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On Algebraic Structures of Petri Net Morphisms based on Place Connectivity

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1 Introduction

A Petri net is a useful mathematical model applied to descriptions of various parallel processing systems. So far, some types of morphisms related to Petri nets (or condition/event net) have been studied in terms of the category theory, in order to investigate the relationship between different Petri nets and understand the concurrency in other computation models [4][10].

Usually such a morphism is defined based on connection of transitions and thier nearby places. It is one of necessary conditions that such morphisms commute with the transition function of a Petri net.

Studying how the structure of Petri nets have an effect on Petri net languages and codes, we often realize that the ratio between the number of tokens in a place and the weights of edges connected to the place is important. We give our definition of morphims between Petri nets focusing on the connection state/level of edges which come in or go out a place. This is an extension of an automorphism which we used to introduce to a net in [5][6].

After summarising the monoid of all surjective morphisms of a Petri net and ideals in the monoid, we state the decomposition of automorphism group $G = \mathbf{Aut}(\mathcal{P})$ of a Petri net $\mathcal{P}$ into $G = KN = NK$, where N is a kind of normal subgroup of G.

2 Preliminaries

Here we give our definition of morphisms of a Petri net and state the properties of some monoids composed of these morphisms.

2.1 Petri Nets and Morphisms

In this section, we give definitions and fundamental properties related to Petri nets. We denote the set of all nonnegative integers by $\boldsymbol{N}_0$, that is, $\boldsymbol{N}_0 = \{0, 1, 2, \ldots\}$.

First of all, a Petri net is viewed as a particular kind of directed graph, together with an initial state μ_0, called the *initial marking*. The underlying graph N of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called *places* and *transitions*, where arcs are either from a place to a transition or from a transition to a place.

DEFINITION 2.1 (Petri net) A *Petri net* is a 4-tuple (P, T, W, μ_0) where
(1) $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places,
(2) $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions,
(3) $W : \boldsymbol{E}(P,T) \to \{0, 1, 2, 3, \ldots\}$, i.e.,$W \in \boldsymbol{N}_0{}^{\boldsymbol{E}(P,T)}$, is a *weight function*, where $\boldsymbol{E}(P,T) = (P \times T) \cup (T \times P)$,
(4) $\mu_0 : P \to \{0, 1, 2, 3, \ldots\}$, i.e., $\mu_0 \in \boldsymbol{N}_0{}^P$, is the initial marking,
(5) $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

A Petri net structure (net, for short) $N = (P, T, W)$ without any specific initial marking is denoted by N, a Petri net with a given initial marking μ_0 is denoted by (N, μ_0). □

In the graphical representation, the places are drawn as circles and the transitions are drawn as bars or boxes. Arcs are labeled with their weights(positive integers), where a k-weighted arc can be interpreted as the set of k parallel arcs. Labels for unity weights are usually omitted. A marking (state) assigns a nonnegative integer k to each place. If a marking assigns a nonnegative integer k to a place p, we say that p is *marked with* k tokens. Pictorially, we put k black dots (tokens) in place p. A marking is denoted by μ, an n-dimensional row vector, where n is the total number of places. The i-th component of μ, denoted by $\mu(p_i)$, is the number of tokens in the i-th place p_i.

EXAMPLE 2.1 Fig. 1 shows a graphical representation of a Petri net $\mathcal{P} = (P, T, W, \mu_0)$. $P = \{\mathbf{a}, \mathbf{b}\}$ and $T = \{\mathbf{t}\}$. $(\mathbf{a}, \mathbf{t})$ and $(\mathbf{t}, \mathbf{b})$ are arcs of weights 2 and 1 respectively. $(\mathbf{t}, \mathbf{a})$ and $(\mathbf{b}, \mathbf{t})$ are arcs of weight 0, which are not usually drawn in the picture. Note that the weight of $(\mathbf{t}, \mathbf{b})$ is omitted since it is unity. That is, $W(\mathbf{a}, \mathbf{t}) = 2, W(\mathbf{b}, \mathbf{t}) = 1, W(\mathbf{t}, \mathbf{a}) = W(\mathbf{b}, \mathbf{t}) = 0$. The initial marking μ_0 with $\mu_0(\mathbf{a}) = 3, \mu_0(\mathbf{b}) = 0$ is often written like a row vector $\mu_0 = (3, 0)$. □

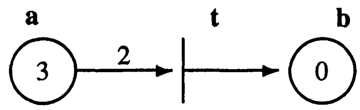

Figure 1. Graphical representation of a Petri net

Now we introduce a Petri net morphism based on place connectivity. We denote the set of all positive rational numbers by $\boldsymbol{Q}_+$.

DEFINITION 2.2 Let $\mathcal{P}_1 = (P_1, T_1, W_1, \mu_1)$ and $\mathcal{P}_2 = (P_2, T_2, W_2, \mu_2)$ be Petri nets. Then a triple $(f, (\alpha, \beta))$ of maps is called a *morphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if the maps $f : P_1 \to \boldsymbol{Q}_+, \alpha : P_1 \to P_2$ and $\beta : T_1 \to T_2$ satisfy the condition that for any $p \in P_1$ and $t \in T_1$,

$$\begin{aligned} W_2(\alpha(p), \beta(t)) &= f(p)W_1(p, t), \\ W_2(\beta(t), \alpha(p)) &= f(p)W_1(t, p), \\ \mu_2(\alpha(p)) &= f(p)\mu_1(p). \end{aligned} \tag{2.1}$$

In this case we write $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$. □

The morphism $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ is called *injective* (resp. *surjective*) if both α and β are injective (resp. surjective). In particular, it is called an *isomorphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if it is injective and surjective. Then $\mathcal{P}_1$ is said to be *isomorphic* to $\mathcal{P}_2$ and we write $\mathcal{P}_1 \simeq \mathcal{P}_2$. Moreover, in case of $\mathcal{P}_1 = \mathcal{P}_2$, an isomorphism is called an *automorphism* of $\mathcal{P}_1$. By $\mathbf{Aut}(\mathcal{P})$ we denote the set of all the automorphisms of $\mathcal{P}$.

For Petri nets $\mathcal{P}_1$ and $\mathcal{P}_2$, we write $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ if there exists a surjective morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$. The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order is regarded as an order by identifying isomorphisms.

PROPOSITION 2.1 *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be Petri nets. Then,*

(1) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_1$.

(2) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ *and* $\mathcal{P}_2 \sqsupseteq \mathcal{P}_1 \iff \mathcal{P}_1 \simeq \mathcal{P}_2$.

(3) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ *and* $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$ *imply* $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$. □

DEFINITION 2.3 (Similar) *Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Two places $p, q \in P$ are said to be similar if there exists some positive rational number r such that $\mu(p) = r\mu(q)$, $W(q, t) = rW(p, t)$ and $W(t, q) = rW(t, p)$ for all $t \in T$. Two transitions $s, t \in T$ are said to be similar if $W(p, s) = W(p, t)$ and $W(s, p) = W(t, p)$ for all $p \in P$.* □

The similarity defined above is obviously an equivalence relation on $P \cup T$. We denote this relation by $\sim_{\mathcal{P}}$ or simply $\sim$ and the $\sim_{\mathcal{P}}$-class of a place or a transition u by $C(u)$. A place (resp. a transition) is said to be *isolated* if it has no connection to any transitions (resp. any places). Especially, a place p is 0-isolated if it is isolated and $\mu(p) = 0$. Note that two 0-isolated places p and q are similar because for any positive rational number r $\mu(p) = 0 = r\mu(q)$, $W(q,t) = 0 = rW(p,t)$ and $W(t,q) = 0 = rW(t,p)$ for all $t \in T$.

2.2 Monoids $\mathcal{S}$ of Surjective Morphisms of Petri Nets

We introduce a composition of morphisms; all the morphisms between Petri nets form a monoid under this composition.

Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1,2,3)$ be Petri nets, $(f,\ (\alpha,\beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ and $(g,\ (\gamma,\delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be morphisms. Then,

$$\begin{aligned}
W_3(\gamma(\alpha(p)), \delta(\beta(t))) &= g(\alpha(p))W_2(\alpha(p), \beta(t)) \\
&= g(\alpha(p))f(p)W_1(p,t), \\
W_3(\delta(\beta(t)), \gamma(\alpha(p))) &= g(\alpha(p))W_2(\beta(t), \alpha(p)) \\
&= g(\alpha(p))f(p)W_1(t,p), \\
\mu_3(\gamma(\alpha(p))) = g(\alpha(p))\mu_2(\alpha(p)) &= g(\alpha(p))f(p)\mu_1(p)
\end{aligned}$$

hold.

In this manuscript, by writing compositions of maps like $g \circ \alpha$, $\gamma \circ \alpha$ and $\delta \circ \beta$ in the form of multiplications like αg, $\alpha\gamma$ and $\beta\delta$ respectively, the *composition* of morphisms is written as $(f \otimes_{P_1} (\alpha g), (\alpha\gamma, \beta\delta))$, where $\otimes_{P_1}$ is the operation in the following fundamental commutative group $({\boldsymbol{Q}_+}^{P_1}, \otimes_{P_1})$.

The set $({\boldsymbol{Q}_+}^{P}, \otimes_P)$ of all maps from a set P to $\boldsymbol{Q}_+$ forms a commutative group under the operation $\otimes_P$ defined by $f \otimes_P g : p \mapsto f(p)g(p)$. $1_{\otimes_P} : P \to \boldsymbol{Q}_+ : p \mapsto 1$ is the identity and $f^{-1} : P \to \boldsymbol{Q}_+ : p \mapsto 1/f(p)$ is the inverse of a $f \in {\boldsymbol{Q}_+}^P$. Whenever it does not cause confusion, we write $\otimes$ instead of $\otimes_P$. Immediately we obtain the following lemma.

LEMMA 2.1 Let α and β be arbitrary maps on P and $f, g : P \to \boldsymbol{Q}_+$. Then the following equations are true.

(1) $(\alpha\beta)f = \alpha(\beta f)$.
(2) $\alpha(f \otimes g) = (\alpha f) \otimes (\alpha g)$.
(3) $\alpha \mathbf{1}_\otimes = \mathbf{1}_\otimes$.
(4) $(\alpha f) \otimes (\alpha f^{-1}) = 1_\otimes$.
(5) $(\alpha f)^{-1} = \alpha f^{-1}$. □

For a surjective morphim $x : \mathcal{P}_1 \to \mathcal{P}_2$, $\mathcal{P}_1$ is called the domain of x, denoted by $Dom(x)$, and $\mathcal{P}_2$ is called the image(or range) of x, denoted by $Im(x)$.

We denote the set of all surjective morphisms between two Petri nets and a zero element 0, by $\mathcal{S}_0$. Especially $Dom(0) = Im(0) = \emptyset$. $\mathcal{S}_0$ forms a semigroup, equipped with the multiplication of $x = (f,\ (\alpha,\beta))$ and $y = (g,\ (\gamma,\delta))$:

$$x \cdot y \stackrel{\text{def}}{=} \begin{cases} (f \otimes_{\mathcal{P}} \alpha\, g, (\alpha\gamma, \beta\delta)) & \text{if} \quad Im(x) = Dom(y). \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{S} = \mathcal{S}_0 \cup \{1\}$ is the monoid obtained from $\mathcal{S}_0$ by adjoining an (extra) identity 1, that is, $1 \cdot s = s \cdot 1 = s$ for all $s \in \mathcal{S}_0$ and $1 \cdot 1 = 1$.

3 Ideals in the monoid $\mathcal{S}$

In this section we consider ideals and Green's relations on the monoid $\mathcal{S}$. At first, we consider some properties of the structure of the automorphism group of a Petri net $\mathcal{P}$.

3.1 Green's equivalences on the monoid $\mathcal{S}$

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a monoid M, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$\begin{aligned} &x\mathcal{L}y \iff Mx = My, \\ &x\mathcal{R}y \iff xM = yM, \\ &x\mathcal{J}y \iff MxM = MyM, \\ &\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \\ &\mathcal{D} = (\mathcal{L} \cup \mathcal{R})^*, \end{aligned}$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. Mx (resp. xM) is called the *principal left* (resp. *right*) *ideal generated by* x and MxM the it principal (two-sided) ideal generated by x. Then, the following facts are generally true[2, 1].

FACT 1 *The following relations are true.*

$$\begin{aligned} &(1)\, \mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L} \\ &(2)\, \mathcal{H} \subset \mathcal{L} \ (\textit{resp.}\mathcal{R}) \ \subset \mathcal{D} \subset \mathcal{J} \end{aligned}$$

FACT 2 *An $\mathcal{H}$-class of a monoid M is a group if and only if it contains an idempotent.*

Now we consider the case of $M = \mathcal{S}$ in the rest of the maniscript. The following lemma is obviously true.

LEMMA 3.1 *Let $x : \mathcal{P}_1 \to \mathcal{P}_2$, $y : \mathcal{P}_3 \to \mathcal{P}_4 \in \mathcal{S}$. Then,*
(1) $x\mathcal{S} \subset y\mathcal{S} \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \sqsubseteq \mathcal{P}_4$.
(2) $\mathcal{S}x \subset \mathcal{S}y \Longrightarrow \mathcal{P}_1 \sqsubseteq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$.
(3) $x\mathcal{S} = y\mathcal{S} \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \simeq \mathcal{P}_4$.
(4) $\mathcal{S}x = \mathcal{S}y \Longrightarrow \mathcal{P}_1 \simeq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$. □

Note that any reverses of the implications above are not necessarily true.

PROPOSITION 3.1 The following conditions are equivalent.
(1) H is an $\mathcal{H}$-class and a group.
(2) $H = \mathbf{Aut}(\mathcal{P})$ for some Petri net $\mathcal{P}$. □

PROPOSITION 3.2 *On the monoid $\mathcal{S}$, $\mathcal{J} = \mathcal{D}$.* □

3.2 Intersection of principal ideals

The aim here is that for given $x, y \in \mathcal{S}$ we find a elements z such that $\mathcal{S}x \cap \mathcal{S}y = \mathcal{S}z$ (resp. $x\mathcal{S} \cap y\mathcal{S} = z\mathcal{S}$). $x\mathcal{S} \cap y\mathcal{S} = \{0\}$ (resp. $\mathcal{S}x \cap \mathcal{S}y = \{0\}$) is a trivial case(i.e., $z = 0$). We should only consider the non-trivial case.

LEMMA 3.2 *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1, 2, 3)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_3$, $y = (g, (\gamma, \delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}$. If $|\alpha^{-1}(p)| \leq |\gamma^{-1}(p)|$ and $|\beta^{-1}(t)| \leq |\delta^{-1}(t)|$ for any $p \in P_3$ and $t \in T_3$, then $\mathcal{S}y \subset \mathcal{S}x$.* □

LEMMA 3.3 *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0, 1, 2)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_0 \to \mathcal{P}_1$, $y = (g, (\gamma, \delta)) : \mathcal{P}_0 \to \mathcal{P}_2$ be elements of $\mathcal{S}$. If for any $p \in P_1$ and $t \in T_1$, there exist $q \in P_2$ and $s \in T_2$ such that $\alpha^{-1}(p) \subset \gamma^{-1}(q)$ and $\beta^{-1}(t) \subset \delta^{-1}(s)$, then $y\mathcal{S} \subset x\mathcal{S}$.* □

PROPOSITION 3.3 (Intersection of Principal Left Ideals) *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1, 2, 3)$ be Petri nets, $x : \mathcal{P}_1 \to \mathcal{P}_3$ and $y : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}$. Then, there exist a Petri net P and a surjective morphism z such that $\mathcal{S}x \cap \mathcal{S}y = \mathcal{S}z$.* □

COROLLARY 3.1 (Diamond Property I) *Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1, 2, 3)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. Then there exists a Petri net $\mathcal{P}$ such that $\mathcal{P} \sqsupseteq \mathcal{P}_1$ and $\mathcal{P} \sqsupseteq \mathcal{P}_2$.* □

PROPOSITION 3.4 (Intersection of Principal Right Ideals) *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0, 1, 2)$ be Petri nets, $x : \mathcal{P}_1 \longrightarrow \mathcal{P}_3$ and $y : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$ be elements of S. Then, there exist a Petri net $\mathcal{P}$ and a surjective morphism z such that $xS \cap yS = zS$.* □

COROLLARY 3.2 (Diamond Property II) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets with $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$. Then there exists a Petri net $\mathcal{P}_3$ such that $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. □

We define the concept of irreducible forms of a Petri net with respect to $\sqsupseteq$ and show the uniqueness of them up to isomophism.

DEFINITION 3.1 (Irreducible) *A Petri net $\mathcal{P}$ is called a $\sqsupseteq$-irreducible if $\mathcal{P} \sqsupseteq \mathcal{P}'$ implies $\mathcal{P} \simeq \mathcal{P}'$ for any Petri net $\mathcal{P}'$. Then $\mathcal{P}$ is called an $\sqsupseteq$-irreducible form.* □

COROLLARY 3.3 Let $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ be Petri nets with $\mathcal{P} \sqsupseteq \mathcal{P}'$ and $\mathcal{P} \sqsupseteq \mathcal{P}''$. If $\mathcal{P}'$ and $\mathcal{P}''$ are $\sqsupseteq$-irreducible, then $\mathcal{P}' \simeq \mathcal{P}''$. □

4 Structure of the automorphism group of a Petri net

Our aim in this section is to decompose the automorphism group $G = \mathbf{Aut}(\mathcal{P})$ of a Petri net $\mathcal{P}$ into $G = KN = NK$, where N is a kind of normal subgroup of G.

At first, we consider some properties of the structure of the automorphism group of a fixed (given) Petri net $\mathcal{P} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, \mu)$.

4.1 The group of automorphisms of a Petri net

Let $\boldsymbol{Q_+}^P \rtimes (P^P \times T^T)$ be the semi-direct product of the group $\boldsymbol{Q_+}^P$ and the monoid $P^P \times T^T$, equipped with the multiplication defined by

$$(f, (\alpha, \beta))(g, (\alpha', \beta')) \overset{\text{def}}{=} (f \otimes \alpha\, g, (\alpha\alpha', \beta\beta')), \tag{4.1}$$

where P^P is the set of all maps from P to P and T^T is the set of all maps from T to T. $\boldsymbol{Q_+}^P \rtimes (P^P \times T^T)$ forms a monoid with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$, where $\mathbf{1}_\otimes$ is the identity of the group $\boldsymbol{Q_+}^P$, $\mathbf{1}_P$ and $\mathbf{1}_T$ are the identity maps on P and T respectively.

Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Now we consider the following set related to the Petri net $\mathcal{P}$.

$$\begin{aligned} \mathbf{Mor}(\mathcal{P}) &: \text{ the set of all the morphisms of } \mathcal{P}. \\ \mathbf{Aut}(\mathcal{P}) &: \text{ the set of all the automorphisms of } \mathcal{P}. \end{aligned}$$

By changing the weight function and the markings of $\mathcal{P}$, we can construct another Petri net $\mathcal{P}_0 = (P, T, 0^{E(P,T)}, 0^P)$ be Petri nets, where 0^P denotes the special marking with $0^P : P \to \boldsymbol{N}_0, p \mapsto 0$ and $0^{E(P,T)}$ the special weight function with $0^{E(P,T)} : E(P,T) \to \boldsymbol{N}_0, e \mapsto 0$. Then the following inclusion relation holds.

PROPOSITION 4.1 *Let $\mathcal{P} = (P, T, W, \mu)$ and $\mathcal{P}_0 = (P, T, 0^{E(P,T)}, 0^P)$ be Petri nets. And let S_P and S_T be the symmetric groups of P and T, respectively.*
(1) The subset $\boldsymbol{Q_+}^P \rtimes (S_P \times S_T)$ of $\boldsymbol{Q_+}^P \rtimes (P^P \times T^T)$ forms a group with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$.
(2) $\mathbf{Mor}(\mathcal{P}_0) = \boldsymbol{Q_+}^P \rtimes (P^P \times T^T)$.
(3) $\mathbf{Mor}(\mathcal{P})$ *is a submonoid of* $\mathbf{Mor}(\mathcal{P}_0)$.
(4) $\mathbf{Aut}(\mathcal{P}_0) = \boldsymbol{Q_+}^P \rtimes (S_P \times S_T)$.
(5) $\mathbf{Aut}(\mathcal{P})$ *is a subgroup of* $\mathbf{Aut}(\mathcal{P}_0)$. □

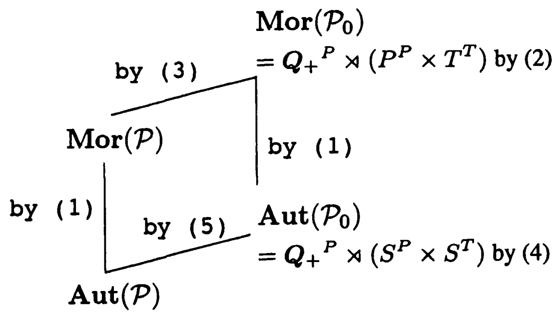

Figure 2. Inclusion relations among monoids of morphisms and groups of automorphisms related to the Petri nets $\mathcal{P}$ and $\mathcal{P}_0$, as a result of Propositon 4.1.

4.2 Similarity and automorphism

Recall that $(Q_+{}^P, \otimes_P)$ is an abelian group and a 0-isolated place does not have any connection to any transition and is marked with 0 tokens.

LEMMA 4.1 *Let P be a nonempty set and P_1, P_2 be subsets of P.*
(1) $Q_+{}^{P_1} = \{f \in Q_+{}^P \mid f(p) = 1, p \in P \setminus P_1\}$ *is a subgroup of* $(Q_+{}^P, \otimes_P)$.
(2) $Q_+{}^{P_1} \otimes_P Q_+{}^{P_2} = Q_+{}^{P_1 \cup P_2}$. □

LEMMA 4.2 (Transposition-type automorphisms) *Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net, $p, q \in P$ be two distinct similar places in P and $s, t \in T$ be two distinct similar transitions in T. Then*
(1) *If p is not 0-isolated, $N_{\{p,q\}} = \langle (f_{p,q}, ((p\ q), 1_T)) \rangle$ is a subgroup of $\mathbf{Aut}(\mathcal{P})$ and its order is 2, where $(p\ q)$ is the transposition of p and q, $f_{p,q}(p) = r, f_{p,q}(q) = 1/r, f_{p,q}(x) = 1$ for $x \in P \setminus \{p, q\}$, and r is the rational number such that $\mu(p) = r\mu(q), W(p,t) = rW(q,t)$ and $W(t,p) = rW(t,q)$ for all $t \in T$.*
(2) *If p is 0-isolated, $N_{\{p,q\}} = Q_+{}^{\{p,q\}} \times \langle ((p\ q), 1_T)) \rangle$ is a subgroup of $\mathbf{Aut}_+(\mathcal{P})$.*
(3) *$N_{\{t,s\}} = \langle (1_{\otimes_P}, (1_P, (s\ t))) \rangle$ is a subgroup of $\mathbf{Aut}(\mathcal{P})$ and its order is 2.* □

For a $\sim_{\mathcal{P}}$-class $C(u)$ of u, the subgruop $N_{C(u)}$ of $\mathbf{Aut}(\mathcal{P})$ is defined as follows:

$$N_{C(u)} = \begin{cases} \langle S_{\{a,b\}} | a, b \in C(u), a \neq b \rangle & \text{if } |C(u)| \geq 2, \\ \{(1_{\otimes_P}, (1_P, 1_T))\} & \text{if } |C(u)| = 1. \end{cases}$$

If u is a 0-isolated place, the $\sim_{\mathcal{P}}$-class $Z = C(u)$ is the set of all 0-isolated places in P and we can easily verify that $N_Z = Q_+{}^Z \times (S_Z \times \{1_T\})$, where S_Z is the symmetric group of Z. The following proposition holds with respect to N_Z.

PROPOSITION 4.2 (Separation of 0-isolated places) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net, $Z \subset P$ be $\sim_{\mathcal{P}}$-class of all the 0-isolated places, $N_Z = Q_+{}^Z \times (S_Z \times \{1_T\})$, $H = \{(f, (\alpha, \beta)) \in (\mathbf{Aut}(\mathcal{P}) | f|_Z = 1_{\otimes_Z}, \alpha|_Z = 1_Z\}$. Then, $\mathbf{Aut}(\mathcal{P}) = N_Z \times H$.

Proof) Here set $G = \mathbf{Aut}(\mathcal{P})$ and $\mathbf{1} = (1_\otimes, (1_P, 1_T))$. What we have to do is to prove that
(a) $G = N_Z H$, (b) $N_Z \cap H = \{\mathbf{1}\}$, and (c) $xy = yx$ for any $x \in N_Z, y \in H$.
(a) Let $(f, (\alpha, \beta))$ be an arbitrary element in G. $f = f_0 \otimes f_1 = f_1 \otimes f_0$ for some $f_0 \in Q_+{}^Z, f_1 \in Q_+{}^{P \setminus Z}$. Since $\alpha(Z) = Z$ and $\alpha(P \setminus Z) = P \setminus Z$ hold, $\alpha = \alpha_0\alpha_1$ for some $\alpha_0 \in S_Z, \alpha_1 \in S_{P \setminus Z}$. Because α_0 and f_1 are constant on $P \setminus Z$ and Z respectively, we have $\alpha_0 f_1 = f_1$ and $(f_0, (\alpha_0, 1_T))(f_1, (\alpha_1, \beta)) = (f_0 \otimes \alpha_0 f_1, (\alpha_0\alpha_1, \beta)) = (f, (\alpha, \beta))$. Therefore $G = N_Z H$.

The condition(b) is trivial by the construction of H. (c) Let $x = (f, (\alpha, \beta)) \in H, y = (g, (\gamma, \mathbf{1}_T)) \in N_Z$. Since α and γ are constant on Z and $P \setminus Z$ respectively, $xy = (f \otimes \alpha g, (\alpha\gamma, \beta)) = (g \otimes \gamma f, (\gamma\alpha, \beta)) = yx$, that is, x and y commute. $\square$

LEMMA 4.3 Let $\mathcal{P} = (P, T, W, \mu)$,$\{p, q\} \subset P$,$\{s, t\} \subset T$ and $C(u)$ be the $\sim_{\mathcal{P}}$-class of $u \in P \cup T$. If $(f, (\alpha, \beta))$ is an automorphism of $\mathcal{P}$, then
(1) $p \sim_{\mathcal{P}} q \iff \alpha(p) \sim_{\mathcal{P}} \alpha(q)$,
(1') $s \sim_{\mathcal{P}} t \iff \beta(s) \sim_{\mathcal{P}} \beta(t)$,
(2) $\alpha(C(p)) = \{\alpha(q) | q \sim_{\mathcal{P}} p\} = C(\alpha(p))$,
(2') $\beta(C(t)) = \{\beta(s) | s \sim_{\mathcal{P}} t\} = C(\beta(t))$,
(3) $\min\{i | C(\alpha^i(u)) = C(u)\} = \min\{i | C(\beta^i(v)) = C(v)\}$ if $u, v \in P \cup T$ are connected,. $\square$

Note that $|C(\alpha(p))| = |C(p)|$ for all $p \in P$ and $|C(\beta(t))| = |C(t)|$ for all $t \in T$.

Let $C_1, C_2, \ldots, C_k$ be the all $\sim_{\mathcal{P}}$-classes on $P \cup T$ and $\pi = \{C_1, C_2, \ldots, C_k\}$ be the partition of $P \cup T$ determinded by $\sim_{\mathcal{P}}$. Then we introduce the permutation group $S_\pi = \{\sigma \in S_{P \cup T} \mid \forall X \in \pi, X^\sigma = X\} = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$, which does not move any elements of π.

PROPOSITION 4.3 (Embedding into a symmetric group) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net without 0-isolated places.
(1) $\phi : \mathbf{Aut}(\mathcal{P}) \to S_{P \cup T}, (f, (\alpha, \beta)) \mapsto (\alpha, \beta)$ is a monomorphims, i.e. $\mathbf{Aut}(\mathcal{P}) \simeq \phi(G) \subset S_{P \cup T}$.
(2) $S_\pi \subset \phi(G)$.
(3) $X \in \pi \Longrightarrow g(X) \in \pi$ for any $g \in \phi(G)$.
(4) S_π is a normal subgroup of $\phi(G)$, that is, $S_\pi \lhd \phi(G)$.
(5) Let $a_1, a_2, \ldots, a_k$ be a system of representatives for S_π of $\phi(G)$ and $A = \langle a_1, a_2, \ldots, a_k \rangle$. Putting $K = \phi^{-1}(A), N = \phi^{-1}(S_\pi), \mathbf{Aut}(\mathcal{P}) = KN = NK$.

Proof) Here set $G = \mathbf{Aut}(\mathcal{P})$ and $\mathbf{1} = (\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$.
(1) ϕ is a homomorphim from G to $S_{P \cup T}$. Indeed, for any $x = (f, (\alpha, \beta)), y = (g, (\gamma, \delta)) \in \mathbf{Aut}_+(\mathcal{P})$, Since $xy = (f \otimes \alpha g, (\alpha\gamma, \beta\delta))$ holds, $\phi(xy) = (\alpha\gamma, \beta\delta) = (\alpha, \beta)(\gamma, \delta) = \phi(x)\phi(y)$. Next, suppose $\phi(x) = (\alpha, \beta) = \mathbf{1}_{P \cup T} = (\mathbf{1}_P, \mathbf{1}_T)$. $x = (f, (\mathbf{1}_P, \mathbf{1}_T))$ must hold. Since $\mathcal{P}$ has no 0-isolated places, $f = \mathbf{1}_\otimes$, that is, $\ker(\phi) = \mathbf{1}$. Therefore ϕ is a monomorphism.
(2) $N = N_{C_1} N_{C_2} \ldots N_{C_k}$ is a subgroup of G.

$$\begin{aligned} \phi(N) &= \phi(N_{C_1})\phi(N_{C_2}) \ldots \phi(N_{C_k}) \\ &= S_{C_1} S_{C_2} \ldots S_{C_k} \\ &= S_\pi \subset \phi(G). \end{aligned}$$

(3) Let $g \in \phi(G)$. By LEMMA4.3 (2) and (2'), if $X = C_i \in \pi \, (1 \le i \le k)$, then $g(X) \in \pi$.
(4) Let $\sigma \in S_\pi$, $g \in \phi(G)$ and x be an arbitrary element of $P \cup T$. Suppose that $x \in X, X \in \pi$. Since $g(x) \in g(X)$ and $g(X) \in \pi$ by (3), $(g\sigma)(X) = g(X)$. $(g\sigma g^{-1})(X) = gg^{-1}(X) = X$ and $g\sigma g^{-1} \in S_{P \cup T}$ imply $g\sigma g^{-1} \in S_\pi$, that is, $gS_\pi g^{-1} \subset S_\pi$. Therefore S_π is a normal subgroup of $\phi(G)$.
(5) It is trivial. $\square$

THEOREM 4.1 *Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net and $C_1, C_2, \ldots, C_k$ be the all $\sim_{\mathcal{P}}$-classes on $P \cup T$. $N = N_{C_1} \times N_{C_2} \times \cdots \times N_{C_k}$ is a normal subgroup of $G = \mathbf{Aut}(\mathcal{P})$ and $K = \langle \{a_i | i \in \Lambda\} \rangle$ is a subgroup generated by $\{a_i | i \in \Lambda\}$ with $G = \bigcup_{i \in \Lambda} a_i N$.*
(1) *If P has no 0-isolated places, $G = KN = NK$.*
(2) *Otherwise, $G = \mathbf{Q}_+{}^Z \times (KN) = (KN) \times \mathbf{Q}_+{}^Z$, where $Z \subset P$ be $\sim_{\mathcal{P}}$-class of a 0-isolated place.*

LEMMA 4.4 (1-step reduction) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net.
(1) $p, q \in P$ be two distinct similar places in P. Then $\mathcal{P} \sqsupseteq \mathcal{P}' = (P', T, W', \mu')$, where $P' = P - \{q\}$, $W' = W | (P' \times T) \cup (T \times P'), \mu' = \mu | P'$.
(2) $s, t \in T$ be two distinct similar transitions in T. Then $\mathcal{P} \sqsupseteq \mathcal{P}' = (P, T', W', \mu)$, where $T' = T - \{s\}$, $W' = W | (P \times T') \cup (T' \times P)$. $\square$

In the lemma above, $|P' \cup T| = |P \cup T'| = |P \cup T| - 1$ holds. So we call such a relation *1-step reduction*, denoted by $\sqsupseteq_1$.

PROPOSITION 4.4 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)(i = 1, 2)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$, $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ be a surjective morphism. If $\mathcal{P}_2$ is a normal form, then
(1) For any $p,\ q \in P, p \sim_{\mathcal{P}} q \iff \alpha(p) = \alpha(q)$,
(2) For any $t,\ s \in T, t \sim_{\mathcal{P}} s \iff \beta(t) = \beta(s)$. □

Proof) (1)(if part) For an arbitrary transition $t \in T$,

$$f(p)W_1(p,t) = W_2(\alpha(p), \beta(t)) = W_2(\alpha(q), \beta(t)) = f(q)W_1(q,t),$$
$$f(p)W_1(t,p) = W_2(\beta(t), \alpha(p)) = W_2(\beta(t), \alpha(q)) = f(q)W_1(t,q), and$$
$$f(p)\mu_1(p) = \mu_2(\alpha(p)) = \mu_2(\alpha(q)) = f(q)\mu_1(q)$$

hold. So setting $r = f^{-1}(p)f(q)$, we have $\mu_1(p) = r\mu_1(q)$ and $W_1(p,t) = rW_1(q,t)$ and $W_1(t,p) = rW_1(t,q)$ for all $t \in T$. Therefore $p \sim_{\mathcal{P}} q$.

(only if part) Suppose that $\alpha(p) \neq \alpha(q)$. Since $p \neq q$, By lemma 4.4 there exists a Petri net $\mathcal{P}_2'$ such that $\mathcal{P}_2 \sqsupseteq_1 \mathcal{P}_2'$ and thus $\mathcal{P}_2 \not\simeq \mathcal{P}_2'$. This contradicts that $\mathcal{P}_2$ is a normal form.
(2) The claim is proved in a similar way to (1). □

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