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SEMIGROUPS PRESENTED BY CONGRUENCE CLASSES OF REGULAR LANGUAGES -SURVEY-

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This is a survey on semigroups presented by congruence classes of regular languages.

1. Semigroups presented by regular congruences

Let A be a finite alphabet and $A^* = A^+ \cup \{\epsilon\}$ the set of all words over A.

A semigroup [monoid] S is presented by *regular congruence classes* if there exists a finite set A and there exists a surjective homomorphism ϕ of $A^+[A^*]$ to S such that for each word $W \in A^+[A^*]$, $\phi^{-1}(\phi(W))$ is a regular language.

Semigroups presented by regular congruence classes has nice properties (residually finiteness, having a soluvable word problem). However, except finite semigroups, there are few examples of semigroups presented by regular congruence classes. In particular, the free Burnside semigroups are typical examples.

2. History of the free Burnside semigroups

Let A be a finite alphabet and $A^* = A^+ \cup \{\epsilon\}$ the set of all words over A.

Definition 1 .
$\mathbb{B}(m,n,t) = < x_1, \cdots, x_t \mid (W^{m+n}, W^m),\ W \in A^+ >$ *is called the Free Burnside semigroup.*

$\mathbb{B}(m,n,1)$ is a cyclic semigroup of order $m+n-1$.

Theorem 1 . (Green and Rees, 1952)

(1) $\mathbb{B}(m,1,t)$ *is finite* $\Longleftrightarrow$ *the Burnside group* $\mathbb{G}(m,t)$ *is finite.*

(2) $\mathbb{B}(2,1,t)$ *is the free band of order* $\sum_{k=0}^{t} \binom{d}{k} \prod_{1 \le i \le k} (k-i+1)^2$.

Conjecture 1 . (J. Brzozowski, 1969) *For each* $W \in A^+$,

$[W] = \{X \in A^+ \mid X = W \text{ in } \mathbb{B}(m,1,t)\}$ *is a regular language.*

Conjecture 2 . (J. McCammond, 1991)

The conjecture for all free Burnside semigroups $\mathbb{B}(m,n,t)$.

Theorem 2 . (De Luca and Varricchio, 1990)

For $m \ge 5$ *and* $n \ge 2$,

the Brzozowski and McCammond's conjecture is true.

Theorem 3 . (J. McCammond, 1991)

For $m \ge 6$ *and* $n \ge 1$,

the Brzozowski and McCammond's conjecture is true.

Theorem 4 . (V. Guba, 1993)

For $m \ge 3$ *and* $n \ge 1$,

the Brzozowski and McCammond's conjecture is true.

Theorem 5 . (do Lago, 1993)

For $m = 2$ *and* $n = 2$,

the McCammond's conjecture is false.

Theorem 6 . **(A. Plyushchenko, 2009)**

The Brzozowski's conjecture holds for $\mathbb{B}(2,1,t)$ *if and only if the Brzozowski's conjecture holds for* $\mathbb{B}(2,1,t)$

3. Key points of the Guba's proof

In [4] and [5], Guba proved that the Brzozowski and McCammond's conjecture is true for $m \geq 3$ and $n \geq 1$. He used the method as follows :

Definition 2 .

$\mathbb{B}_k(m,n,t) = < x_1, \cdots, x_t \mid (W^{m+n}, W^m),\ W \in A^k >$ *is called the Free Burnside semigroup.*

Lemma 1 .

(*a*) *Let* $W (\in A^+)$ *be a long* k*-periodic word.*

Then $T^{sn}W = W$ *in* $\mathbb{B}(m,n,t)$.

(*b*) $X = VC$ *is a right* k*-extension of* V *if and only if* ${}^{\exists}C$: *the reduced form of* V *equals to one of* VCD.

Lemma 2 . *Let* X, Y, Z *be reduced word (no occurrence of* T^nW *).*

Then $XZ = Y$ *in* $\mathbb{B}_k(m,n,t)$ *if and only if* ${}^{\exists}V$: $Y \in VA^*$ *and* X *is a right* k*- extension of* V.

Definition 3 . (1) W *is a long* k*-periodic word if* ${}^{\exists}X, Y$: $T^m \leq_{subword} XWY <_{subword} T^{m+s}$ (T : primitive word), XW *is a left* k-1-*extension of* W *and* WY *is a right* k-1-*extension of* W.

(2) $XW(WY)$ *is a left* k*-extension of* W *if* ${}^{\exists}i_1 < \cdots < i_r \leq k\text{-}1$: $X = Z_1 \cdots Z_r W$, *each* $Z_j Z_{j+1} \cdots Z_r W$ *is an immediate left* i_j*- extension of* $Z_{j+1} \cdots Z_r W$ $(1 \leq j \leq r)$.

(3) $XW(WY)$ *is an immediate left* k*-extension of* W *if* ${}^{\exists}C, D$: $W = CD$, XC *is* k *-periodic,* C *is a long* k*-periodic word.*

For any word W, construction of a finite automaton $\mathcal{A}(\mathcal{W})$:

States : $[X](X$ is a right $k-$extension of some prefix of $W)$ (for $^{\forall}k \in \mathbb{N}$)

The number of the states is finite (since we have only to choose $X = VZ_1 \cdots Z_r$, each $VZ_r \cdots Z_{j+1}Z_j$ is an immediate right i_j-extension of $VZ_r \cdots Z_{j+1}$ $(1 \leq j \leq r)$ and $|Z_j| \leq (m+n-1)i_j$.)

Initial States $[a](a \in A \cap \{\text{Prefixes of } W\})$, Terminal State $[W]$

Edges : $[X] \overset{a}{\Rightarrow} [Xa]$ ($a \in A$)

Then the set of accepted words of $\mathcal{A}(\mathcal{W})$ is equal to $[W]$.

4. Rewriting systems of the free Burnside semigroups and do Lago's theorem

In [9], do Lago innovated new rewriting systems to analyze sturucture of the free Burnside semigroups.

Definition 4 . *Let R be a rewriting system over A. Then*

(1) *A relator (l,r) of R with $|l| \geq |r|$ is stable if $\dfrac{|l|-|r|}{n}$ is the smallest period of r.*

(2) *A rewriting system R is stable if every relator of R istable.*

Theorem 7 (De Largo, 1996). *For $m \geq 3, n \geq 1$,*

(1) *there exists a subset Σ of $\pi = \{(W^{m+n}, W^m) \ |W \in A^+\}$ which is stable and congruences generated by Σ or by π equal to each other.*

(2) *Σ is a complete rewriting system.*

(3) *$\mathbb{B}(m,n,t)$ is $\mathcal{J}$-above finite.*

(4) *the Brzozowski and McCammond's conjecture is true for $\mathbb{B}(m,n,t)$.*

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