functors.

(iii) Pushouts exist in $C$.

(iv) Finite simply connected limits exist in $C$. 

These conditions imply that all morphisms of \( C \) are epimorphisms.

Remark. "(iii) \( \Rightarrow \) (iv)" is true for a general \( C \) ([Paré]).

7. Category with pushouts

We here give an example of a category with pushouts.

Let \( P \) be a partially ordered set. Suppose that a group \( G \) acts on \( P \):

\[
\sigma \in G, x \in P \rightsquigarrow x^\sigma \in P.
\]

The category \( PG \) is defined as follows.

(object) Objects of \( PG \) are elements of \( P \).

(morphism) For \( x, y \in P \)

\[
\text{Hom}_{PG}(x, y) = \{ \sigma | \sigma \in G, x \leq y^\sigma \}.
\]

(composition) Composition is given by multiplication in \( G \).

Proposition. If \( P \) has pushouts, then so does \( PG \).

That \( P \) has pushouts means that if \( z \leq x, z \leq y \), then there exists \( \sup(x, y) \).

Suppose that for each \( x \in P \) a subgroup \( K_x \) of \( G \) is given. Assume the following conditions hold.

(i) \( \sigma \in K_x \Rightarrow x^\sigma = x \)

(ii) \( x \leq y \Rightarrow K_x \leq K_y \)

(iii) \( K_x^\sigma = K_{x^\sigma} \)

We then define the category \( D \) as follows.

(object) Objects of \( D \) are elements of \( P \).

(morphism) For \( x, y \in P \) we set

\[
\text{Hom}_{D}(x, y) = \text{Hom}_{PG}(x, y)/K_y.
\]

Here \( K_y \) acts on \( \text{Hom}_{PG}(x, y) \) by multiplication in \( G \).

(composition) The composition of \( D \) is induced by that of \( PG \).

Proposition. If \( P \) has pushouts, then so does \( D \).

8. Adjoint

We recall the definition of adjoint in terms of \( \text{hom} \)-functor. Let \( F: B \to C \) and \( G: C \to B \) be functors. "\( G \) is a right adjoint of \( F \)" means

\[
\text{Hom}_C(F(X), Y) \cong \text{Hom}_B(X, G(Y))
\]

(naturally in \( X, Y \)).

This isomorphism, \( X \) viewed a variable, is written as

\[
\text{Hom}_C(F(-), Y) \cong h_{G(Y)}
\]

(naturally in \( Y \)).
\text{Hom}_C(F(-), Y) = h_Y \circ F \text{ denoted by } F^*(h_Y), \text{ this is written as } \\
F^*(h_Y) \cong h_{G(Y)}. \\
Thus \\
F \text{ has a right adjoint } \\
\iff F^*(h_Y) \text{ are representable for all } Y \in C. \\

We next aim to replace representability in the right-hand side by familial repre- 
sentability.

9. Discrete fibration

Recall that a functor $F: B \to C$ is called a discrete fibration if the following 
condition holds.

\forall g: F(X) \to Y' \text{ morphism of } C, \\
\exists! f: X \to X' \text{ morphism of } B, \\
F(f) = g.

If $F: B \to C$ is a discrete fibration, then \\

$$F^*(h_Y) \cong \coprod_{X \in \text{p-1}(Y)} h_X$$

for every $Y \in C.$

Proposition. Let $F: B \to C$ be a functor. The following are equivalent.

(i) $F^*(h_Y)$ are familialey representable for all $Y \in C.$

(ii) There exists a factorization

$$
\begin{array}{ccc}
C' & \to & C' \\
\uparrow & & \downarrow \pi \\
F' \nearrow & & \searrow \\
B & \to & C \\
\end{array}
$$

such that $F'$ has a right adjoint and $\pi$ is a discrete fibration.

10. Condition (G)

Here we aim to replace representability in the definition of adjoint by being isomorphic to a sum of quotients of hom-functors.

Let $F: B \to C$ be a functor. We introduce the condition (G) for $F.$ It consists of the following:

(i) \\

$$g: F(X) \to Y' \\
\implies \exists f: X \to X', F(f) = g.$$
(ii)

\[ f_1: X \to X_1', \; f_2: X \to X_2', \; F(f_1) = F(f_2) \]
\[ \implies \exists u: X_1' \to X_2', \; F(u) = 1, \; f_2 = uf_1. \]

If condition (G) holds, then \( F^*(h_Y) \) is isomorphic to a sum of quotients of hom-functors for every \( Y \in C \).

**Theorem 4.** Let \( F: B \to C \) be a functor. Assume that \( C \) is finite. The following are equivalent.

(i) \( F^*(h_Y) \) are isomorphic to sums of quotients of hom-functors for all \( Y \in C \).

(ii) There exists a commutative diagram

\[
\begin{array}{ccc}
B' & \xrightarrow{F'} & C' \\
\nu \downarrow & & \downarrow \pi \\
B & \xrightarrow{\pi} & C
\end{array}
\]

such that \( F' \) has a right adjoint, \( \nu \) is full and dense, and \( \pi \) satisfies condition (G).

**References**


