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Improvement on Searching Minimum Dominating Vertex Sets of Complete Grid Graphs using an IP Solver

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Abstract

In the game of Go, a ren is a group of connected stones of the same color. In a legal stone arrangement on a Go board, one of stones in a ren must be adjacent to an empty point called a liberty. The maximum number of rens in legal stone arrangements on a 19×19 Go board was obtained with an IP (Integer Programming) solver. Finding the number is equivalent to finding the size of a minimum dominating set of the 19×19 grid graph. A formula that, for any integers m and n with $16 \leqq m \leqq n$, calculate the size of a minimum dominating set of $m \times n$ grid graphs has been presented. To prove the validity of the formula, an enormous amount of computation was carried out. In this paper, the method of finding minimum dominating sets of complete grid graphs using an IP solver is developed. The size of a minimum dominating set of 20×20 grid graph can be obtained using an IP solver.

KEYWORDS. minimum dominating set, complete grid graph, Integer Programming solver, game of Go.

1 Introduction

A *ren* is a group of connected stones of the same color. In a legal stone arrangement on a Go board, one of stones in a ren must be adjacent to an empty point called a *liberty*. Miyashiro et al found the maximum number of rens in legal stone arrangements on a 19×19 Go board using an IP (Integer Programming) solver[4]. They also shown that problem of finding a legal stone arrangement of the maximum number of rens is equivalent to finding the minimum dominating sets of complete grid graphs. In fact, the 19×19 Go board is corresponding to complete grid graph $P_{19} \times P_{19}$. We also have found a minimum dominating set of complete grid graph $P_{20} \times P_{20}$ using an IP solver in two strategies.

The *infinite grid graph* (V, E) is a graph whose vertex set V is the set of all pairs of integers, that is all grid points on the x-y plane, and whose edge set E is the set of all pairs of vertices corresponding to segments of length 1 that connects two grid points. A *grid graph* is a finite induced subgraph of the infinite grid graph. Furthermore, A *complete grid graph* is a grid graph whose vertex set is corresponding to a set of grid points in a rectangle whose edges are parallel to the x or y axis. An $m \times n$ grid graph is a complete grid graph, denoted by $P_m \times P_n$, whose vertices compose a rectangular grid with m rows and n columns. Let $\gamma_{m,n}$ denote the size of a minimum dominating set of $P_m \times P_n$.

As a matter of fact, D. Gonçalves et al proved that the following formula is valid for every integer pair (m, n) satisfying $16 \leqq m \leqq n$[3]:

$$\gamma_{m,n} = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4 \ .$$

Before that, Chang proved that the right hand side expression is an upper bound on $\gamma_{m,n}$. In a later section, we provide an explicit instance of dominating set of $P_m \times P_n$ with size $\lfloor (m+2)(n+2)/5 \rfloor - 4$ for every integer pair (m, n) satisfying $16 \leqq m \leqq n$.

In the next section, we shall describe two strategies using an IP solver in which a minimum dominating set of $P_{20} \times P_{20}$ is derived. In Section 3, we shall provide an explicit instance of dominating set of $P_m \times P_n$ with size $\lfloor (m+2)(n+2)/5 \rfloor - 4$ for every integer pair (m, n) satisfying $16 \leqq m \leqq n$. In Section 4, we shall make concluding remarks.

2 Two Strategies to Solve Minimum Dominating Set Problems of complete grid graphs

Minimum dominating set problems of complete grid graphs can be converted into integer programming problems straightforwardly. Each vertex $v(i, j)$ of $P_m \times P_n$ is corresponding to binary variable $x(i, j)$. Each $x(i, j)$ is interpreted as follows:

$$x(i,j) = 1 \ \Leftrightarrow \ v(i,j) \text{ is a dominating vertex.}$$

The problem is minimization one. The objective function is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x(i,j).$$

The constraints for variables $x(i, j)$ are as follows:

$$x(1,1) + x(2,1) + x(1,2) \geqq 1,$$

$$x(1,n) + x(2,n) + x(1,n-1) \geqq 1,$$

$$x(m,1) + x(m-1,1) + x(m,2) \geqq 1,$$

$$x(m,n) + x(m-1,n) + x(m,n-1) \geqq 1;$$

for $1 < i < m$,

$$x(i,1) + x(i,2) + x(i-1,1) + x(i+1,1) \geqq 1,$$
$$x(i,n) + x(i,n-1) + x(i-1,n) + x(i+1,n) \geqq 1;$$

for $1 < j < n$,

$$x(1,j) + x(2,j) + x(1,j-1) + x(1,j+1) \geqq 1,$$
$$x(m,j) + x(m-1,j) + x(m,j-1) + x(m,j+1) \geqq 1;$$

for $1 < i < m$ and $1 < j < n$,

$$x(i,j) + x(i-1,j) + x(i+1,j) + x(i,j-1) + x(i,j+1) \geqq 1.$$

As above, each constraints is corresponding to a vertex of the complete grid graph, and the vertices are classified into three parts, (1) the four corners, (2) the border except the four corners, and (3) the rest, that is the interior vertices.

The IP solver used in this research is SCIP[1], and the performance of the computer we used is as follows:

CPU: Intel® Core™ i7 CPU 960 (Clock frequency 3.20GHz),
Memory size: 12GB.

SCIP cannot solve the problems created above for $m = n = 20$ on our hardware in several days. We deduce from the output of SCIP that the difficulty of the computation is chiefly due to a huge number of optimum or nearly optimum solutions. Thus, determining $\gamma_{19,19}$ by solving IP problems directly is limit to our computing environment.

Since, for any integers m and n satisfying $16 \leqq m \leqq n$, we can present a dominating set of $P_m \times P_n$ with $\lfloor (m+2)(n+2)/5 \rfloor - 4$ vertices, see the next section, to prove $\gamma_{20,20} = 92$, it suffices to show that no dominating set of $P_{20} \times P_{20}$ of size 91 exists. We shall show it in two strategies. The first one is partitioning $P_{20} \times P_{20}$ into two $P_{20} \times P_{10}$'s, creating IP problems corresponding to the two complete grid graphs, solving the two IP problems, and merging the solutions. The second one is setting a number of areas in $P_{20} \times P_{20}$, adding constraints on the number of dominating vertices in the areas, and solving the modified IP problem.

2.1 The Strategy of Dividing the IP Problem

We first partition $P_{20} \times P_{11}$ into three parts A, B, and C as in Figure 1. Part A, B, and C are $P_{20} \times P_9$, $P_{20} \times P_1$, and $P_{20} \times P_1$, respectively. For every vertex in part C, any constraint that forces the vertex to be covered is not assigned.

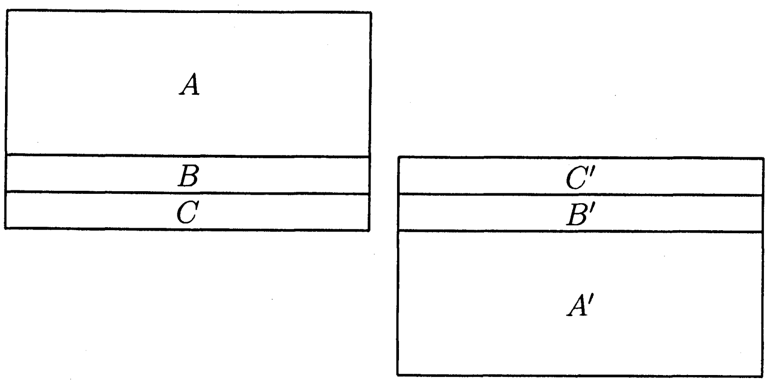

Figure 1: Partitioning $P_{20} \times P_{11}$ into three parts.

Let $N(A \cup B)$, $N(B)$, and $N(C)$ denote the number of dominating vertices in $A \cup B$, B, and C, respectively. We obtain the following values by solving corresponding IP problems.

(a) The minimum value of $N(A \cup B)$.

(b) The minimum and maximum values of $N(B)$ under the constraint $N(A \cup B) = v$, for given v.

(c) The minimum and maximum values of $N(C)$ under the constraint $N(A \cup B) = w$, for given w.

Since the value of (a) is 43, $N(A \cup B) \geqq 43$ holds. If $\gamma_{20,20} \leqq 91$, then there must be a feasible solution such that $N(A \cup B) = 43$, 44, or 45. From the value of (c) for $w = 43$, $N(C) \geq 14$ follows. From the value of (b) for $v = 48$, $N(B) \leq 9$ follows. It is therefore impossible that both $N_{\text{upper}} = 43$ and $N_{\text{lower}} = 48$ hold simultaneously, where N_{upper} and N_{lower} are the number of dominating vertices in upper and lower half of $P_{20} \times P_{20}$, respectively. By similar arguments, it follows that it is impossible for both $N_{\text{upper}} = 44$ and $N_{\text{lower}} = 47$ to hold simultaneously. Furthermore, if there exists a dominating set of $P_{20} \times P_{20}$ consisting of 91 vertices then there are two feasible solutions X and Y such that X satisfies $N(A \cup B) = 45$, $3 \leq N(B) \leqq 4$, and $5 \leq N(C) \leq 6$, Y satisfies $N(A \cup B) = 46$, $3 \leq N(B) \leq 4$, and $5 \leq N(C) \leq 6$, the part B of X is equal to the part C of Y, and the part C of X is equal to the part B of Y. To find X and Y above, we had SCIP collect all feasible solutions for eight cases of constraints: four constraints $N(A \cup B) = 45$, $N(B) \in \{3, 4\}$, and $N(C) \in \{5, 6\}$, and other four constraints $N(A \cup B) = 46$, $N(B) \in \{5, 6\}$, and $N(C) \in \{3, 4\}$. The total computation time is at most 13 hours, and the total number of feasible solutions is 84111.

2.2 The Strategy of Adding Constraints

Let N, S, W, and E be the number of dominating vertices in the upper, lower, left, and right half of $P_{20} \times P_{20}$, respectively. First, we obtain IP problem P by adding the following constraints to the IP problem directly corresponding to the minimum dominating set problem of $P_{20} \times P_{20}$:

$$N = W = 45 \quad \text{and} \quad S = E = 46.$$

Next, for each row r_i, we find the maximum and the minimum values of the number of dominating vertices in r_i by solving the corresponding IP problems. For each column c_j, we similarly find the maximum and the minimum values of the number of dominating vertices in c_j. Next, we obtain IP problem P' by adding the constraints derived from the maximum and the minimum values above to IP problem P. Finally, we verify that there is no feasible solution in P' by solving it using an IP solver, namely SCIP.

The total computation time is about 32 hours.

3 Construction of Dominating Sets of Size Equal to the Given Upper Bound

Let $f(m, n)$ denote the function defined as:

$$f(m, n) = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4 .$$

Chang proved that $f(m,n)$ is an upper bound on the size of a minimum dominating set of $P_{20} \times P_{20}$[2]. In this section, we explicitly describe the structure of a dominating set of $P_m \times P_n$ of size equal to $f(m,n)$ for arbitrary integer pair (m,n) satisfying $16 \geqq m \geqq n$.

Let $P_m \times P_n$ be considered as a subgraph of the infinite grid graph. Fill the infinite grid graph with X-pentominoes, and let $S_{m,n}(k)$ denote the set of vertices at the center of a X-pentomino in $P_m \times P_n$, where $k \in \{0,1,2,3,4\}$ represent the position of the most left vertex in the highest row in $S_{m,n}(k)$. For example, the set of vertices at $\times$ marks in Figure 2 is denoted by $S_{m,n}(0)$ All of the vertices of $P_m \times P_n$ except the ones on the border are dominated by $S_{m,n}(k)$.

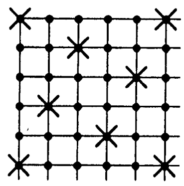

Figure 2: Example of $S_{m,n}(0)$.

If $k \neq 1$ then two undominated vertices vanish by moving and adding a dominating vertex on the border near the NW corner. For example, if $k = 0$, then two undominated vertices near the NW corner vanish by moving the dominating vertex at the NW corner one vertex below and adding a dominating vertex two vertices right from the NW corner, see Figure 3. We omit the case of $k \in \{2,3,4\}$.

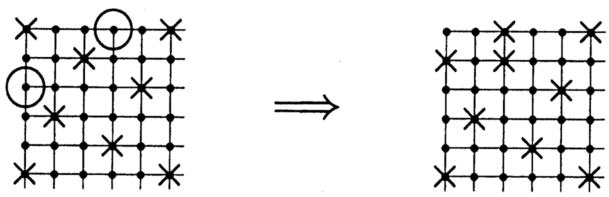

Figure 3: Deleting undominated vertices near the NW corner.

By applying the process above to all corners, and adding undominated vertices on the border to $S_{m,n}(k)$ as dominating vertices, we obtain a dominating set of $P_m \times P_n$, denoted by $\overline{S_{m,n}(k)}$. For integers m and n with $m \geqq 16$ and $n \geqq 16$, let $k(m,n)$ denote the minimum element in $\{0,1,2,3,4\}$ such that

$$\left|\overline{S_{m,n}(k(m,n))}\right| = \min\left\{\left|\overline{S_{m,n}(k)}\right| \mid k \in \{0,1,2,3,4\}\right\}.$$

We write $\overline{S_{m,n}(k(m,n))}$ as $S_{m,n}$ for short. By counting the size of the dominating set $S_{m,n}$, we have

$$|S_{m,n}| = f(m,n) \tag{1}$$

for any m and n in $\{16,17,18,19,20\}$. Let $g(m,n)$ denote the function defined as

$$g(m,n) = \left\lfloor \frac{m}{5} \right\rfloor n + m \left\lfloor \frac{n}{5} \right\rfloor - 5 \left\lfloor \frac{m}{5} \right\rfloor \left\lfloor \frac{n}{5} \right\rfloor.$$

Since $\left|\overline{S_{m,n}(k)}\right| - g(m,n)$, where $k \in \{0,1,2,3,4\}$, and $f(m,n) - g(m,n)$ are all periodic in the both variables m and n with period 5, it follows that equation (1) holds for any integers m and n with $m \geqq 16$ and $n \geqq 16$.

4 Concluding Remarks

It seems to be intractable to carry out computations used to prove $\gamma_{m,n} = f(m,n)$ by Gonçalves et al using an IP solver in practical time[3], where $f(m,n)$ is defined in the previous section. It will therefore be an interesting challenge to prove above equation using only an IP solver. Notice that the performance of a notable commercial IP solver is much higher than the one of SCIP. We will use a commercial IP solver, e.g. ILOG CPLEX, to address the challenge.

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