Locally freely productable groups and the primitivity of their group rings

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Let $R$ be a ring with the identity element. $R$ is (right) primitive provided there exists a faithful irreducible (right) $R$-module. A group $G$ is LFP (locally freely productable) provided for each finitely generated subgroup $H = \langle g_1, \ldots, g_n \rangle$ of $G$, either $H$ is a non-trivial free product of groups both of which are not isomorphic to $\mathbb{Z}_2$ or there exists an element $x \in G$ with $x \neq 1$ such that $H \ast \langle x \rangle$ is free product. In this note, we shall introduce the primitivity of group rings of LFP groups. And as a result, we state that every group ring of a one-relator group with torsion is primitive. In order to prove primitivity of group rings, we shall need the graph theoretic approach used in [5] which extends the Formanek’s method in [3].

1 Graph theoretic approach

Let $KG$ be the group ring of a group $G$ over a field $K$, and let $a = \sum_{i=1}^{n} \alpha_i f_i$ and $b = \sum_{i=1}^{n} \beta_i g_i$ be in $KG$ ($\alpha_i \neq 0, \beta_i \neq 0$). If $ab = 0$ then for each $f_ig_j$, there exists $f_pg_q$ such that $f_ig_j = f_pg_q$. Suppose that the following $k$ equations hold:

\[ f_1g_1 = f_2g_2, \ f_3g_2 = f_4g_3, \ldots, f_{2k-3}g_{k-1} = f_{2k-2}g_{k} \text{ and } f_{2k-1}g_{k} = f_{2k}g_1. \]

Then we can regard the above equations as forming a kind of cycle, and they imply $f_1^{-1}f_2 \cdots f_{2k-1}^{-1}f_{2k} = 1$. That is, the above equations give us a information on supports of $a$. We can use this idea for a more general case; $a_1 b_1 + \cdots + a_n b_n \in K$ for $a_i, b_i \in KG$ with $a_i = \sum \alpha_{ij} f_{ij}$ and $b_i = \sum \beta_{ij} g_{ij}$. In order to do this, regarding the elements $f_{ij}g_{ik}$ appeared in $a_i b_i$ as vertices and the equalities of their elements as edges, we use a graph-theoretic method.

Throughout this section, $\mathcal{G} = (V, E)$ denotes a simple graph; a finite undirected graph which has no multiple edges or loops, where $V$ is the set of vertices and $E$ is the set of edges. A finite sequence $v_0e_1v_1 \cdots e_pv_p$ whose terms are alternately elements $e_q$'s in $E$ and $v_q$'s in $V$ is called a path of length $p$ in $\mathcal{G}$ if $v_{q-1}v_q = e_q \in E$ and $v_q \neq v_q'$ for any $q, q' \in \{0, 1, \ldots, p\}$ with $q \neq q'$; simply denoted by $v_0v_1 \cdots v_p$.

Two vertices $v$ and $w$ of $\mathcal{G}$ are said to be connected if there exists a path from $v$ to $w$ in $\mathcal{G}$. Connection is an equivalence relation on $V$, and so there exists a decomposition of $V$ into subsets $C_i$'s ($1 \leq i \leq m$) for some $m > 0$ such that $v, w \in V$ are connected if and only if both $v$ and $w$ belong to the same set $C_i$.

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The subgraph generated by $C_i$ is called a (connected) component of $G$. Any graph is a disjoint union of components.

**Definition 1.1** Let $G = (V, E)$ and $H = (V, F)$ be simple graphs with the same vertex set $V$. For $v \in V$, let $U(v)$ be the set consisting of all neighbours of $v$ in $H$ and $v$ itself: $U(v) = \{ w \in V \mid vw \in F \} \cup \{ v \}$. A triple $(V, E, F)$ is an SR-graph (for a sprint relay like graph) if it satisfies the following conditions:

(i) $G$ is a clique graph; thus $uv, vu \in E$ implies $uv \in E$.

(ii) If $C$ is a component of $G$ and $v, w \in C$ with $v \neq w$, then $U(v) \cap U(w) = \emptyset$.

If $G$ has no isolated vertices, that is, if $v \in V$ then $vw \in E$ for some $w \in V$, then SR-graph $(V, E, F)$ is called a proper SR-graph.

Fig 1 shows an example of an SR-graph, in which edges in $E$ and $F$ are respectively denoted by solid lines and dotted lines. In what follows, solid lines and dotted lines denote edges in $E$ and $F$, respectively. In the above definition, the condition (i) means that every component of $G$ is a complete graph, and (ii) does that each $U(v)$ has at most one vertex from each component of $G$. Hence, under the assumption (i), (ii) is equivalent to the condition that if $w, u \in U(v)$ then $uv \not\in E$. That is, (i) and (ii) implies that there exists no subgraph of types appeared in Fig 2.

We call $U(v)$ the SR-neighbour set of $v \in V$, and set $\mathfrak{U}(V) = \{ U(v) \mid v \in V \}$. For $v, w \in V$ with $v \neq w$, it may happen that $U(v) = U(w)$, and so $|\mathfrak{U}(V)| \leq |V|$ generally. Let $S = (V, E, F)$ be an SR-graph. We say $S$ is connected if the graph $(V, E \cup F)$ in which there is no distinction between $E$ and $F$ is connected.

![Fig 1. An example of an SR-graph: Solid lines are edges in $E$ and dotted lines are edges in $F$. Sequences $(e_1, f_1, \cdots, e_p, f_p, e_1)$, $(e_1, e_p, f_1, f_p, e_1)$ and $(e_1, f_p, e_1)$ are SR-cycles.](image1)

![Fig 2. Prohibits: It is not allowed to exist the above two subgraphs in an SR-graph.](image2)

**Definition 1.2** Let $S = (V, E, F)$ be an SR-graph and $p > 1$. Then a path $v_1w_1v_2w_2, \cdots, v_pw_pv_{p+1}$ in the graph $(V, E \cup F)$ is called a SR-path of length $p$ in $S$ if either $v_qw_q \in E$ and $w_qv_{q+1} \in F$ or $v_qw_q \in F$ and $w_qv_{q+1} \in E$ for $1 \leq q \leq p$; simply denoted by $(e_1, f_1, \cdots, e_p, f_p)$ or $(f_1, e_1, \cdots, f_p, e_p)$, respectively, where $e_q \in E$ and $f_q \in F$. If, in addition, it is a cycle in $(V, E \cup F)$, that is, $v_{p+1} = v_1$, then it is an SR-cycle of length $p$ in $S$. 
That is, for \( e_q \in E \) and \( f_q \in F \), an SR-cycle \((e_1, f_1, \cdots, e_p, f_p)\) means that it is a cycle in \((V, E \cup F)\) which consists alternately solid lines and dotted lines (see Fig1).

In what follows, let \( S = (V, E, F) \) be an SR-graph with \( G = (V, E) \) and \( H = (V, F) \). \( \mathcal{C}(V) \) denotes the set of components of \( V \) on \( H = (V, F) \). In addition, we set \( \mathfrak{N}(S) = \{ U \in \mathfrak{U}(V) \mid |U| = 1 \}, \mathfrak{M}(S) = \{ U \in \mathfrak{U}(V) \mid |U| = 2 \} \) and \( \mathcal{L}(S) = \{ U \in \mathfrak{U}(V) \mid |U| > 2 \}. \)

We would like to know when \( S \) has an SR-cycle. We first consider the somewhat trivial case of \( S \) in which \( H = (V, F) \) is also a clique graph. In this case, \( \mathfrak{U}(V) \) coincides with \( \mathcal{C}(V) \). We have the next theorem:

**Theorem 1.3** Let \( S = (V, E, F) \) be an SR-graph and let \( \omega_E \) and \( \omega_F \) be, respectively, the number of components of \( G = (V, E) \) and \( H = (V, F) \). Suppose that \( H = (V, F) \) is a clique graph and \( S \) is connected. Then \( S \) has an SR-cycle if and only if \( \omega_E + \omega_F < |V| + 1 \).

In particular, if \( S \) is proper and \( |\mathfrak{N}(S)| \leq |\mathcal{L}(S)| \) then \( S \) has an SR-cycle.

In the above theorem, every component is a complete graph. We next consider the case that every component \( G_i = (V_i, E_i) \) is a complete \( k \)-partite graph \( K_{m_1, \cdots, m_k} \). Let \( \mu(V_i) \) be the maximum number in \( \{m_1, \cdots, m_k\} \). For \( v \in V \), let \( d_G(v) \) be the degree of \( v \) in \( G \); thus the number of edges of \( G \) incident with \( v \). \( I(V) \) denotes the set of isolated vertices in \( G \); thus \( I(V) = \{ v \in V \mid d_G(v) = 0 \} \). Then we have

**Theorem 1.4** Let \( S = (V, E, F) \) be an SR-graph and \( \mathcal{C}(V) = \{ V_1, \cdots, V_n \} \) with \( n > 1 \). Suppose that every component \( G_i = (V_i, E_i) \) of \( G \) is a complete \( k \)-partite graph. If \( |V_i| > 2\mu(V_i) \) for each \( i \in \{ 1, \cdots, n \} \) and \( |I(V)| \leq n \) then \( S \) has an SR-cycle.

We can prove two theorems above by a similar argument in [5].

## 2 LFP groups

**Definition 2.1** A group \( G \) is LFP provided for each finitely generated subgroup \( H = \langle g_1, \cdots, g_n \rangle \) of \( G \), either \( H \) is a non-trivial free products of groups both of which are not isomorphic to \( \mathbb{Z}_2 \) or there exists an element \( x \in G \) with \( x \neq 1 \) such that \( H * \langle x \rangle \) is free product.

It is obvious that a locally free group is LFP and so is free group. Moreover, by the Kurosh Subgroup Theorem for free products, we can see that the non-trivial free product \( A * B \) of groups \( A \) and \( B \) is LFP provided both of \( A \) and \( B \) are not isomorphic to \( \mathbb{Z}_2 \).
By making use of theorems in the previous section, we can state the following theorem:

**Theorem 2.2** If $G$ is LFP, then the group ring $KG$ is primitive for any field $K$.

3 Primitivity of group rings of one-relator groups with torsion

Let $\langle X \rangle$ be the free group with the base $X$. For a word $R$ in $\langle X \rangle$, $G = \langle X \mid R \rangle$ denotes the one-relator group with a generating set $X$ of $G$ and a defining relation $R = 1$. If $W$ is a cyclically reduced word in $\langle X \rangle$ and $R = W^n$ ($n > 1$), then $G$ is called a one-relator group with torsion. The class of one-relator groups with torsion has been well studied, in particular, on residual finiteness (for instance, [2], [7], [8], [1]).

In this section, by making use of the Theorem 2.2, we shall show the next theorem:

**Theorem 3.1** The group ring $KG$ of $G = \langle X \mid W^n \rangle$ over a field $K$ is primitive provided $n > 1$ and $|X| > 1$, where $W$ is a cyclically reduced word in $\langle X \rangle$.

In what follows, let $F = \langle X \rangle$ be the free group with the base $X = \{x_1, \ldots, x_m\}$. $\langle g_1, \ldots, g_m \rangle_G$ denotes the subgroup of a group $G$ generated by $g_1, \ldots, g_m \in G$. If $W \in F$, then $N_F(W)$ denotes the normal closure of $W$ in $F$. For a cyclically reduced word $W$, $W_F(W)$ denotes the set of all cyclically reduced conjugates of both $W$ and $W^{-1}$. If $W_1, \ldots, W_t$ are reduced words in $F$ and $W = W_1 \cdots W_t$ is also reduced, that is, there is no cancellation in forming the product $W_1 \cdots W_t$, then we write $W \equiv W_1 \cdots W_t$.

**Lemma 3.2** Let $m, n > 1$ and $W_0 = W_0(x_1, \ldots, x_m)$ be a cyclically reduced word in $F$ which involves all $x_i$'s in $X$. Suppose that $V \in N_F(R_0)$, where $R_0 = W_0^n$. If $V \equiv V_1V_2$, then every generator in $X$ appears either in $V_1$ or in $V_2$.

**Proof.** By the well-known the Newman-Gurevich Spelling Theorem([6], cf. [4]), $V$ contains a subword $S^{n-1}S_0$, where $S \equiv S_0S_1 \in W_F(W_0)$ and every generator in $X$ appears in $S_0$. Hence either $V_1$ or $V_2$ contains the subword $S_0$, and the assertion follows.

**Lemma 3.3** For $m > 1$, $n > 1$ and $X = \{x_1, \ldots, x_m\}$, let $G = \langle X \mid R \rangle$, where $R = W^n$ and $W$ is a cyclically reduced words in the free group $\langle X \rangle$ with the base $X$. If $S, T \subseteq X$, then $\langle S \rangle_G \cap \langle T \rangle_G = \langle S \cap T \rangle_G$. 


Proof. It is obvious that $\langle S \rangle_G \cap \langle T \rangle_G \supseteq \langle S \cap T \rangle_G$. Suppose, to the contrary, that $\langle S \rangle_G \cap \langle T \rangle_G \neq \langle S \cap T \rangle_G$. Then there exist reduced words $u = u(s, a, \ldots, b)$ in $\langle S \rangle \setminus \langle S \cap T \rangle$ and $v = v(t, c, \ldots, d)$ in $\langle T \rangle \setminus \langle S \cap T \rangle$ such that $uv \in N_F(R)$, where $a, \ldots, b \in S$, $c, \ldots, d \in T$, $s \in S \setminus \langle S \cap T \rangle$ and $t \in T \setminus \langle S \cap T \rangle$. Let $w$ be the reduced word for $uv$, say $w \equiv u_1v_1$, where $u \equiv u_1u_2$ and $v \equiv u_2^{-1}v_1$. Then $w \in N_F(R)$, however, $u_1$ involves $s$ but not $t$, and $v_1$ involves $t$ but not $s$, which contradicts the assertion of Lemma 3.2.

Let $X = \{a_i, b_i, \ldots \mid i \in \mathbb{Z}\}$ and $W_i$ $(i \in \mathbb{Z})$ cyclically reduced words in the free group $\langle X \rangle$ with the base $X$ such that

$$W_i = W_i(a_{j_{a1}+i}, \ldots, a_{j_{as}+i}, b_{j_{b1}+i}, \ldots, b_{j_{bt}+i}, \ldots),$$

where $j_{a1} < j_{a2} < \ldots < j_{as}$ and $j_{b1} < j_{b2} < \ldots < j_{bt}$ and $\ldots$. Let $\alpha_*, \beta_*, \ldots$ be the minimum subscripts on $a, b, \ldots$ occurring in $W_0$, respectively, and $\alpha^*, \beta^*, \ldots$ be the maximum subscript on $a, b, \ldots$ occurring in $W_0$, respectively. That is, $\alpha_* = j_{a1}$, $\alpha^* = j_{as}$ and $\beta_* = j_{b1}$, $\beta^* = j_{bt}$ and $\ldots$. We set $A = \{a_i \mid i \in \mathbb{Z}\}$, $B = \{b_i \mid i \in \mathbb{Z}\}$, \ldots; in this case, $X = A \cup B \cup \ldots$.

Let

$$G_\infty = \langle X \mid R_t(i \in \mathbb{Z}) \rangle \text{ with } R_i = W_i^n(n > 1).$$

In $G_\infty$, we set subgroups $Q_t$ and $P_t$ of $G_\infty$ for all $t \in \mathbb{Z}$, as follows:

$$\begin{cases}
\text{For } N \neq 0, \\
Q_t = \langle a_{i+t}, b_{j+t}, \ldots \mid \alpha_* \leq i \leq \alpha^*, \beta_* \leq j \leq \beta^*, \ldots \rangle_{G_\infty}, \\
P_t = \langle a_{i+t}, b_{j+t}, \ldots \mid \alpha_* \leq i \leq \alpha^*-1, \beta_* \leq j \leq \beta^*-1, \ldots \rangle_{G_\infty}.
\end{cases}$$

(2)

for

$$\begin{cases}
\text{For } N = 0, \\
Q_t = \langle a_{\alpha^*+t}, b_{\beta^*+t}, \ldots \rangle_{G_\infty}, \\
P_t = 1.
\end{cases}$$

(3)

where $N$ is the maximum number in $\{\alpha^* - \alpha_*, \beta^* - \beta_*, \ldots\}$.

Then $P_t \leq Q_t$ and $Q_t \simeq \langle a_{\alpha^*+t}, \ldots, a_{\alpha^*+t}, b_{\beta^*+t}, \ldots, b_{\beta^*+t}, \ldots \mid R_t \rangle$. By the Magnus' method for Freiheitssatz, we may identify $G_\infty$ as the union of the chain of the following $G_i$'s:

$$G_\infty = \bigcup_{i=0}^{\infty} G_i, \text{ where }$$

$$G_0 = Q_0, \quad G_{2i} = Q_{-i} \ast P_{i+1} G_{2i-1} \quad \text{and} \quad G_{2i+1} = G_{2i} \ast P_{i+1} Q_{i+1}. \quad (3)$$

Generally, for each $k \in \mathbb{Z}$, set

$$G_0 = Q_k, \quad G_{2i} = Q_{-i+k} \ast P_{i+k+1} G_{2i-1} \quad \text{and} \quad G_{2i+1} = G_{2i} \ast P_{i+k+1} Q_{i+k+1}, \quad (4)$$

and we can also identify $G_\infty$ as $\bigcup_{i=0}^{\infty} G_i$. Then we have

$$\begin{align*}
G_0 &= Q_k = \langle a_{\alpha^*+k}, \ldots, a_{\alpha^*+k}, b_{\beta^*+k}, \ldots, b_{\beta^*+k}, \ldots \rangle_{G_\infty} \\
G_{2i} &= \langle a_{\alpha^*+k-i}, \ldots, a_{\alpha^*+k+i}, b_{\beta^*+k-i}, \ldots, b_{\beta^*+k+i}, \ldots \rangle_{G_\infty} \\
G_{2i+1} &= \langle a_{\alpha^*+k-i}, \ldots, a_{\alpha^*+k+i+1}, b_{\beta^*+k-i}, \ldots, b_{\beta^*+k+i+1}, \ldots \rangle_{G_\infty} \quad (5)
\end{align*}$$
Lemma 3.4 Let $H$ be a subgroup of $G_{\infty}$ generated by a finite subset $Y$ of $X$; thus $H = \langle Y \rangle_{G_{\infty}}$. Set $I = \{i \in \mathbb{Z} \mid a_i \in A \cap Y \text{ or } \cdots \text{ or } b_i \in B \cap Y\}$, and let $i^*$ (resp. $i_*$) be the maximum number (resp. the minimum number) in $I$ and $M_*$ (resp. $m^*$) the maximum number (resp. the minimum number) in $\{\alpha_*, \beta_*, \cdots\}$ (resp. $\{\alpha^*, \beta^*, \cdots\}$).

If $N < t$ and $N + i^* - i_* + M_* - m^* < t$, then $H \cap P_t = 1$.

Proof. If $N = 0$ then the assertion of the Lemma is trivial, and so we suppose $N \neq 0$, and also suppose, to the contrary, there exists $t \in \mathbb{Z}$ such that

$$N < t, \quad N + i^* - i_* + M_* - m^* < t \quad \text{and} \quad H \cap P_t \neq 1.$$ 

If we set $k = \mu = i_* - M_*$ in (4) just above this lemma, then

$$G_0 = Q_{\mu}, \quad \text{and} \quad G_{2i} = Q_{-i+\mu} * p_{-t+\mu+1} G_{2i-1}.$$ 

Moreover, let $\tau$ be the largest number between $0$ and $i^* - \mu - m^*$. If we set $i = \tau$ in the above, then we can see that $G_{2\tau} \supseteq H$ and $\alpha^* + \tau < \alpha_* + t, \beta^* + \tau < \beta_* + t$, $\cdots$.

In fact, if $\tau = 0$, then $\alpha^* + \tau = \alpha^* \leq \alpha_* + N < \alpha_* + t$, because of $N < t$. On the other hand, if $\tau \neq 0$, then $\tau = i^* - (i_* - M_*) - m^*$, and so,

$$\alpha^* + \tau \leq \alpha_* + N + \tau = \alpha_* + N + i^* - i_* + M_* - m^* < \alpha_* + t,$$

because of $N + i^* - i_* + M_* - m^* < t$. We similarly obtain that $\beta^* + \tau < \beta_* + t$, $\cdots$.

Next, we shall show $G_{2\tau} \supseteq H$. To see this, since

$$G_{2\tau} = \langle a_{\alpha_*+\mu-\tau}, \cdots, a_{\alpha^*+\mu+\tau}, b_{\beta_*+\mu-\tau}, \cdots, b_{\beta^*+\mu+\tau}, \cdots \rangle_{G_{\infty}},$$

it suffices to show that $\alpha_* + \mu - \tau \leq i_*$, $\beta_* + \mu - \tau \leq i_*$, $\cdots$, and $\alpha^* + \mu + \tau \geq i^*$, $\beta^* + \mu + \tau \geq i_*$, $\cdots$. Note that $\mu + \tau = i^* - m^*$ if $\tau \neq 0$ and $\mu \geq i^* - m^*$ if $\tau = 0$.

In fact, if $\tau \neq 0$, then $\mu + \tau = \mu + i^* - \mu - m^* = i^* - m^*$, and if $\tau = 0$, then $i^* - \mu - m^* \leq 0$ and so $i^* - m^* \leq \mu$.

Since $\tau \geq 0$ and $\alpha_* - M_* \leq 0$ by definitions, we have

$$\alpha_* + \mu - \tau \leq \alpha_* + \mu = i_* + \alpha_* - M_* \leq i_*.$$ 

We similarly obtain that $\beta_* + \mu - \tau \leq i_*$, $\cdots$. Moreover, as mentioned above, if $\tau = 0$, then $\mu \geq i^* - m^*$, and so we have that

$$\alpha^* + \mu + \tau \geq \alpha^* + i^* - m^* \geq \alpha^* + i^* - \alpha^* = i^*$$

because $m^* \leq \alpha^*$. If $\tau \neq 0$, since $\mu + \tau = i^* - m^*$, we also have

$$\alpha^* + \mu + \tau = \alpha^* + i^* - m^* \geq \alpha^* + i^* - \alpha^* = i^*.$$
We have thus seen $\alpha^* + \mu + \tau \geq i^*$ for either cases, and similarly we have $\beta^* + \mu + \tau \geq i^*$, $\cdots$, as desired.

In the above, replacing $\alpha_* + \mu$ with $\alpha_*$, $\alpha^* + \mu$ with $\alpha^*$, $\beta_* + \mu$ with $\beta_*$, $\cdots$, and $\tau$ with $k$, we may assume that $G_\infty = \bigcup_{i=0}^{\infty} G_i$ with the presentation (4) and there exists $k \geq 0$ such that $G_{2k} \supseteq H$ and

$$\alpha^* + k < \alpha_* + t, \beta^* + k < \beta_* + t, \cdots.$$  \hfill (6)

Now, let $n = \beta^* - \beta_*$, and we may here assume $N = \alpha^* - \alpha_* \geq \cdots \geq \beta^* - \beta_*$. For $j \in \{0, 1, \cdots, N\}$, we define $P_t^{(j)}$'s so as to satisfy

$$P_t = P_t^{(N)} \supset P_t^{(1)} \supset \cdots \supset P_t^{(0)} = 1$$

as follows:

\[
\begin{align*}
P_t &= P_t^{(N)} = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-1}, b_{\beta_*+t}, \cdots, b_{\beta^*+t-1}, \cdots \rangle_{G_\infty}, \\
P_t^{(N-1)} &= \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-2}, b_{\beta_*+t}, \cdots, b_{\beta^*+t-2}, \cdots \rangle_{G_\infty}, \\
\vdots & \quad \vdots \\
P_t^{(N-n+1)} &= \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-n}, b_{\beta_*+t}, \cdots \rangle_{G_\infty}, \\
P_t^{(N-n)} &= \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-n-1}, \cdots \rangle_{G_\infty}, \\
\vdots & \quad \vdots \\
P_t^{(1)} &= \langle a_{\alpha_*+t} \rangle_{G_\infty}, \\
P_t^{(0)} &= 1.
\end{align*}
\]

By our assumption, $H \cap P_t \neq 1$, that is, there exists $u \in H \cap P_t$ such that $u \neq 1$.

Then there exists $l \in \{0, 1, \cdots, N - 1\}$ such that $u \in P_t^{(N-l)}$ and $u \not\in P_t^{(N-l-1)}$. We shall show that this is impossible. In fact, we shall show that $u \in P_t^{(N-l)}$ implies $u \in P_t^{(N-l-1)}$, and this completes the proof of the Lemma.

By (6), $\alpha^* + k \leq \alpha_* + t - 1$, and so $k \leq -N + t - 1 \leq -l + t - 2$, which implies

$$H \subseteq G_{2(t-l-2)}$$ \hfill (7)

because $H \subseteq G_{2k} \subseteq G_{2(t-l-2)}$. By way of construction of $P_t^{(N-l)}$, we have

$$P_t^{(N-l)} = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-l-1}, b_{\beta_*+t}, \cdots, b_{\beta^*+t-l-1}, \cdots \rangle_{G_\infty},$$

where $b_{\beta^*+t-l-1} = b_{\beta^*+t-1}$ if $l < n$ and $b_{\beta^*+t-l-1} = 1$ if $l \geq n$. By (2), we also have

$$Q_{t-l-1} = \langle a_{\alpha_*+t-l-1}, \cdots, a_{\alpha^*+t-l-1}, b_{\beta_*+t-l-1}, \cdots, b_{\beta^*+t-l-1}, \cdots \rangle_{G_\infty},$$

and therefore we see that $P_t^{(N-l)} \subseteq Q_{t-l-1}$. Combining this with (7), it follows that $u \in G_{2(t-l-2)} \cap Q_{t-l-1}$. Since $G_{2(t-l-2)} \cap Q_{t-l-1} = P_{t-l-1}$, we have $u \in P_{t-l-1}$, and thus $u \in P_{t-l-1} \cap P_t^{(N-l)}$.\[37\]
On the other hand, \( P_{t-l-1} = \langle S \rangle Q_{t-l-1} \) and \( P_{t}^{(N-l)} = \langle T \rangle Q_{t-l-1} \) in \( Q_{t-l-1} \), where

\[
S = \{ a_{\alpha^*+t-l-1}, \ldots, a_{\alpha^*+t-l-2}, b_{\beta^*+t-l-1}, \ldots, b_{\beta^*+t-l-2}, \ldots \}
\]

and \( T = \{ a_{\alpha^*+t}, \ldots, a_{\alpha^*+t-l-1}, b_{\beta^*+t}, \ldots, b_{\beta^*+t-l-1}, \ldots \} \).

Then it is easily seen that \( \langle S \cap T \rangle Q_{t-l-1} = P_{t}^{(N-l-1)} \). We can here identify \( Q_{t-l-1} \) as the one-relator group with torsion, and therefore it follows form Lemma 3.3 that

\[
u \in P_{t-l-1} \cap P_{t}^{(N-l)} = \langle S \rangle Q_{t-l-1} \cap \langle T \rangle Q_{t-l-1} = \langle S \cap T \rangle Q_{t-l-1} = P_{t}^{(N-l-1)};
\]

thus \( u \in P_{t}^{(N-l-1)} \), as desired.

By the proof of the above Lemma, we have

**Corollary 3.5** If \( H \) be a subgroup of \( G_\infty \) generated by a finite subset \( Y \) of \( X \), then there exists a positive integer \( t \) such that \( H \subseteq G_{2t-l} \) and \( H \cap P_{t} = 1 \).

**Lemma 3.6** If \( G_\infty \) and \( W_{t} \) are as in (1), then for each finite elements \( g_{1}, \ldots, g_{m} \) in \( G_\infty \), there exists an integer \( i \) such that \( \langle g_{1}, \ldots, g_{m}, W_{t} \rangle_{G_\infty} \) is the free product \( \langle g_{1}, \ldots, g_{m} \rangle_{G_\infty} \ast \langle W_{t} \rangle_{G_\infty} \).

**Proof.** Let \( G_\infty \) be as in (3) and \( Y \) the set of generators which appear in \( g_{i} \)'s. By virtue of Corollary 3.5, for \( H = \langle Y \rangle_{G_\infty} \), there exists \( t > 0 \) such that \( H \subseteq G_{2t-l} \) and \( H \cap P_{t} = 1 \).

Now, by (3), \( G_{2t-1} = G_{2(t-l)} \ast_{P_{t}} Q_{t} \), where

\[
Q_{t} = \langle a_{\alpha^*+t}, \ldots, a_{\alpha^*+t-l}, b_{\beta^*+t}, \ldots, b_{\beta^*+t-l}, \cdots | R_{t} \rangle,
\]

and either \( P_{t} = \langle a_{i+t}, b_{j+t}, \ldots | \alpha_{i} \leq i \leq \alpha^* - 1, \beta_{j} \leq j \leq \beta^* - 1, \cdots \rangle_{G_\infty} \) or \( P_{t} = 1 \). We see then that \( W_{t} \in Q_{t} \). As is well known, \( W_{t}^{m} \neq 1 \) if \( 1 \leq m < n \) because \( R_{t} = W_{t}^{n} \) and \( n > 1 \). Moreover, if \( W_{t}^{m} \in P_{t} \), then \( (W_{t}^{m})^{n} \neq 1 \) because \( P_{t} \) is a free subgroup in \( Q_{t} \) by Freiheitssatz, which implies contradiction. Hence we have that \( \langle W_{t} \rangle \cap P_{t} = 1 \). Combining this with \( H \cap P_{t} = 1 \), we see that \( \langle Y, W_{t} \rangle_{G_{2t-1}} = \langle Y \rangle_{G_{2t-1}} \ast \langle W_{t} \rangle_{G_{2t-1}} = H \ast \langle W_{t} \rangle_{G_\infty} \). Since \( \langle g_{1}, \ldots, g_{m} \rangle_{G_\infty} \subseteq H \), we have that \( \langle g_{1}, \ldots, g_{m}, W_{t} \rangle_{G_\infty} \subseteq \langle g_{1}, \ldots, g_{m} \rangle_{G_\infty} \ast \langle W_{t} \rangle_{G_\infty} \).

We are now in a position to prove Theorem 3.1.

**Proof of Theorem 3.1** If there exists \( x \in X \) such that \( W \) contains none of \( x \) or \( x^{-1} \), then \( G \) is a non-trivial free product of groups both of which are not isomorphic to \( \mathbb{Z}_{2} \). Hence we may assume that \( X = \{ x_{1}, \ldots, x_{m} \} \) \((m > 1)\) and \( W \) contains either \( x_{i} \) or \( x_{i}^{-1} \) for all \( i \in \{1, \ldots, m\} \).

If \( W \) has no zero exponent sum \( \sigma_{x}(W) \) on \( x \) for all \( x \in X \), say \( \sigma_{x_{1}}(W) = \alpha \) and \( \sigma_{x_{2}}(W) = \beta \), then \( G \simeq \langle a^{\beta}, x_{2}, \ldots, x_{m} | R \rangle \subseteq E \), by the Magnus’ method
for Freiheitssatz, where $R = W^n(a^\beta, x_2, \ldots, x_m)$ and $E = \langle a, x_2, \ldots, x_m \mid R \rangle$.

Let $N = \mathcal{N}_F(x_2a^\alpha x_3 \cdots, x_m)$, where $F = \langle x_1, \ldots, x_m \rangle$. Then we have that $N \supset \mathcal{N}_F(R)$ and $N/\mathcal{N}_F(R) \simeq G_{\infty}$, where $G_{\infty}$ is as in (1), and so we may let $G_{\infty} = N/\mathcal{N}_F(R)$.

Let $F_G = \langle a^\beta, x_2, \ldots, x_m \rangle$ and $H = (N \cap F_G)/\mathcal{N}_{F_G}(R)$. Then we can easily see that $H$ can be isomorphically embedded in $G_{\infty}$ and that $G$ is a cyclic extension of $H$. Since $W_i \in H$, it follows from Lemma 3.6 that $H$ is LFP. Hence $KH$ is primitive for any field $K$ by Theorem 2.2. Since $G/H$ is cyclic, by [9, Theorem 1], we have that $KG$ is also primitive.

If $W$ has a zero exponent sum $\sigma_x(W)$ on $x$ for some $x \in X$, say $\sigma_{x_1}(W) = 0$, then we set $N = \mathcal{N}_F(x_2, x_3 \cdots, x_m)$. Since $N/\mathcal{N}_F(R) \simeq G_{\infty}$ and $G$ is a cyclic extension of $N/\mathcal{N}_F(R)$, the result is similarly obtained as above. This completes the proof of the theorem.

References


