Hypergeometric type generating functions of several variables associated with the Lerch zeta-function (summarized version) *

Masanori KATSURADA
Department of Mathematics, Hiyoshi Campus, Keio University

Abstract

This is a summarized version of the forthcoming paper [10]. Let \( s, z \) and \((z_0, z) = (z_0, z_1, \ldots, z_n)\) be complex variables, and \( \zeta(s, z, \lambda) \) denote the Lerch zeta-function defined by (1.1) below. We introduce in the present article a class of generating functions and their confluent analogues, denoted by \( \mathcal{Z}^{(n)}(s, \beta; z_0, z) \) and \( \hat{\mathcal{Z}}^{(n)}(s, \beta_{n-1}; z_0, z) \) respectively (see (2.1) and (2.3)), in the forms of the fourth Lauricella hypergeometric type (of several variables) associated with \( \zeta(s, z, \lambda) \). It is shown that complete asymptotic expansions of \( \mathcal{Z}^{(n)}(s, \beta; z_0, z) \) exit when \( z_0 \to 0 \) (Theorem 1) as well as when \( z_0 \to \infty \) (Theorem 2) through the sectorial region \(| \arg z - \theta_0 | < \pi/2\) with any fixed angle \( \theta_0 \in [-\pi/2, \pi/2] \), while other \( z_j \)’s move through the same sector satisfying the conditions \( z_j \ll z_0 \) \((j = 1, \ldots, n)\). Similar asymptotic results also hold for \( \hat{\mathcal{Z}}^{(n)}(s, \gamma; z_0, z) \) (Theorems 3 and 4) through the confluence operation in (2.3). Our main formulae (3.1) and (3.4) (resp. (3.7) and (3.10)) first assert that \( \mathcal{Z}^{(n)}(s, \beta; z_0, z) \) (resp. \( \hat{\mathcal{Z}}^{(n)}(s, \beta_{n-1}; z_0, z) \)) can be continued to a meromorphic function of \( s \) over the whole \( s \)-plane, to the whole polysector \(| \arg z_j | < \pi \) \((j = 0, 1, \ldots, n)\), and for all \((\beta, \gamma) \in \mathbb{C}^n \times (\mathbb{C} \setminus \{0, \ldots, 1\})\) (resp. for all \((\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times (\mathbb{C} \setminus \{0, -1, \ldots\})\)). We can further apply (3.1) and (3.4) to deduce complete asymptotic expansions of \( \partial/\partial s \) of \( \mathcal{Z}^{(n)}(s, \beta; z_0, z) \) \((m = 1, 2, \ldots)\) at any integer arguments \( s = l \in \mathbb{Z} \) when \((z_0, z)\) becomes small (Corollary 6) and large (Corollary 8) under the same settings as in Theorems 1 and 2. Furthermore, several applications of Theorems 1–4 in the cases of \( n = 1 \) and 2 are finally presented.

Introduction

Throughout this article, \( s = \sigma + \sqrt{-1}t, z \) and \((z_0, z) = (z_0, z_1, \ldots, z_n)\) are complex variables with \(| \arg z | < \pi \) and \(| \arg z_j | < \pi \) \((j = 0, 1, \ldots, n)\), and \( a \) and \( \lambda \) real parameters with \( a > 0 \). We hereafter set \( e(\lambda) = e^{2\pi\lambda\sqrt{-1}} \), use the vectorial notation \( x = (x_1, \ldots, x_m) \) with the abbreviation

\[
(x) = x_1 + \cdots + x_m
\]

for any \( m \geq 1 \) and any complex \( x_i \) \((i = 1, \ldots, m)\), and further write \( x_{m-1} = (x_1, \ldots, x_{m-1}) \) and

\[
\frac{x}{y} = \left( \frac{x_1}{x}, \ldots, \frac{x_m}{y} \right)
\]

for any \( y \neq 0 \). The Lerch zeta-function \( \zeta(s, z, \lambda) \) is defined by the Dirichlet series

\[
\zeta(s, z, \lambda) = \sum_{l=0}^{\infty} e(\lambda l)(l + z)^{-s} \quad (\sigma = \Re s > 1),
\]

and its meromorphic continuation over the whole \( s \)-plane; this is an entire function when \( \lambda \in \mathbb{R} \setminus \mathbb{Z} \), while if \( \lambda \in \mathbb{Z} \) it reduces to the Hurwitz zeta-function \( \zeta(s, a) \), and so \( \zeta(s) = \)

2010 Mathematics Subject Classification. Primary 11E45; Secondary 33C65.

*Key words and phrases. Lerch zeta-function, Lauricella hypergeometric function, Mellin-Barnes integral, asymptotic expansion.
\( \zeta(s, 1) \) is the Riemann zeta-function. We remark here that the notation (1.1) differs from the original \( \phi(z, \lambda, s) \) due to Lerch [13], in order to retain notational consistency with other terminology.

It is the principal aim of the present article to introduce a class of generating functions and their confluent analogues, denoted by \( \mathcal{E}_{\alpha, \lambda}^{(n)}(\tau; z, \mathbf{0}, \mathbf{0}) \) and \( \mathcal{E}_{\alpha, \lambda}^{(n)}(\tau; \mathbf{0}, z, \mathbf{0}) \) respectively (see (2.1) and (2.4) below), in the forms of the (fourth) Lauricella hypergeometric type (of several variables) associated with \( \zeta(s, \lambda, z) \). We shall first show that complete asymptotic expansions of \( \mathcal{E}_{\alpha, \lambda}^{(n)}(\tau; z, \mathbf{0}) \) and \( \mathcal{E}_{\alpha, \lambda}^{(n)}(\tau; \mathbf{0}, z, \mathbf{0}) \) exist when \( (z, \mathbf{0}) \) becomes small (Theorems 1 and 3) and large (Theorems 2 and 4) under certain settings on the movement of \( (z, \mathbf{0}) \). Several applications of Theorems 1–4 will further be presented. Before stating our main results, some necessary notations and terminology will be prepared.

Let \( \Gamma(s) \) be the gamma function, \( \langle s \rangle = \Gamma(s + k)/\Gamma(s) \) for any \( k \in \mathbb{Z} \) the shifted factorial of \( s \), and write

\[
\Gamma\left(\frac{\mathbf{\mu}}{\mathbf{\nu}}\right) = \frac{\prod_{i=1}^{h} \Gamma(\mu_{i})}{\prod_{j=1}^{k} \Gamma(\nu_{j})}
\]

for complex vectors \( \mathbf{\mu} = (\mu_{1}, \ldots, \mu_{h}) \) and \( \mathbf{\nu} = (\nu_{1}, \ldots, \nu_{k}) \). In the sequel, the sets of non-negative and non-positive integers are respectively denoted by \( \mathbb{N}_{0} = \mathbb{N} \cup \{0\} \) and \( -\mathbb{N}_{0} = \{-k \mid k \in \mathbb{N}_{0}\} \). The (fourth) Lauricella hypergeometric function of \( m \)-variables \( x_{i} \) \( (i = 1, \ldots, m) \) is defined by the \( m \)-ple power series

\[ (1.2) \quad F_{D}^{(m)}(\alpha, \beta_{1}, \ldots, \beta_{m}; x_{1}, \ldots, x_{m}) = \sum_{k_{1}, \ldots, k_{m}=0}^{\infty} \frac{(\alpha)_{k_{1}+\cdots+k_{m}} \ldots (\beta_{1})_{k_{1}} \cdots(\beta_{m})_{k_{m}}}{(\gamma)_{k_{1}+\cdots+k_{m}} k_{1}! \cdots k_{m}!} x_{1}^{k_{1}} \cdots x_{m}^{k_{m}}, \]

for complex parameters \( \alpha, \beta_{i} \) \( (i = 1, \ldots, m) \) and \( \gamma \neq -k \) \( (k \in \mathbb{N}_{0}) \), where the series converges absolutely in the poly-disk \( |x_{i}| < 1 \) \( (i = 1, \ldots, m) \); this is continued to a one-valued holomorphic function \( (\alpha, \beta, \gamma, x) \) for all \( (\alpha, \beta, \gamma) \in \mathbb{C}^{m+1} \times \{ \mathbb{C} \setminus (-\mathbb{N}_{0}) \} \), and \( x \) in the poly-sector \( |\arg(1 - x_{i}) - (1 + \varphi_{0}) < \pi/2 \) \( (i = 1, \ldots, m) \) for any angle fixed with \( \varphi_{0} \in [-\pi/2, \pi/2] \) (cf. [1]). Note that (1.2) reduces when \( m = 1 \) to Gauss\' hypergeometric function \( F_{1}(\alpha, \beta; z) \), and when \( m = 2 \) to (the first) Appell's hypergeometric function \( F_{2}(\alpha, \beta; x, z) \). The abbreviations

\[
(\beta)_{k} = (\beta_{1})_{k_{1}} \cdots(\beta_{m})_{k_{m}}, \quad k! = k_{1}! \cdots k_{m}!,
\]

\[
x^{k} = x_{1}^{k_{1}} \cdots x_{m}^{k_{m}}
\]

for \( k = (k_{1}, \ldots, k_{m}) \) and \( x = (x_{1}, \ldots, x_{m}) \) allow to rewrite (1.2) in a more concise form

\[
F_{D}^{(m)}(\alpha, \beta; x) = \sum_{k \geq 0} \frac{(\alpha)_{(k)}(\beta)_{k}}{(\gamma)_{(k)} k!} x^{k},
\]

where (and hereafter) the summation condition \( k \geq \mu \) means that the sum runs over all indices \( k \) with \( k_{j} \geq h_{j} \) \( (j = 1, \ldots, n) \). Furthermore, a new class of \( m \)-variable hypergeometric functions \( \mathcal{E}_{\alpha, \lambda}^{(n)}(\tau; \mathbf{x}, \mathbf{0}) \) is obtained from \( F_{D}^{(m)}(\alpha, \beta; x) \) through the confluence operation

\[ (1.3) \quad \frac{F_{D}^{(m)}(\alpha, \beta_{m-1}; x_{m-1}, x_{m})}{\beta_{m}} (\beta_{m} \to +\infty) \mathcal{E}_{\alpha, \lambda}^{(n)}(\tau; \mathbf{0}, \mathbf{0}) \]

\[
= \sum_{k \geq 0} \frac{(\alpha)_{(k)}(\beta_{m-1})_{k} x^{k}}{(\gamma)_{(k)} k!}.
\]
Note that the case \( m = 1 \) of (1.3) gives Kummer's hypergeometric function

\[
\hat{F}_{D}^{(1)}(\alpha; \gamma; x) = _{1}F_{1}(\alpha; \gamma; x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha)_k}{\Gamma(\gamma)_k k!} x^k
\]

for \(|x| < +\infty\), while \( m = 2 \) the confluent form \( F_{1}(\alpha, \beta_1, \beta_2; x_1, x_2) \), defined by

\[
\hat{F}_{D}^{(2)}(\alpha, \beta_1; x_1, x_2) = \Phi_1(\alpha, \beta_1; x_1, x_2) = \sum_{k_1, k_2=0}^{\infty} \frac{\Gamma(\alpha)_{k_1+k_2}(\beta_1)_{k_1}}{(\gamma)_{k_1+k_2} k_1! k_2!} x_1^{k_1} x_2^{k_2}
\]

for \(|x_1| < 1\) and \(|x_2| < +\infty\) (cf. [4]).

**Main objects**

We can now introduce the hypergeometric type generating function \( \mathcal{Z}_{a, \lambda}^{(n)}(s, \beta; z, z) \) of \((n + 1)\)-variables \((z_0, z) = (z_0, z_1, \ldots, z_n)\) associated with \( \zeta(s, a + z_0, \lambda) \), defined by the \( n \)-ple power series

\[
(2.1) \quad \mathcal{Z}_{a, \lambda}^{(n)}(s, \beta; z_0, z) = \sum_{k_1, \ldots, k_n=0}^{\infty} \frac{\Gamma(s+\langle k \rangle) \Gamma(\beta)_{\langle k \rangle}}{\Gamma(\gamma)_{\langle k \rangle} k!} \zeta(s+\langle k \rangle, a+z_0, \lambda) (-z)^{\langle k \rangle}
\]

which converges absolutely in the domain \(|z_j| < |\text{Im} z_0| (j = 1, \ldots, n)\). The change of the order of summations in (2.1) readily implies that

\[
(2.2) \quad \mathcal{Z}_{a, \lambda}^{(n)}(s, \beta; z_0, z) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} \hat{F}_{D}^{(n)}(s, \beta; \frac{z}{a+l+z_0})
\]

for \( \sigma > 1 \); the cases \( \beta = 0 \) and \( z = 0 \) of (2.2) both reduce to

\[
\mathcal{Z}_{a, \lambda}^{(n)}(s, 0; z_0, z) = \mathcal{Z}_{a, \lambda}^{(n)}(s, \beta; z_0, 0) = \zeta(s, a + z_0, \lambda),
\]

while the cases \( n = 1 \) and \( n = 2 \) respectively to

\[
\mathcal{Z}_{a, \lambda}^{(1)}(s, \beta; z_0, z_1) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} \hat{F}_{D}^{(1)}(s, \beta; \frac{z_1}{a+l+z_0}),
\]

\[
\mathcal{Z}_{a, \lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} \times \hat{F}_{D}^{(2)}(s, \beta_1, \beta_2; \frac{z_1}{a+l+z_0}, \frac{z_2}{a+l+z_0})
\]

for \( \sigma > 1 \). It is further possible to obtain a new class of generating functions, denoted by \( \hat{\mathcal{Z}}_{a, \lambda}^{(n)}(s, \beta_{n-1}; z_0, z) \), from \( \mathcal{Z}_{a, \lambda}^{(n)}(s, \beta; z_0, z) \) through the confluence operation

\[
(2.3) \quad \mathcal{Z}_{a, \lambda}^{(n)}(s, \beta_{n-1}; z_0, z_{n-1}, \overline{z_n \rightarrow +\infty}) \rightarrow \hat{\mathcal{Z}}_{a, \lambda}^{(n)}(s, \beta_{n-1}; z_0, z) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} \hat{F}_{D}^{(n)}(s, \beta_{n-1}; \frac{z}{a+l+z_0})
\]
for \( \sigma > 1 \), where the change of the order of summations in the last expression gives

\[
(2.4) \quad \mathcal{Z}_{a,\lambda}^{(n)}(s, \gamma; z_{0}, z) = \sum_{k \geq 0} \frac{(s)_{\langle k \rangle} (\beta_{n-1})_{k_{n-1}}}{(\gamma)_{\langle k \rangle} k!} \zeta(s + \langle k \rangle, a + z_{0}, \lambda)(-z)^{k}
\]

in the domain \(|z_{j}| < |\text{Im } z|\) (\(j = 1, \ldots, n\)).

We shall give in the remaining of this section a brief overview of the history of research related to various generating functions associated with specific values of zeta-functions. Several power series involving the particular values of \( \zeta(s, a) \) were first studied by Srivastava \cite{18,19,20}, while Klusch \cite{11} treated the Taylor series for \( \zeta(s, a + z, \lambda) \) in the variable \( z \in \mathbb{C} \), and gave its many interesting applications. Hypergeometric type generating functions of \( \zeta(s) \) were first introduced and studied by Raina-Srivastava \cite{17} and the author \cite{6,7}, independently of each other; we refer the reader to the comprehensive account \cite{21} into this direction. Hikami-Kirillov \cite{5} more recently investigated hypergeometric generating functions of various \( L \)-function values in connection with \( q \)-hypergeometric series and quantum invariants. Hypergeometric type generating functions associated with \( \zeta_{\nu}(s, a, w) \) (a weighted extension of \( \zeta(s, a, \lambda) \)) were first introduced and studied by Bin-Saad and Al-Gonah \cite{3} and further by Bin-Saad \cite{2}. Li-Kanemitsu-Tsukada \cite{14} made Majer's \( G \)-function theoretic interpretation of the results in \cite{6,7}, while similar \( G \)-function theoretic study on the results in \cite{8} was made by Kuzumaki \cite{12}. We next mention several relevant asymptotic aspects into this direction. Complete asymptotic expansions of \( \zeta(s, a + z, \lambda) \) for small and large \( z \in \mathbb{C} \) in the sector \(|\arg z| < \pi\) was established by the author \cite{8}. Matsumoto \cite{15} investigated complete asymptotic expansions of \( \zeta_{2}(s, a | (1, w)) \) (Barnes' double zeta-function) for small and large basis parameter \( w \in \mathbb{C} \) in \(|\arg w| < \pi\). Onodera \cite{16} more recently studied complete asymptotic expansions of \( \zeta_{m}(s, a + x | \omega) \) (Barnes' multiple zeta-function) for small and large \( x \in \mathbb{R}_{+} \) and one of \( \omega_{i}\)'s in the basis parameters \( \omega = (\omega_{1}, \ldots, \omega_{m}) \in \mathbb{R}_{+}^{m} \), where \( \mathbb{R}_{+} \) denotes the set of positive real numbers.

**Asymptotic expansions for small and large \((z_{0}, z)\)**

To describe our results we introduce the generalized Bernoulli polynomials \( B_{k}(x, y) \) \((k \in \mathbb{N}_{0})\) for any parameters \( x, y \in \mathbb{C} \) by the power series

\[
\frac{xe^{xz}}{ye^{z} - 1} = \sum_{k=0}^{\infty} \frac{B_{k}(x, y)}{k!} z^{k},
\]

centered at \( z = 0 \); this in particular gives

\[
B_{0}(x, y) = \begin{cases} 1 & \text{if } y = 1; \\ 0 & \text{if } y \neq 1. \end{cases}
\]

Note that \( B_{k}(x) = B_{k}(x, 1) \) are the usual Bernoulli polynomials, and so \( B_{k} = B_{k}(0) \) are the usual Bernoulli numbers. The vertical straight path from \( u - i \infty \) to \( u + i \infty \) (with \( u \in \mathbb{R} \)) is hereafter denoted by \((u)\).

We first state the asymptotic expansion of \( \mathcal{Z}_{a,\lambda}^{(n)}(s, \gamma; z_{0}, z) \) when \((z_{0}, z)\) becomes small.

**Theorem 1.** Let \( \theta_{0} \) be any angle fixed with \( \theta_{0} \in [-\pi/2, \pi/2] \). Then for any integer \( K \geq 0 \), in the region \( \sigma > 1 - K \) except at \( s = 1 \) the formula

\[
(3.1) \quad \mathcal{Z}_{a,\lambda}^{(n)}(s, \gamma; z_{0}, z) = S_{a,\lambda,K}^{+}(s, \gamma; z_{0}, z) + R_{a,\lambda,K}^{+}(s, \gamma; z_{0}, z)
\]

\[\text{[1]}\text{The author would like to make apology for insufficiency (in many respects) of the present survey.}\]
holds for all \((z_0, z)\) in the poly-sector \(|\arg z_j - \theta_0| < \pi/2\) \((j = 0, 1, \ldots, n)\) and for all \((\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}\). Here

\[
S_{a,\lambda,K}^{+} (s, \beta, \gamma; z_0, z) = \sum_{k=0}^{K-1} (-1)^k (s)_k F_D^{(n)} \left( -k, \beta, \gamma; -\frac{z}{z_0} \right) \zeta(s + k, a, \lambda) z_0^k,
\]

and \(R_{a,\lambda,K}^{+}\) is the remainder term expressed as

\[
R_{a,\lambda,K}^{+} (s, \beta, \gamma; z_0, z) = \frac{1}{2\pi \sqrt{-1}} \int_{(u_K^{+})} F_D^{(n)} \left( -w, \beta, \gamma; -\frac{z}{z_0} \right) \zeta(s+w, a, \lambda) z_0^w dw,
\]

where \(u_K^+\) is a constant satisfying \(\max(1-\sigma, K-1) < u_K^+ < K\). Formula (3.1) further provides the analytic continuation of \(Z_{a,\lambda}^{(n)} (s, \beta, \gamma; z_0, z)\) over the whole s-plane except at \(s = 1\), to the poly-sector \(|\arg z_j - \theta_0| < \pi/2\) \((j = 0, 1, \ldots, n)\), and for all \((\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}\). Moreover if \((z_0, z)\) is in \(|\arg z_j - \theta_0| \leq \pi/2 - \delta\) with any small \(\delta > 0\) \((j = 0, 1, \ldots, n)\), and satisfies

\[|z_j| \leq c|z_0|\]

for some constant \(c > 0\), then the estimates

\[
F_D^{(n)} \left( -k, \beta, \gamma; -\frac{z}{z_0} \right) = O(1) \quad \text{and} \quad R_{a,\lambda,K}^{+} (s, \beta, \gamma; z_0, z) = O(|z_0|^K)
\]

follow for all \(K \geq k \geq 0\) as \(z_0 \rightarrow 0\) through \(|\arg z_0 - \theta_0| \leq \pi/2 - \delta\), in the same region of \((s, \beta, \gamma)\) above, where the constants implied in the \(O\)-symbols may depend on \(a, K, c, s, \beta, \gamma\) and \(\delta\); this shows that (3.1) with (3.2) and (3.3) gives a complete asymptotic expansion in the ascending order of \(z_0\) as \(z_0 \rightarrow 0\) through the sector \(|\arg z_0 - \theta_0| < \pi/2\).

It can be seen that \(\lim_{K \rightarrow +\infty} R_{a,\lambda,K}^{+} (s, \beta, \gamma; z_0, z) = 0\) for \(|z_j| < a\) \((j = 0, 1, \ldots, n)\); this yields the following corollary.

**Corollary 1.** Let \((s, \beta, \gamma)\) be as in Theorem 1. Then the infinite series

\[
Z_{a,\lambda}^{(n)} (s, \beta, \gamma; z_0, z) = \sum_{k=0}^{\infty} (-1)^k (s)_k F_D^{(n)} \left( -k, \beta, \gamma; -\frac{z}{z_0} \right) \zeta(s + k, a, \lambda) z_0^k
\]

holds for all \((z_0, z)\) in the poly-disk \(|z_j| < a\) \((j = 0, 1, \ldots, n)\).

**Corollary 2.** Function \(Z_{a,\lambda}^{(n)} (s, \beta, \gamma; z_0, z)\) is continued to a one-valued meromorphic function of \(s\) over the whole \(s\)-plane, to the whole poly-sector \(|\arg z_j| < \pi\) \((j = 0, 1, \ldots, n)\), and for all \((\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}\); its only singularity, as a function of \(s\), is a (possible) simple pole at \(s = 1\) with the residue \(B_0(a, e(\lambda))\).

We next state the asymptotic expansion of \(Z_{a,\lambda}^{(n)} (s, \beta, \gamma; z_0, z)\) when \((z_0, z)\) becomes large.

**Theorem 2.** Let \(\theta_0\) be any angle fixed with \(\theta_0 \in [-\pi/2, \pi/2]\). Then for any integer \(K \geq 0\), in the region \(\sigma > -K\) except the point at \(s = 1\) the formula

\[
Z_{a,\lambda}^{(n)} (s, \beta, \gamma; z_0, z) = S_{a,\lambda,K}^{-} (s, \beta, \gamma; z_0, z) + R_{a,\lambda,K}^{-} (s, \beta, \gamma; z_0, z)
\]
holds for all \((z_0, z)\) in the poly-sector \(|\arg z_j - \theta_0| < \pi/2\) \((j = 0, 1, \ldots, n)\) and for all \((\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}\). Here

\[
S_{a,\lambda,K}^{-}(s, \beta, \beta; z_0, z) = \sum_{k=0}^{K-1} \frac{(-1)^{k+1}(s)_k}{(k+1)!} F_D^{(n)}(s + k, \beta; -\frac{z}{z_0}) B_{k+1}(a, e(\lambda)) z_0^{-s-k},
\]
and \(R_{a,\lambda,K}^{-}\) is the remainder term expressed as

\[
R_{a,\lambda,K}^{-}(s, \beta, \beta; z_0, z) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_k}{(k+1)!} F_D^{(n)}(s + k, \beta; -\frac{z}{z_0}) B_{k+1}(a, e(\lambda)) z_0^{-s-k},
\]

where \(u_K^-\) is a constant satisfying \(-\sigma - K < u_K^- < \min(-\sigma - K + 1, 0)\). Formula (3.4) further provides the analytic continuation of \(Z_{a,\lambda}^{(n)}(s, \beta; z_0, z)\) over the whole \(s\)-plane except at \(s = 1\), to the poly-sector \(|\arg z_j - \theta_0| < \pi/2\) \((j = 0, 1, \ldots, n)\), and for all \((\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}\). Moreover if \((z_0, z)\) is in \(|\arg z_j - \theta_0| \leq \pi/2 - \delta\) with any small \(\delta > 0\) \((j = 0, 1, \ldots, n)\), and satisfies

\[|z_j| \leq c|z_0| \quad (j = 1, \ldots, n)\]

for some constant \(c > 0\), then the estimates

\[F_D^{(n)}(s + k, \beta; -\frac{z}{z_0}) = O(1) \quad and \quad R_{a,\lambda,K}^{-}(s, \beta, \beta; z_0, z) = O(|z_0|^{-\sigma-K})\]

follow for all \(K > k \geq 0\) as \(z_0 \to \infty\) through \(|\arg z_0 - \theta_0| \leq \pi/2 - \delta\), in the same region of \((s, \beta, \gamma)\) as above, where the constants implied in the \(O\)-symbols may depend on \(a, K, c, s, \beta, \gamma\) and \(\delta\); this shows that (3.4) with (3.5) and (3.6) gives a complete asymptotic expansion in the descending order of \(z_0\) as \(z_0 \to \infty\) through the sector \(|\arg z_0 - \theta_0| < \pi/2\).

The cases \(s = -l\) \((l \in \mathbb{N}_0)\) of Theorem 2 reduce to the evaluations in finite closed form of \(Z_{a,\lambda}^{(n)}(s, \beta; z_0, z)\).

Corollary 3. Let \((\beta, \gamma)\) be as in Theorem 2, and \((z_0, z)\) in the poly-sector \(|\arg z_j - \theta_0| < \pi\) \((j = 0, 1, \ldots, n)\) with any angle fixed with \(\theta_0 \in [-\pi/2, \pi/2]\). Then for any \(l \in \mathbb{N}_0\) we have

\[
Z_{a,\lambda}^{(n)}(-l, \beta; \gamma; z_0, z) = \frac{1}{l+1} \sum_{k=1}^{l} \left( \begin{array}{c} l+1 \\ k+1 \end{array} \right) F_D^{(n)}(k-l, \beta; -\frac{z}{z_0}) B_{k+1}(a, e(\lambda)) z_0^{l-k}.
\]

The asymptotic expansions of \(Z_{a,\lambda}^{(n)}(s, \beta_n-1; z_0, z)\) can be derived from our main formulæ (3.1) and (3.4) through the confluence operation in (2.3); this asserts the following Theorems 3 and 4.

Theorem 3. Let \(\theta_0\) be any angle fixed with \(\theta_0 \in [-\pi/2, \pi/2]\). Then for any integer \(K \geq 0\), in the region \(\sigma > 1 - K\) except at \(s = 1\) the formula

\[
Z_{a,\lambda}^{(n)}(s, \beta_n-1; z_0, z) = S_{a,\lambda,K}^{+}(s, \beta_n-1; z_0, z) + R_{a,\lambda,K}^{+}(s, \beta_n-1; z_0, z)
\]
holds for all \((z_0, z)\) in the poly-sector \(|\arg z_j - \theta_0| < \pi/2\) \((j = 0, 1, \ldots, n)\) and for all \((\beta_n-1, \gamma) \in \mathbb{C}^{n-1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}\). Here

\[
S_{a,\lambda,K}^{+}(s, \beta_n-1; z_0, z) = \sum_{k=0}^{K-1} \frac{(-1)^{k}(s)_k}{k!} F_D^{(n)}(-k, \beta_n-1; -\frac{z}{z_0}) \zeta(s+k, a, \lambda) z_0^k,
\]

for all \((z_0, z)\) in the poly-sector \(|\arg z_j - \theta_0| < \pi/2\) \((j = 0, 1, \ldots, n)\) and for all \((\beta_n-1, \gamma) \in \mathbb{C}^{n-1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}\). Here
and $\hat{R}_{a,\lambda,K}^{+}$ is the remainder term expressed as

$$
\hat{R}_{a,\lambda,K}^{+} \left( s, \beta_{n-1}; z_{0}, z \right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_{K}^{+})} \Gamma(s+w, -w) \hat{F}_{D}^{(n)}(-w, \beta_{n-1}; -\frac{z}{z_{0}}) \times \zeta(s+w, a, \lambda) z_{0}^{w} dw,
$$

where $u_{K}^{+}$ is a constant satisfying $\max(1-\sigma, K-1) < u_{K}^{+} < K$. Formula (3.7) further provides the analytic continuation of $\hat{Z}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_{0}, z)$ over the whole $s$-plane except at $s=1$, to the poly-sector $|\arg z_{j} - \theta_{0}| < \pi/2 (j=0,1,\ldots,n)$, and for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{ \mathbb{C} \setminus (-\mathbb{N}_{0}) \}$. Moreover if $(z_{0}, z)$ is in $|\arg z_{j} - \theta_{0}| \leq \pi/2 - \delta$ with any small $\delta > 0$ $(j=0,1,\ldots,n)$, and satisfies $|z_{j}| \leq c|z_{0}| (j=1, \ldots, n)$ for some constant $c > 0$, then the estimates $\hat{F}_{D}^{(n)}(-k, \beta_{n-1}; -\frac{z}{z_{0}}) = O(1)$ and $\hat{R}_{a,\lambda,K}^{+} \left( s, \beta_{n-1}; z_{0}, z \right) = O(|z_{0}|^{K})$ follow for all $K > k \geq 0$ as $z_{0} \rightarrow 0$ through $|\arg z_{0} - \theta_{0}| < \pi/2 - \delta$ in the same region of $(s, \beta_{n-1}, \gamma)$ above, where the constants implied in the $O$-symbols may depend on $a, K, c, s, \beta_{n-1}, \gamma$ and $\delta$; this shows that (3.7) with (3.8) and (3.9) gives a complete asymptotic expansion in the ascending order of $z_{0}$ as $z_{0} \rightarrow 0$ through the sector $|\arg z_{0} - \theta_{0}| < \pi/2$.

**Corollary 4.** Let $(s, \beta_{n-1}, \gamma)$ be as in Theorem 3. Then the infinite series

$$
\hat{Z}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_{0}, z) = \sum_{k=0}^{\infty} \frac{(-1)^{k}(s)_{k}}{k!} \hat{F}_{D}^{(n)}(-k, \beta_{n-1}; -\frac{z}{z_{0}}) \zeta(s+k, a, \lambda) z_{0}^{k}
$$

holds for all $(z_{0}, z)$ in the poly-disk $|z_{j}| < a (j=0,1,\ldots,n)$.

**Theorem 4.** Let $\theta_{0}$ be any angle fixed with $\theta_{0} \in [-\pi/2, \pi/2]$. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s=1$ the formula

$$
\hat{Z}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_{0}, z) = \hat{S}_{a,\lambda,K}^{-}(s, \beta_{n-1}; z_{0}, z) + \hat{R}_{a,\lambda,K}^{-}(s, \beta_{n-1}; z_{0}, z)
$$

holds for all $(z_{0}, z)$ in the poly-sector $|\arg z_{j} - \theta_{0}| < \pi/2 (j=0,1,\ldots,n)$ and for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{ \mathbb{C} \setminus (-\mathbb{N}_{0}) \}$. Here

$$
\hat{S}_{a,\lambda,K}^{-} \left( s, \beta_{n-1}; z_{0}, z \right) = \sum_{k=-1}^{K-1} \frac{(-1)^{k}(s)_{k}}{k!} \hat{F}_{D}^{(n)}(s+k, \beta_{n-1}; -\frac{z}{z_{0}}) \times B_{k+1}(a, e(\lambda)) z_{0}^{-s-k},
$$

and $\hat{R}_{a,\lambda,K}^{-}$ is the remainder term expressed as

$$
\hat{R}_{a,\lambda,K}^{-} \left( s, \beta_{n-1}; z_{0}, z \right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_{K}^{-})} \Gamma(s+w, -w) \hat{F}_{D}^{(n)}(-w, \beta_{n-1}; -\frac{z}{z_{0}}) \times \zeta(s+w, a, \lambda) z_{0}^{w} dw,
$$

where $u_{K}^{-}$ is a constant satisfying $-\sigma - K < u_{K}^{-} < \min(-\sigma, K+1, 0)$. Formula (3.10) further provides the analytic continuation of $\hat{Z}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_{0}, z)$ over the whole $s$-plane.
except at $s = 1$, to the poly-sector $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$, and for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{ \mathbb{C} \setminus \{-\mathbb{N}_0\}\}$. Moreover, if $(z_0, z)$ is in $|\arg z_j - \theta_0| \leq \pi/2 - \delta$ with any small $\delta > 0$ $(j = 0, 1, \ldots, n)$, and satisfies

$$|z_j| \leq c|z_0| \quad (j = 1, \ldots, n)$$

for some constant $c > 0$, then the estimates

$$\hat{F}_D^{(n)}(s + k, \beta_{n-1}; -\frac{z}{z_0}) = O(1) \quad \text{and} \quad \hat{R}_{a, \lambda, K}^{-}(s, \beta; -\frac{z}{z_0}) = O(|z_0|^{-\sigma-K})$$

follow for all $K > k \geq 0$ as $z_0 \to \infty$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$ in the same region of $(s, \beta_{n-1}, \gamma)$ above, where the constants implied in the $O$-symbols may depend on $a, K, c, s, \beta_{n-1}, \gamma$ and $\delta$; this shows that (3.10) with (3.11) and (3.12) gives a complete asymptotic expansion in the descending order of $z_0$ as $z_0 \to \infty$ through the sector $|\arg z_0 - \theta_0| < \pi/2$.

**Corollary 5.** Let $(\beta, \gamma)$ be as in Theorem 4, and $(z_0, z)$ in the poly-sector $|\arg z_j - \theta_0| < \pi$ $(j = 0, 1, \ldots, n)$ with any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any $l \in \mathbb{N}_0$ we have

$$\hat{Z}_{a, \lambda}^{(n)}(-l, \beta; -\frac{z}{z_0}) = -\frac{1}{l+1} \sum_{k=-1}^{l} \binom{l+1}{k+1} \hat{F}_D^{(n)}(k-l, \beta_{n-1}; -\frac{z}{z_0}) B_{k+1}(a, e(\lambda)) z_0^{l-k}.$$

### Asymptotics for derivatives

We define the generalized Euler-Stieltjes constants $\gamma_m(a, e(\lambda)) (m \in \mathbb{N}_0)$ and the modified Stirling polynomials $\sigma_{m,n}(x) (m, n \in \mathbb{N}_0)$ respectively by the power series

$$\zeta(s, a, \lambda) = \frac{B_0(a, e(\lambda))}{s-1} + \sum_{m=0}^{\infty} \gamma_m(a, e(\lambda))(s-1)^m$$

centered at $s = 1$, and

$$\frac{1}{m!}(1-z)^{-x}\{-\log(1-z)\}^m = \sum_{n=0}^{\infty} \sigma_{m,n}(x)z^n$$

centered at $z = 0$. Note that $\sigma_{m,n}(x) = 0$ for $0 \leq n < m$. We further set

$$C_{k,l,m}(a, e(\lambda)) = \sum_{j=0}^{m} \frac{m!}{(m-j)!}\sigma_{j,k}(l) \left(\frac{\partial}{\partial s}\right)^{m-j} \zeta(s, a, \lambda)\bigg|_{s=l+k}$$

for any $k, l, m \in \mathbb{N}_0$. Then Theorem 1 yields:

**Corollary 6.** Let $(\beta, \gamma, z)$ be as in Theorem 1. For any integer $K \geq 1$ the following asymptotic expansions hold as $z_0 \to 0$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$ with any $\delta > 0$, while other $z_j$'s move through the same sector satisfying the conditions $|z_j| \leq c|z_0|$ $(j = 1, \ldots, n)$ with some constant $c > 0$:

1) when $s \to 1$,

$$\lim_{s \to 1} \left(\frac{\partial}{\partial s}\right)^m \left\{ Z_{a, \lambda}^{(n)}(s, \beta; z_0, z) - \frac{B_0(a, e(\lambda))}{s-1} \right\} = m!\gamma_m(a, e(\lambda)) + \sum_{k=1}^{K-1} \frac{(-1)^k}{k!} C_{k,1,m}(a, e(\lambda)) \hat{F}_D^{(n)}(-k, \beta; -\frac{z}{z_0}) z_0^k + O(|z_0|^K);$$

2) when $s \to \infty$,

$$\lim_{s \to \infty} \left(\frac{\partial}{\partial s}\right)^m \left\{ Z_{a, \lambda}^{(n)}(s, \beta; z_0, z) - \frac{B_0(a, e(\lambda))}{s} \right\} = m!\gamma_m(a, e(\lambda)) + \sum_{k=1}^{K-1} \frac{(-1)^k}{k!} C_{k+1,m}(a, e(\lambda)) \hat{F}_D^{(n)}(-k+1, \beta; -\frac{z}{z_0}) z_0^{k+1} + O(|z_0|^K);$$
ii) When \( s = l \) \( (l = 2, 3, \ldots) \),

\[
\left( \frac{\partial}{\partial s} \right)^m \mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, z) \bigg|_{s = l} = \sum_{k=0}^{K-1} \frac{(-1)^k}{k!} C_{k,l,m}(a, e(\lambda)) F_{D}^{(n)}(-k, \beta; \frac{z}{z_0}) z_0^k + O(|z_0|^K).
\]

It is known that \( \lim_{s \to 1} \{ \zeta(s, z) - 1/(s-1) \} = \gamma_0(z) = -\psi(z) = -(\Gamma'/\Gamma)(z) \). The case \((\lambda, \beta) = (0, 0)\) above reduces to the classical Taylor series expansion of \( \psi(a + z) \) (cf. [4]).

**Corollary 7.** For \( |z| < a \) we have

\[
\psi(a + z) = \psi(a) - \sum_{k=1}^{\infty} \left\{ \left( \sum_{h=1}^{k} \frac{1}{h} \right) \zeta(1+k, a) + \zeta'(1+k, a) \right\} z^k.
\]

We next define the polynomials \( \mathcal{P}_{l,m}, \mathcal{Q}_{k,l,m} \in \mathbb{C}[[x]][y] \) \( (k, l, m \in \mathbb{N}_0) \) by

\[
\mathcal{P}_{l,m}(\beta; x, y) = \sum_{j=0}^{m} m! \left\{ \sum_{i=0}^{j} \frac{(l+1)^{i-j-1}}{(j-i)!} \right\} (-y)^{m-j},
\]

\[
\mathcal{Q}_{k,l,m}(\beta; x, y) = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left\{ \sum_{i=0}^{j} \frac{\sigma_{i,j}(-l)}{(j-i)!} \right\} (-y)^{m-j},
\]

**Corollary 8.** Let \((\beta, \gamma, z)\) be as in Theorem 2, and \( l, m \in \mathbb{N}_0 \) arbitrary. Then for any integer \( K \geq l + 1 \) the asymptotic expansion

\[
\left( \frac{\partial}{\partial s} \right)^m \mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, z) \bigg|_{s = -l} = -B_0(a, e(\lambda)) z_0^{l+1} \mathcal{P}_{l,m}(\beta; \frac{z}{z_0}, \log z_0)
\]

\[
+ \sum_{k=0}^{K-1} \frac{(-1)^k}{k!(k+1)!} B_{k+1}(a, e(\lambda)) z_0^{l-k} \mathcal{Q}_{k,l,m}(\beta; \frac{z}{z_0}, \log z_0)
\]

\[
+ O(|z_0|^{l-K} \log^m |z_0|)
\]

holds as \( z_0 \to \infty \) through \( |\arg z_0 - \theta_0| \leq \pi/2 - \delta \) with any \( \delta > 0 \), while other \( z_j \)'s move through the same sector satisfying the conditions \( |z_j| \leq c|z_0| (j = 1, \ldots, n) \) with some constant \( c > 0 \).

It is known that \( \partial/\partial s \zeta(s, z) \big|_{s=0} = \log \{ \Gamma(z)/\sqrt{2\pi} \} \) (cf. [4]). The case \((n, \beta) = (2, 0)\) and \( \lambda \in \mathbb{Z} \) above reduces to the following variant of Stirling's formula (cf. [4]).

**Corollary 9.** For any integer \( K \geq 0 \) the asymptotic expansion

\[
\log \Gamma(a + z) = \left( z + a - \frac{1}{2} \right) \log z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a)}{k(k+1)} z^{-k}
\]

\[
+ O(|z|^{-K} \log |z|)
\]

holds as \( z \to \infty \) through \( |\arg z| \leq \pi - \delta \) with any small \( \delta > 0 \).
Applications of our main formulae with \( n = 2 \)

One can observe that the case \((n, \gamma) = (2, s)\) of (2.2) and (2.4) reduce respectively to the expressions

\[
Z_{a,\lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2) = \sum_{l=0}^{\infty} e(\lambda l)(a + l + z_0)^{-s} \left(1 + \frac{z_1}{a + l + z_0}\right)^{-\beta_1} \left(1 + \frac{z_2}{a + l + z_0}\right)^{-\beta_2},
\]

and

\[
\hat{Z}_{a,\lambda}^{(2)}(s, \nu; z_0, z_1, z_2) = \sum_{l=0}^{\infty} e(\lambda l)(a + l + z_0)^{-s} \left(1 + \frac{z_1}{a + l + z_0}\right)^{-\beta_1} \exp\left(-\frac{z_2}{a + l + z_0}\right).
\]

Theorems 1 and 2 in particular assert on (5.1) and (5.2) the following corollaries.

**Corollary 10.** Let \( \theta_0 \) be as in Theorem 1. The for any integer \( K \geq 0 \), in the region \( \sigma > 1 - K \) except at \( s = 1 \) Function \( Z_{a,\lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2) \) is represented as (3.1) in the poly-sector \( |\arg z_j - \theta_0| < \pi/2 \) \( (j=0,1, \ldots, n) \) and for all \( (\beta_1, \beta_2) \in \mathbb{C}^2 \), where

\[
S_{a,\lambda,K}^{+}(s, \beta_1, \beta_2; z_0, z_1, z_2) = \sum_{k=0}^{K-1} \frac{(-1)^k(s)_k}{k!} F_1\left(-k, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k,
\]

and

\[
R_{a,\lambda,K}^{+}(s, \beta_1, \beta_2; z_0, z_1, z_2) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^+)} \Gamma(s+w, \lambda) F_1\left(-w, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \zeta(s+w) z_0^w dw.
\]

These formulae give a complete asymptotic expansion of \( Z_{a,\lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2) \) as \( z_0 \to 0 \) through \( |\arg z_0 - \theta_0| < \pi/2 \) in the sense of Theorem 1.

**Corollary 11.** Let \( \theta_0 \) be as in Theorem 1. Then for any integer \( K \geq 0 \), in the region \( \sigma > -K \) except at \( s = 1 \) Function \( Z_{a,\lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2) \) is represented as (3.4) in the poly-sector \( |\arg z_j - \theta_0| < \pi/2 \) \( (j=0,1, \ldots, n) \) and for all \( (\beta_1, \beta_2) \in \mathbb{C}^2 \), where

\[
S_{a,\lambda,K}^{-}(s, \beta_1, \beta_2; z_0, z_1, z_2) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_k}{(k+1)!} F_1\left(s+k, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) z_0^{-s-k},
\]

and

\[
R_{a,\lambda,K}^{-}(s, \beta_1, \beta_2; z_0, z_1, z_2) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma(s+w, \lambda) F_1\left(-w, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \zeta(s+w) z_0^w dw.
\]

These formulae give a complete asymptotic expansion of \( Z_{a,\lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2) \) as \( z_0 \to \infty \) through \( |\arg z_0 - \theta_0| < \pi/2 \) in the sense of Theorem 2.
Corollary 12. Let  be as in Theorem 1. Then for any integer 0, in the region  except at  Function  is represented as (3.7) in the poly-sector  and for all  Here

\[
\hat{S}_{a,\lambda,K}^{+}\left(s, \beta_{1}; z_{0}, z_{1}, z_{2}\right) = \sum_{k=0}^{K-1} \frac{(-1)^{k}(s)_{k}}{k!} \Phi_{1}\left(-k, \beta_{1}, \frac{-z_{1}}{z_{0}}, \frac{-z_{2}}{z_{0}}\right) \zeta(s+k, a, \lambda) z_{0}^{k},
\]

and

\[
\hat{R}_{a,\lambda,K}^{+}\left(s, \beta_{1}; z_{0}, z_{1}, z_{2}\right) = \frac{1}{\sqrt{-1}} \int_{(u_{K}^{+})} \Gamma\left(s+w, -w\right) \Phi_{1}\left(-w, \beta_{1}, \frac{-z_{1}}{z_{0}}, \frac{-z_{2}}{z_{0}}\right) \zeta(s+w, a, \lambda) z_{0}^{w} dw.
\]

These formulae give a complete asymptotic expansion of  as  through in the sense of Theorem 3.

Corollary 13. Let  be as in Theorem 1. Then for any integer 0, in the region  except at  Function  is represented as (3.10) in the poly-sector  and for all  Here

\[
\hat{S}_{a,\lambda,K}^{-}\left(s, \beta_{1}; z_{0}, z_{1}, z_{2}\right) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_{k}}{(k+1)!} \Phi_{1}\left(s+k, \beta_{1}, \frac{-z_{1}}{z_{0}}, \frac{-z_{2}}{z_{0}}\right) B_{k+1}(a, e(\lambda)) z_{0}^{-s-k},
\]

and

\[
\hat{R}_{a,\lambda,K}^{-}\left(s, \beta_{1}; z_{0}, z_{1}, z_{2}\right) = \frac{1}{\sqrt{-1}} \int_{(u_{K}^{-})} \Gamma\left(s+w, -w\right) \Phi_{1}\left(-w, \beta_{1}, \frac{-z_{1}}{z_{0}}, \frac{-z_{2}}{z_{0}}\right) \zeta(s+w, a, \lambda) z_{0}^{w} dw.
\]

These formulae give a complete asymptotic expansion of  as  through in the sense of Theorem 4.

Further applications

We define for  and for  the functions

\[
C_{a,\lambda}(s, \beta; x, y) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+x)^{-s} \frac{\cos\left\{\beta \arctan\left(\frac{y}{a+l+x}\right)\right\}}{1 + \left(\frac{y}{a+l+x}\right)^{2}}^{\beta/2},
\]

\[
S_{a,\lambda}(s, \beta; x, y) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+x)^{-s} \frac{\sin\left\{\beta \arctan\left(\frac{y}{a+l+x}\right)\right\}}{1 + \left(\frac{y}{a+l+x}\right)^{2}}^{\beta/2},
\]

and their confluent forms

\[
\hat{C}_{a,\lambda}(s; x, y) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+x)^{-s} \cos\left(\frac{y}{a+l+x}\right),
\]

and
\[
S_{a,\lambda}(s; x, y) = \sum_{l=0}^{\infty} e(\lambda l) (a + l + x)^{-s} \sin \left( \frac{y}{a + l + x} \right).
\]

It is in fact possible to show that Theorems 1 and 2 are valid when \( n = 1 \) in a wider sector
\[
\max(-\pi, \arg z_0 - \pi) < \arg z_1 < \min(\pi, \arg z_0 + \pi),
\]
and this allows us to take \( z_0 = x \) and \( z_1 = e^{\pm \pi i/2} y \) with \( \arg x = 0 \) and \( \arg y = 0 \); the following Corollaries 14 and 15 are derived.

**Corollary 14.** Let \((s, \beta)\) be as in Theorem 1. Then for any \( s \in \mathbb{C} \) except at \( s = 1 - k \) \((k \in \mathbb{N}_0)\), and any \( x, y \in \mathbb{R} \) with \( |x|, |y| < a \) the following formulae hold:

\[
C_{a,\lambda}(s, \beta; x, y) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k}(s)_{k}}{k!} \left\{ {}_{2}F_{1}(^{-k, \beta_{;}}s \frac{iy}{x}) + {}_{2}F_{1}(^{-k, \beta_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{k},
\]

\[
\hat{C}_{a,\lambda}(s; x, y) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k}(s)_{k}}{k!} \left\{ {}_{1}F_{1}(^{-k_{;}}s \frac{iy}{x}) + {}_{1}F_{1}(^{-k_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{k},
\]

and similarly,

\[
S_{a,\lambda}(s, \beta; x, y) = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(-1)^{k}(s)_{k}}{k!} \left\{ {}_{2}F_{1}(^{s+k, \beta_{;}}s \frac{iy}{x}) - {}_{2}F_{1}(^{s+k, \beta_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{k},
\]

\[
\hat{S}_{a,\lambda}(s; x, y) = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(-1)^{k}(s)_{k}}{k!} \left\{ {}_{1}F_{1}(^{s+k_{;}}s \frac{iy}{x}) - {}_{1}F_{1}(^{s+k_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{k}.
\]

**Corollary 15.** Let \((s, \beta)\) be as in Theorem 2. Then for any integer \( K \geq 0 \) in the region \( \sigma > -K \) except at \( s = 1 - k \) \((k \in \mathbb{N}_0)\) the following asymptotic expansions hold as \( x \to +\infty \), while \( y \) satisfies \( y \ll x \):

\[
C_{a,\lambda}(s, \beta; x, y) = \frac{1}{2} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_{k}}{(k+1)!} \left\{ {}_{2}F_{1}(^{s+k, \beta_{;}}s \frac{iy}{x}) + {}_{2}F_{1}(^{s+k, \beta_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{-s-k} + O(x^{-\sigma-K}),
\]

\[
\hat{C}_{a,\lambda}(s; x, y) = \frac{1}{2} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_{k}}{(k+1)!} \left\{ {}_{1}F_{1}(^{s+k_{;}}s \frac{iy}{x}) + {}_{1}F_{1}(^{s+k_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{-s-k} + O(x^{-\sigma-K}),
\]

and similarly,

\[
S_{a,\lambda}(s, \beta; x, y) = \frac{1}{2i} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_{k}}{(k+1)!} \left\{ {}_{2}F_{1}(^{s+k, \beta_{;}}s \frac{iy}{x}) - {}_{2}F_{1}(^{s+k, \beta_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{-s-k} + O(x^{-\sigma-K}),
\]

\[
\hat{S}_{a,\lambda}(s; x, y) = \frac{1}{2i} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_{k}}{(k+1)!} \left\{ {}_{1}F_{1}(^{s+k_{;}}s \frac{iy}{x}) - {}_{1}F_{1}(^{s+k_{;}}s \frac{-iy}{x}) \right\} \zeta(s+k, a, \lambda) x^{-s-k} + O(x^{-\sigma-K}).
\]
References


[13] M. Lerch, *Note dur la fonction $K(w, x, s) = \sum_{n \geq 0} \exp\{2\pi inx\}(n+w)^{-s}$*, Acta Math. 11 (1887), 19–24.


Department of Mathematics, Hiyoshi Campus, Keio University, 4–1–1 Hiyoshi, Kouhoku-ku, Yokohama 223–8521, Japan

E-mail address: katsurad@z3.keio.jp