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A Conjecture of Ducci Sequences and the Aspects

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Abstract

Since the last paper [1], we have argued a conjecture of Ducci sequences. Our aim of the issue is how to decide the recursive operational steps under certain conditions. And the results we got was somewhat surprising us because the conjecture or theorem of the recursive steps contains also a power of two as well as the well-known fact of Ducci sequences that ***if the number N of edges of the polygon is a power of two, the recursive procedure will always terminate in finite steps.*** However, the conjecture we offer faced a serious problem because we found a counter example for it after the paper was accepted. We eventually could patch the flaw and it therefore resulted in the conjecture with obscure conditions, although the general algebraic operations obviously make sense yet. We discuss the issues, aspects, and significant possibilities for the conjecture.

Keywords: Ducci sequences, polygon subtraction, recursive procedure, binary operation.

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1. Introduction

First of all, our initial conjecture is following: Let A, B, and C be positive integers on consecutive vertices on a polygon with N vertices, where N is a power of two. If the conditions A≤C≤B, A≥C≥B, B≤A≤C, or B≥A≥C hold throughout the recursive procedure defined below, then the procedure will always terminate by the power of two steps. In what follows we give basic concepts and fundamental properties, in order to formulate our basic conjecture on the above mentioned problem.

Definition 1.1. We denote by [x,y] the difference of two arguments x and y:

$$[x,y]=|x-y| \quad (1)$$

Let $a_1, a_2, a_3, \ldots, a_{N-1}, a_N$ be non-negative integers associated with the vertices of a polygon with N vertices. We transform this sequence as follows:

$$[a_1, a_2], [a_2, a_3], [a_3, a_4], \ldots, [a_{N-1}, a_N], \ldots, [a_N, a_1]$$

By applying this procedure recursively, can we say that all components of the sequence will eventually become zero? If so, how many steps will be needed for that?

Remark: a couple of members in square brackets are commutative.

Lemma 1.1. Let A, B, and C be non-negative integers on consecutive vertices on polygon. If A≤C≤B, A≥C≥B, B≤A≤C, or B≥A≥C, then

$$[[A, B], [B, C]]=[A, C] \quad (2)$$

Proof.

If A≤C≤B, then

$$[[A,B],[B,C]]=(B-A)-(B-C)=C-A=[A,C]$$

If A≥C≥B, then

$$[[A,B],[B,C]]=(A-B)-(C-B)=A-C=[A,C]$$

If B≤A≤C, then

$$[[A,B],[B,C]]=(C-B)-(A-B)=C-A=[A,C]$$

If B≥A≥C,

$$[[A,B],[B,C]]=(B-C)-(B-A)=A-C=[A,C]$$ □

Now, let us set the positive integers around all the vertices of square (; N=4). We assume that the condition of the lemma 1.1. will be satisfied throughout the recursive procedure. In what follows, we describe the numbers around the square at each step.

In some cases, we use ***much stronger condition***, which allows us to apply the equality [[A,B], [B. C]]=[A, C] even when A, B, and C are ***not*** the adjacent numbers.

Step1

$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_1]$

Step2

$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3], [[a_2, a_3], [a_3, a_4]]=[a_2, a_4], [[a_3, a_4], [a_4, a_1]]=[a_3, a_1], [[a_4, a_1], [a_1, a_2]]=[a_4, a_2]$

Step3

$[[a_1, a_3], [a_2, a_4]]$, $[[a_2, a_4], [a_3, a_1]]=[[a_1, a_3], [a_2, a_4]]$, $[[a_3, a_1], [a_4, a_2]]=[[a_1, a_3], [a_2, a_4]]$,
$[[a_4, a_2], [a_1, a_3]]=[[a_1, a_3], [a_2, a_4]]$

Step4

0 allout.

Therefore, we get the valid results, compatible with our conjecture. Likewise, we shall verify the recursive operations from N=2 to 8. The results are as follows.

First of all, if N=2 (a polygon with two vertices and double sides).

Step1

$[a_1, a_2], [a_2, a_1]$

Step2

$[[a_1, a_2], [a_2, a_1]]=[a_1, a_1]=0$

N=3,

Step1

$[a_1, a_2], [a_2, a_3], [a_3, a_1]$

Step2

$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$, $[[a_2, a_3], [a_3, a_1]]=[a_2, a_1]$, $[[a_3, a_1], [a_1, a_2]]=[a_3, a_2]$

Step3

$[[a_1, a_3], [a_2, a_1]]=[[a_2, a_1], [a_1, a_3]]=[a_2, a_3]$, $[[a_2, a_1], [a_3, a_2]]= [[a_3, a_2], [a_2, a_1]]=[a_3, a_1]$,
$[[a_3, a_2], [a_1, a_3]]= [[a_1, a_3], [a_3, a_2]]= [a_1, a_3]$

This sequence is nothing but a "cyclic shift" of the result of step 1. Therefore, the values never become all zeros, unless $a_1=a_2=a_3$.

N=4: it is as calculated above.

N=5,

Step1

$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_1]$

Step2

$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$, $[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$, $[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$, $[[a_4, a_5], [a_5, a_1]]=[a_4, a_1]$,
$[[a_5, a_1], [a_1, a_2]]=[a_5, a_2]$

Step3

$[[a_1, a_3], [a_2, a_4]]$, $[[a_2, a_4], [a_3, a_5]]$, $[[a_3, a_5], [a_4, a_1]]$, $[[a_4, a_1], [a_5, a_2]]$, $[[a_5, a_2], [a_1, a_3]]$

Step4 (***Remark: any integer*** at vertices follows the condition of lemma 1.1.)

$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]$($=[[A, B], [B, C]]$)$=[[a_1, a_3], [a_3, a_5]]$,
$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_1]]]=[[a_2, a_4], [a_4, a_1]]$

$[[[a_3, a_5], [a_4, a_1]], [[a_4, a_1], [a_5, a_2]]]=[[a_3, a_5], [a_5, a_2]]$

$[[[a_4, a_1], [a_5, a_2]], [[a_5, a_2], [a_1, a_3]]]=[[a_4, a_1], [a_1, a_3]]$

$[[[a_5, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_5, a_2], [a_2, a_4]]$

By applying the ***stronger condition***, these are converted to the following numbers:

$[a_1, a_5], [a_2, a_1], [a_3, a_2], [a_4, a_3], [a_5, a_4],$

Step5: using the results under the stronger condition, we obtain a cyclic permutation of the result of Step 2:

$[[a_1, a_5], [a_2, a_1]]=[[a_5, a_1], [a_1, a_2]]=[a_5, a_2]$, $[[a_2, a_1], [a_3, a_2]]=[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$,
$[[a_3, a_2], [a_4, a_3]]=[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$, $[[a_4, a_3], [a_5, a_4]]=[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$,
$[[a_5, a_4], [a_1, a_5]]=[[a_4, a_5], [a_5, a_1]]=[a_4, a_1]$

The values will never become all-zero, unless all components are identical.

N=6,

Step1

$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_6], [a_6, a_1]$

Step2

$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$, $[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$, $[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$, $[[a_4, a_5], [a_5, a_6]]=[a_4, a_6]$,

$[[a_5, a_6], [a_6, a_1]]=[a_5, a_1]$, $[[a_6, a_1], [a_1, a_2]]=[a_6, a_2]$

Step3

$[[a_1, a_3], [a_2, a_4]]$, $[[a_2, a_4], [a_3, a_5]]$, $[[a_3, a_5], [a_4, a_6]]$, $[[a_4, a_6], [a_5, a_1]]$, $[[a_5, a_1], [a_6, a_2]]$,
$[[a_6, a_2], [a_1, a_3]]$

Step4 (***Remark: any integer*** at vertices follows the condition of lemma 1.1, as well as the stronger condition.)

$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]=[[a_1, a_3], [a_3, a_5]]=[a_1, a_5]=[a_5, a_1]$,

$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_6]]]=[[a_2, a_4], [a_4, a_6]]=[a_2, a_6]=[a_6, a_2]$,

$[[[a_3, a_5], [a_4, a_6]], [[a_4, a_6], [a_5, a_1]]]=[[a_3, a_5], [a_5, a_1]]=[a_3, a_1]=[a_1, a_3]$,

$[[[a_4, a_6], [a_5, a_1]], [[a_5, a_1], [a_6, a_2]]]=[[a_4, a_6], [a_6, a_2]]=[a_4, a_2]=[a_2, a_4]$,

$[[[a_5, a_1], [a_6, a_2]], [[a_6, a_2], [a_1, a_3]]]=[[a_5, a_1], [a_1, a_3]]=[a_5, a_3]=[a_3, a_5]$,

$[[[a_6, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_6, a_2], [a_2, a_4]]=[a_6, a_4]=[a_4, a_6]$

Step5: Return to a cyclic permutation of the result of Step3.

N=7,

Step1

$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_6], [a_6, a_7], [a_7, a_1]$

Step2

$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$, $[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$, $[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$, $[[a_4, a_5], [a_5, a_6]]=[a_4, a_6]$,

$[[a_5, a_6], [a_6, a_7]]=[a_5, a_7]$, $[[a_6, a_7], [a_7, a_1]]=[a_6, a_1]$, $[[a_7, a_1], [a_1, a_2]]=[a_7, a_2]$

Step3

$[[a_1, a_3], [a_2, a_4]]$, $[[a_2, a_4], [a_3, a_5]]$, $[[a_3, a_5], [a_4, a_6]]$, $[[a_4, a_6], [a_5, a_7]]$, $[[a_5, a_7], [a_6, a_1]]$,
$[[a_6, a_1], [a_7, a_2]]$, $[[a_7, a_2], [a_1, a_3]]$

Step4 (***Remark:*** ***any integer*** at vertices follows the condition of lemma 1.1, as well as the stronger condition.)

$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]=[[a_1, a_3], [a_3, a_5]]=[a_1, a_5]$
$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_6]]]=[[a_2, a_4], [a_4, a_6]]=[a_2, a_6]$
$[[[a_3, a_5], [a_4, a_6]], [[a_4, a_6], [a_5, a_7]]]=[[a_3, a_5], [a_5, a_7]]=[a_3, a_7]$
$[[[a_4, a_6], [a_5, a_7]], [[a_5, a_7], [a_6, a_1]]]=[[a_4, a_6], [a_6, a_1]]=[a_4, a_1]$
$[[[a_5, a_7], [a_6, a_1]], [[a_6, a_1], [a_7, a_2]]]=[[a_5, a_7], [a_7, a_2]]=[a_5, a_2]$
$[[[a_6, a_1], [a_7, a_2]], [[a_7, a_2], [a_1, a_3]]]=[[a_6, a_1], [a_1, a_3]]=[a_6, a_3]$
$[[[a_7, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_7, a_2], [a_2, a_4]]=[a_7, a_4]$

Step5

$[[a_1, a_5], [a_2, a_6]]$, $[[a_2, a_6], [a_3, a_7]]$, $[[a_3, a_7], [a_4, a_1]]$, $[[a_4, a_1], [a_5, a_2]]$, $[[a_5, a_2], [a_6, a_3]]$,
$[[a_6, a_3], [a_7, a_4]]$, $[[a_7, a_4], [a_1, a_5]]$

Step6 (***Remark:*** ***any integer*** at vertices follows the condition of lemma 1.1, as well as the stronger condition.)

$[[[a_1, a_5], [a_2, a_6]], [[a_2, a_6], [a_3, a_7]]]=[[a_1, a_5], [a_3, a_7]]$,
$[[[a_2, a_6], [a_3, a_7]], [[a_3, a_7], [a_4, a_1]]]=[[a_2, a_6], [a_4, a_1]]$,
$[[[a_3, a_7], [a_4, a_1]], [[a_4, a_1], [a_5, a_2]]]=[[a_3, a_7], [a_5, a_2]]$,
$[[[a_4, a_1], [a_5, a_2]], [[a_5, a_2], [a_6, a_3]]]=[[a_4, a_1], [a_6, a_3]]$,
$[[[a_5, a_2], [a_6, a_3]], [[a_6, a_3], [a_7, a_4]]]=[[a_5, a_2], [a_7, a_4]]$,
$[[[a_6, a_3], [a_7, a_4]], [[a_7, a_4], [a_1, a_5]]]=[[a_6, a_3], [a_1, a_5]]$,
$[[[a_7, a_4], [a_1, a_5]], [[a_1, a_5], [a_2, a_6]]]=[[a_7, a_4], [a_2, a_6]]$

Step7

$[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_1]]]$, $[[[a_2, a_6], [a_4, a_1]], [[a_3, a_7], [a_5, a_2]]]$,
$[[[a_3, a_7], [a_5, a_2]], [[a_4, a_1], [a_6, a_3]]]$, $[[[a_4, a_1], [a_6, a_3]], [[a_5, a_2], [a_7, a_4]]]$,
$[[[a_5, a_2], [a_7, a_4]], [[a_6, a_3], [a_1, a_5]]]$, $[[[a_6, a_3], [a_1, a_5]], [[a_7, a_4], [a_2, a_6]]]$,
$[[[a_7, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]$

Step8 (***Remark:*** ***any integer*** at vertices follows the condition of lemma 1.1, as well as the stronger condition.)

$[[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_1]]], [[[a_2, a_6], [a_4, a_1]], [[a_3, a_7], [a_5, a_2]]]]$
$=[[[a_1, a_5], [a_3, a_7]], [[a_3, a_7], [a_5, a_2]]]= [[a_1, a_5], [a_5, a_2]]=[a_1, a_2]$,
$[[[[a_2, a_6], [a_4, a_1]], [[a_3, a_7], [a_5, a_2]]], [[[a_3, a_7], [a_5, a_2]], [[a_4, a_1], [a_6, a_3]]]]=[a_2, a_3]$,
$[[[[a_3, a_7], [a_5, a_2]], [[a_4, a_1], [a_6, a_3]]], [[[a_4, a_1], [a_6, a_3]], [[a_5, a_2], [a_7, a_4]]]]=[a_3, a_4]$,
$[[[[a_4, a_1], [a_6, a_3]], [[a_5, a_2], [a_7, a_4]]], [[[a_5, a_2], [a_7, a_4]], [[a_6, a_3], [a_1, a_5]]]]=[a_4, a_5]$,

$[[[[a_5, a_2], [a_7, a_4]], [[a_6, a_3], [a_1, a_5]]], [[[a_6, a_3], [a_1, a_5]], [[a_7, a_4], [a_2, a_6]]]]=[a_5, a_6]$,

$[[[[a_6, a_3], [a_1, a_5]], [[a_7, a_4], [a_2, a_6]]], [[[a_7, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]]=[a_6, a_7]$,

$[[[[a_7, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]], [[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_1]]]]=[a_7, a_1]$

Step9

Return to Step2.

N=8

Step1

$[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_5, a_6], [a_6, a_7], [a_7, a_8], [a_8, a_1]$

Step2

$[[a_1, a_2], [a_2, a_3]]=[a_1, a_3]$, $[[a_2, a_3], [a_3, a_4]]=[a_2, a_4]$, $[[a_3, a_4], [a_4, a_5]]=[a_3, a_5]$, $[[a_4, a_5], [a_5, a_6]]=[a_4, a_6]$,

$[[a_5, a_6], [a_6, a_7]]=[a_5, a_7]$, $[[a_6, a_7], [a_7, a_8]]=[a_6, a_8]$, $[[a_7, a_8], [a_8, a_1]]=[a_7, a_1]$, $[[a_8, a_1], [a_1, a_2]]=[a_8, a_2]$

Step3

$[[a_1, a_3], [a_2, a_4]]$, $[[a_2, a_4], [a_3, a_5]]$, $[[a_3, a_5], [a_4, a_6]]$, $[[a_4, a_6], [a_5, a_7]]$, $[[a_5, a_7], [a_6, a_8]]$,

$[[a_6, a_8], [a_7, a_1]]$, $[[a_7, a_1], [a_8, a_2]]$, $[[a_8, a_2], [a_1, a_3]]$

Step4 (***Remark: any integer*** at vertices follows the condition of lemma 1.1, as well as the stronger condition.)

$[[[a_1, a_3], [a_2, a_4]], [[a_2, a_4], [a_3, a_5]]]=[[a_1, a_3], [a_3, a_5]]=[a_1, a_5]$

$[[[a_2, a_4], [a_3, a_5]], [[a_3, a_5], [a_4, a_6]]]=[[a_2, a_4], [a_4, a_6]]=[a_2, a_6]$

$[[[a_3, a_5], [a_4, a_6]], [[a_4, a_6], [a_5, a_7]]]=[[a_3, a_5], [a_5, a_7]]=[a_3, a_7]$

$[[[a_4, a_6], [a_5, a_7]], [[a_5, a_7], [a_6, a_8]]]=[[a_4, a_6], [a_6, a_8]]=[a_4, a_8]$

$[[[a_5, a_7], [a_6, a_8]], [[a_6, a_8], [a_7, a_1]]]=[[a_5, a_7], [a_7, a_1]]=[a_5, a_1]$

$[[[a_6, a_8], [a_7, a_1]], [[a_7, a_1], [a_8, a_2]]]=[[a_6, a_8], [a_8, a_2]]=[a_6, a_2]$

$[[[a_7, a_1], [a_8, a_2]], [[a_8, a_2], [a_1, a_3]]]=[[a_7, a_1], [a_1, a_3]]=[a_7, a_3]$

$[[[a_8, a_2], [a_1, a_3]], [[a_1, a_3], [a_2, a_4]]]=[[a_8, a_2], [a_2, a_4]]=[a_8, a_4]$

Step5

$[[a_1, a_5], [a_2, a_6]]$, $[[a_2, a_6], [a_3, a_7]]$, $[[a_3, a_7], [a_4, a_8]]$, $[[a_4, a_8], [a_5, a_1]]$, $[[a_5, a_1], [a_6, a_2]]$,

$[[a_6, a_2], [a_7, a_3]]$, $[[a_7, a_3], [a_8, a_4]]$, $[[a_8, a_4], [a_1, a_5]]$

Step6 (***Remark: any integer*** at vertices follows the condition of lemma 1.1, as well as the stronger condition.)

$[[[a_1, a_5], [a_2, a_6]], [[a_2, a_6], [a_3, a_7]]]=[[a_1, a_5], [a_3, a_7]]$

$[[[a_2, a_6], [a_3, a_7]], [[a_3, a_7], [a_4, a_8]]]=[[a_2, a_6], [a_4, a_8]]$

$[[[a_3, a_7], [a_4, a_8]], [[a_4, a_8], [a_5, a_1]]]=[[a_3, a_7], [a_5, a_1]]$

$[[[a_4, a_8], [a_5, a_1]], [[a_5, a_1], [a_6, a_2]]]=[[a_4, a_8], [a_6, a_2]]$

$[[[a_5, a_1], [a_6, a_2]], [[a_6, a_2], [a_7, a_3]]]=[[a_5, a_1], [a_7, a_3]]$

$[[[a_6, a_2], [a_7, a_3]], [[a_7, a_3], [a_8, a_4]]]=[[a_6, a_2], [a_8, a_4]]$

$[[[a_7, a_3], [a_8, a_4]], [[a_8, a_4], [a_1, a_5]]]=[[a_7, a_3], [a_1, a_5]]$

$[[[a_8, a_4], [a_1, a_5]], [[a_1, a_5], [a_2, a_6]]]=[[a_8, a_4], [a_2, a_6]]$

Step7

$[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

$[[[a_2, a_6], [a_4, a_8]], [[a_3, a_7], [a_5, a_1]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

$[[[a_3, a_7], [a_5, a_1]], [[a_4, a_8], [a_6, a_2]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

$[[[a_4, a_8], [a_6, a_2]], [[a_5, a_1], [a_7, a_3]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

$[[[a_5, a_1], [a_7, a_3]], [[a_6, a_2], [a_8, a_4]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

$[[[a_6, a_2], [a_8, a_4]], [[a_7, a_3], [a_1, a_5]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

$[[[a_7, a_3], [a_1, a_5]], [[a_8, a_4], [a_2, a_6]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

$[[[a_8, a_4], [a_2, a_6]], [[a_1, a_5], [a_3, a_7]]]=[[[a_1, a_5], [a_3, a_7]], [[a_2, a_6], [a_4, a_8]]]$

Step8

0 allout

From these results noted above, we assumed a conjecture as follows.

Conjecture. ***The recursive procedure by the binary operation [x, y] over the polygon whose number of vertices is a power of two will always terminate by the power of two steps, provided that the stronger condition, that is, in every step of the calculation, we can assume that [[A,B], [B, C]]=[A, B].***

For example, under the stronger condition, if N=2, 4, and 8, then the operation terminates by *2, 4, and 8 steps* as calculated above.

In the beginning, we assumed only the condition described in lemma 1.1. However, we detected a counter example to the case of N=8 after the vast data verification. It is as follows.

	1	2	3	4	5	6	7	8
S	0	0	0	0	1	1	3	3
1	0	0	0	1	0	2	0	3
2	0	0	1	1	2	2	3	3
3	0	1	0	1	0	1	0	3
4	1	1	1	1	1	1	3	3
5	0	0	0	0	0	2	0	2
6	0	0	0	0	2	2	2	2
7	0	0	0	2	0	0	0	2
8	0	0	2	2	0	0	2	2
9	0	2	0	2	0	2	0	2

10	2	2	2	2	2	2	2	2
11	0	0	0	0	0	0	0	0

In this case, the recursive operation takes 11 steps. Throughout this process, the condition of lemma 1.1 is always satisfied, but the stronger condition is violated at Step 4. We do not know any easy way to check the condition of lemma 1.1 by seeing only the starting sequence of non-negative integers. It will be much more difficult to check the stronger condition, since it involves the calculation described above. However, if we limit the problem to binary numbers (0, 1), the stronger condition will be automatically satisfied, since for any A, B, and C, we have A≤C≤B or B≥A≥C for B=1, and B≤C≤A or C≥A≥B for B=0. The following is an elementary proof for the fact that if the numbers are limited to binary (0 and 1) and the number N of vertices is a power of 2, the recursive procedure will eventually terminate by the power of two steps.

Proof.

For $x, y \in Z_2=\{0, 1\}$, the operation [x, y] on the finite field Z_2 is equivalent with the addition in modulo 2 as follows.

$$|x-y| = x+y \in \{0,1\} \pmod{2} \qquad (3)$$

From the equation (3), the recursive operation to the sequence a_j's can be written as follows.

$$\Delta(a,b,c,\ldots,e,f) = (a+b, b+c, \ldots, e+f, f+a),$$

$$\Delta^2(a,b,c,\ldots,e,f) = (a+2b+c, b+2c+d, \ldots, f+2a+b),$$

$$= (a+c, b+d, \ldots, e+f, f+b),$$

$$\Delta^3(a,b,c,\ldots,e,f) = (a+3b+3c+d, \ldots, f+3a+3b+c),$$

$$= (a+b+c+d, \ldots, f+a+b+c),$$

$$\Delta^4(a,b,c,\ldots,e,f) = (a+4b+6c+4d+e, \ldots, f+4a+6b+4c+d),$$

$$= (a+e, \ldots, f+d),$$

$$\vdots$$

Δ^k denotes k-times recursive operation. From the results above, we can find the binomial theorem from each member of Δ^k. Since each member in modulo 2 results in that those applied binomial coefficients in each member can be replaced by 1 if the coefficient is odd or by 0 if the coefficient is even. Thus, each member of Δ^k equals to summation of $k+1$ element.

Fact 1.1. If $m = 2^k - 1$, all the applied binomial coefficient ${}_mC_j$ are odd number. And if $N = 2^k$, any member of $\Delta^{k-1}\alpha$ is

$$s = a_0 + a_1 + \ldots + a_{N-1}$$

where $\alpha = (a_0, a_1, \ldots, a_{N-1})$, $a_j \in Z_2$ and s is summation of all elements at the initial sequence.

Proof of Fact 1.1. If $N = 2^k$, any of the binomial coefficient for $m = N-1 = 2^k - 1$ is odd number. As noted above, any element of $\Delta^{N\text{-}1}\alpha$ is summation of α. For example, if $N = 4$,

$$\Delta^3(a,b,c,\dots e,f) = (a+3b+3c+d,\dots b+3c+3d+a, c+3d+3a+b, d+3a+3b+c),$$

$$= (a+b+c+d, b+c+d+a, c+d+a+b, d+a+b+c)$$

Theorem 1.1. If $N = 2^k$ for the pattern on the field of Z_2, $\alpha = (a_0, a_1, \dots a_{N-1})$, and $a_j \in Z_2$, the recursive operation terminates by N steps at most.

Proof.

$$\Delta^N \alpha = \Delta \cdot \Delta^{N-1}\alpha = \Delta(s, s, \dots, s) = 0$$ □

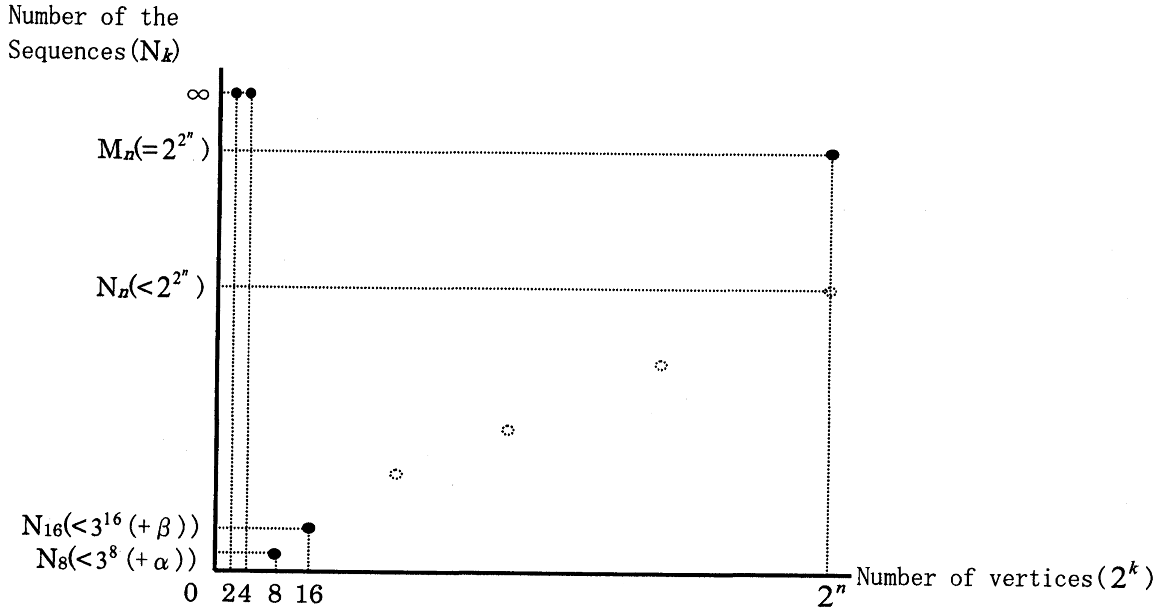
So, we could say the conjecture in other words:

Assume that a recursive operational steps of Ducci sequences with a power of two vertices of polygon has upper bound at the power of two. How is the distribution of the number of such the sequences terminated at the upper bound?

For example, as shown in the graph below, such a distribution mentioned above could be assumed with our discussion in this paper. In this graph, equivalence integers that are elements of any sequence ***should be*** identified as (0,1,2,…,n). Because, as an example, (0,1,1,0)~(2,4,4,2) in the binary recursive operation. Let the number of the sequences that consists only of the elements be and terminates ***by*** the power of two be Mn and ***at*** the power of two Nn , and the unknown number of unknown sequences through our data analyses based on our discussion be α and β. The graph assumed by data verification suggests an equation as follows,

$$\lim_{n\to\infty} \frac{\log_2 N_n}{2^n} \prec 1$$

where N_n is the number the sequences.



Distribution of the Sequences (assumption, by single logarithmic chart)

The results of data verifications are stored in our SkyDrive:
https://skydrive.live.com/redir?resid=F8889838437A1529!731&authkey=!APDxSTqxse6klL4

About supplementary data of the algebraic recursive operations (N=16), click on the link below:
https://skydrive.live.com/redir?resid=F8889838437A1529!463&authkey=!ABdW_FCVJExSuig

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