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On the simultaneous equations $\sigma(2^a)=p^{f_1}q^{g_1}, \sigma(3^b)=p^{f_2}q^{g_2}, \sigma(5^c)=p^{f_3}q^{g_3}$ *†

Tomohiro Yamada

Abstract

We shall solve the simultaneous equations $\sigma(2^a)=p^{f_1}q^{g_1}, \sigma(3^b)=p^{f_2}q^{g_2}, \sigma(5^c)=p^{f_3}q^{g_3}$.

1 Introduction

We denote by $\sigma(N)$ the sum of divisors of N.

In the preprint [14], the author has shown that there are only finitely many odd superperfect numbers (i.e. the number satisfying $\sigma(\sigma(N))=2N$) with bounded number of distinct prime factors. In this preprint, we showed that the simultaneous equation $\sigma(p_i^{e_i})=q_1^{f_{1i}}\cdots q_k^{f_{ki}}$ for $2k+1$ prime powers $p_i^{e_i}(i=1,2,\ldots,2k+1)$ cannot have small solutions $p_1,\cdots,p_{2k+1}$.

Here we use the method in the preprint to solve the simultaneous equations $\sigma(2^a)=p^{f_1}q^{g_1}, \sigma(3^b)=p^{f_2}q^{g_2}, \sigma(5^c)=p^{f_3}q^{g_3}$.

Wakulicz[12] has shown that all solutions of $2^n-5^m=3$ are $(n,m)=(2,0),(3,1)$ and $(7,3)$, from which Makowski and Schinzel[6] derived that $\sigma(2^a)=\sigma(5^c)$ have only the solution $(a,c)=(4,2)$. We note that it is easy to show that $\sigma(2^a)=\sigma(3^b)$ has no nontrivial solution and $\sigma(3^b)=\sigma(5^c)$ also has no nontrivial solution.

Bugeaud and Mignotte[3] has shown that neither of $\sigma(2^a),\sigma(3^b),\sigma(5^c)$ can be perfect power except $\sigma(3^4)=11^2$. Moreover, they have shown that the only perfect powers $\frac{x^n-1}{x-1}$ with $x=z^t, z\le 10$ are $\frac{3^5-1}{3-1}=11^2$ and $\frac{7^4-1}{7-1}=20^2$.

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Now we shall state our result.

Theorem 1.1. *The simultaneous equations* $\sigma(2^a) = p^{f_1}q^{g_1}, \sigma(3^b) = p^{f_2}q^{g_2}, \sigma(5^c) = p^{f_3}q^{g_3}$ *with* $a, b, c > 0, f_1, f_2, f_3, g_1, g_2, g_3 \geq 0$ *has only the following solutions:*

1. $(a, b, c) = (1, 1, 1)$.
2. $(a, b, c) = (4, 1, 2)$,
3. $(a, b, c) = (4, 4, 2)$ *and*
4. $(a, c) = (4, 2)$ *and* $\sigma(3^b)$ *is prime.*

Our results are related to the Nagell-Ljunggren equation

$$\frac{x^n - 1}{x - 1} = y^m, x \geq 2, y \geq 2, n \geq 3, q \geq 2, \tag{1}$$

which has been conjectured to have only finitely many solutions. Some of recent remarkable results are [2], [3], [8] and [9].

Now we are led to conjecture that there exists an integer n_0 such that the equation

$$\frac{x^n - 1}{x - 1} = y^m z^l, x \geq 2, y \geq 2, z \geq 2, n, m, l \geq n_0 \tag{2}$$

has only finitely many solutions. Theorem 1.1 can be seen to support this conjecture.

2 Preliminary Lemmas

In this section, we introduce some preliminary lemmas. One is Matveev's lower bound for linear forms of logarithms [7].

Lemma 2.1. *Let* $a_1, a_2, \ldots, a_n$ *be nonzero integers such that* $\log a_1, \ldots, \log a_n$ *are not all zero. For each* $j = 1, \ldots, n$, *let* $A_j \geq \max\{0.16, \log a_j\}$.

Put

$$\begin{aligned} B &= \max\{1, |b_1| A_1/A_n, |b_2| A_2/A_n, \ldots, |b_n|\}, \\ \Omega &= A_1 A_2 \ldots A_n, \\ C_0 &= 1 + \log 3 - \log 2, \\ C_1(n) &= \frac{16}{n!} e^n (2n+3)(n+2)(4(n+1))^{n+1}(\frac{1}{2}en)(4.4n + 5.5\log n + 7) \end{aligned} \tag{3}$$

and

$$\Lambda = b_1 \log a_1 + \ldots + b_n \log a_n. \tag{4}$$

Then we have

$$\log |\Lambda| > -C_1(n)(C_0 + \log B) \max\left\{1, \frac{n}{6}\right\} \Omega. \tag{5}$$

The others concern to some arithmetical properties of values of cyclotomic polynomials. Lemma 2.2 is a basic and well-known result of this area. Lemma 2.2 has been proved by Zsigmondy[15] and rediscovered by many authors such as Dickson[4] and Kanold[5]. See also Theorem 6.4A.1 in [11]. Lemma 2.3 is proved in [3], as mentioned above.

Lemma 2.2. *If $a > b \geq 1$ are coprime integers, then $a^n - b^n$ has a prime factor which does not divide $a^m - b^m$ for any $m < n$, unless $(a, b, n) = (2, 1, 6)$ or $a - b = n = 1$, or $n = 2$ and $a + b$ is a power of 2.*

Lemma 2.3. *Let a, e, x, f be positive integers with $a, x, f > 1$ and $e > 2$. The equation $(a^e - 1)/(a - 1) = x^f$ has no solution but $(a, e, x, f) = (3, 5, 11, 2), (7, 4, 20, 2)$ in integers $2 \leq a \leq 10, e > 2, x > 1, f > 1$.*

Using Lemmas 2.2 and 2.3, we can prove the following lemma.

Lemma 2.4. *If $(a^e - 1)/(a - 1) = p^{f_1} q^{f_2}$ for some integers a, e, f_1, f_2 and prime $p < q$, then we have $(a, e, p, q, f_1, f_2) = (2, 6, 3, 7, 2, 1)$, $e = r$ or $e = r^2$ for some prime r. Moreover, in the case $e = r$, then we have $p \geq r$. In the case $e = r^2$, we have $(p, q, f_1, f_2) = ((a^r - 1)/(a - 1), (a^{r^2} - 1)/(a^r - 1), 1, 1)$ or $(a, e, p, f_1) = (2^m - 1, 4, 2, m + 1)$ for some integer m.*

3 Main Theory

For convenience, we put $a_1 = 2, a_2 = 3, a_3 = 5$ and $e_1 = a+1, e_2 = b+1, e_3 = c + 1$.

Lemma 3.1. *For each $i = 1, 2, 3$, we have*

$$e_i \log a_i < E_i = C_i \log p \log q(\log \log p + C_{i+3}), \tag{6}$$

where $C_1 = 1.5 \times 10^{10}, C_2 = 1.3 \times 10^{12}, C_3 = 1.9 \times 10^{12}, C_4 = 1.3 \times 10^{10}, C_5 = 1.1 \times 10^{12}, C_6 = 1.6 \times 10^{12}$.

Proof. We may assume that $e_1, e_2, e_3 > 10^{10} \log q$ and $q > 10$.

Let $\Lambda_i = f_1 \log a_i + g_1 \log q + \log(a_i - 1) - e_i \log 2 = \log(1 - a_i^{-e_i})$ for $i = 1, 2, 3$.

Matveev's theorem gives

$$-\log|\Lambda_1| < C(3)(C_0 + \log(e_1 \log 2/\log q)) \log 2 \log p \log q, \tag{7}$$

$$-\log|\Lambda_2| < C(4)(C_0 + \log(e_2 \log 3/\log q)) \log 2 \log 3 \log p \log q \tag{8}$$

and

$$-\log|\Lambda_3| < C(4)(C_0 + \log(e_3 \log 5/\log q)) \log 2 \log 5 \log p \log q. \tag{9}$$

Now we shall show (6) in the case $i = 1$. Since $0 < |\Lambda_1| = -\log(1 - 2^{-e_1}) < \frac{1}{2^{e_1}-1}$, we have $-\log|\Lambda_1| > \log(2^{e_1} - 1) \geq (1 - 10^{-10}) e_1 \log 2$.

Combining upper and lower bounds for Λ_1, we obtain

$$\frac{e_1 \log 2}{\log q} < (1+10^{-10}) \frac{C_0 + \log(10^{10})}{C_0} C(3) \log 2 \log(e_1 \log 2/\log q) \log p. \tag{10}$$

This gives (6) in the case $i = 1$.

Next we shall prove (6) in the case $i = 2$. Since $0 < |\Lambda_2| = -\log(1 - 3^{-e_2}) < \frac{1}{3^{e_2}-1}$, we have $-\log|\Lambda_2| > \log(3^{e_2} - 1) \geq (1 - 10^{-10}) e_2 \log 3$.

Combining upper and lower bounds for Λ_1, we obtain

$$\frac{e_2 \log 3}{\log q} < (1+10^{-10}) \frac{C_0 + \log(10^{10})}{C_0} C(4) \times \log 2 \log 3 \log(e_2 \log 3/\log q) \log p. \tag{11}$$

Since $0 < |\Lambda_2| = -\log(1 - 3^{-e_2}) < \frac{1}{3^{e_2}-1}$, we have $-\log|\Lambda_2| > \log(3^{e_2} - 1) \geq (1 - 10^{-10}) e_2 \log 3$ and therefore

$$\frac{e_2 \log 3}{\log q} < (1 + 10^{-10}) \frac{C_0 + \log(10^{10})}{C_0} C(4) \log 2 \log 3 \log(e_2 \log 3/\log q) \log p. \tag{12}$$

This gives (6) in the case $i = 2$.

A similar argument yields (6) in the case $i = 3$. This completes the proof of the lemma. □

Next, we shall show that we cannot have all of $a_i^{e_i}$'s small.

Lemma 3.2. *Let x be the smallest among $a_i^{e_i}$'s. Let $h_1 = f_2g_3 - f_3g_2, h_2 = f_3g_1 - f_1g_3$ and $h_3 = f_1g_2 - f_2g_1$ and $H = \max|h_i|$. Then*

$$\log x \leq \log(7H/4) + C(3)(C_0 + \log((e_1+2)H)) \log 2 \log 3 \log 5. \qquad (13)$$

Proof. We begin by observing that

$$(2^{e_1}-1)^{h_1}(\frac{3^{e_2}-1}{2})^{h_2}(\frac{5^{e_3}-1}{4})^{h_3} = 1. \qquad (14)$$

Now we put

$$\begin{aligned} \Lambda &= (e_1h_1 - h_2 - 2h_3)\log 2 + e_2h_2 \log 3 + e_3h_3 \log 5 \\ &= h_1 \log \frac{2^{e_1}}{2^{e_1}-1} + h_2 \log \frac{3^{e_2}}{3^{e_2}-1} + h_3 \log \frac{5^{e_3}}{5^{e_3}-1}. \end{aligned} \qquad (15)$$

Then we have

$$0 < |\Lambda| \leq H(\frac{1}{2^{e_1}-1} + \frac{1}{3^{e_2}-1} + \frac{1}{5^{e_3}-1}) \leq \frac{7H}{4x} \qquad (16)$$

and therefore

$$\log|\Lambda| \leq -\log x + \log(7H/4). \qquad (17)$$

It follows from the assumption $e_i > 0$ that $\Lambda \neq 0$. Hence Matveev's lower bound gives

$$\log|\Lambda| \geq -C(3)(C_0 + \log((e_1+2)H)) \log 2 \log 3 \log 5. \qquad (18)$$

Combining (17) and (18), we obtain (13). □

The third step is to obtain upper bounds for each e_i.

Lemma 3.3. *We have $e_1 < 1.1 \times 10^{59}, e_2 < 10^{63}$ and $e_3 < 1.5 \times 10^{63}$.*

Proof. We begin by considering the case $q \mid x$. In this case, we have $\log q < \log x < \log(7H/4) + C(3)(C_0 + \log((e_1+2)H)) \log 2 \log 3 \log 5$. We note that $H \leq C_2C_3 \log p \log q(\log\log p + C_5)(\log\log p + C_6)$. By Lemma 3.1, we have $f_i \leq C_i \log q(\log\log p + C_{i+3})$, $g_i \leq C_i \log p(\log\log p + C_{i+3})$ and therefore $H < C_2C_3(\log q)^2(\log\log q + C_5)(\log\log q + C_6)$. Hence we obtain $\log p < \log q < 5.8 \times 10^{12}$.

Now we consider the case $q \nmid x$. Put i to be the index such that $x = (a_i^{e_i}-1)/(a_i-1)$, j, k be the others and

$$\begin{aligned} \Lambda' &= e_jh_j \log a_j + e_kh_k \log a_k - h_j \log(a_j - 1) - h_j \log(a_j-1) + h_3 \log x \\ &= h_j \log \frac{a_j^{e_j}}{a_j^{e_j}-1} + h_k \log \frac{a_k^{e_k}}{a_k^{e_k}-1}. \end{aligned} \qquad (19)$$

Now Lemma 2.3 implies that $(a^e - 1)/(a - 1) = p^f$ with $a \in \{2, 3, 5\}$ implies that $f = 1$ unless $(a, e, p, f) = (3, 5, 11, 2)$. Therefore we see that $x = p_1$ or $(p_1, x) = (11, 11^2)$, and $(a_j^{e_j} - 1)/(a_j - 1)$ and $(a_k^{e_k} - 1)/(a_k - 1)$ must be divisible by p_2.

Then we have

$$0 < \Lambda' < H(\frac{1}{a_1^{e_1} - 1} + \frac{1}{a_2^{e_2} - 1}) \leq \frac{3H}{2q}. \tag{20}$$

Similarly to the above, Matveev's theorem now gives

$$\log |\Lambda'| \geq -C(4)(C_0 + \log(E_3 H/\log x)) \log 2 \log 3 \log 5 \log x. \tag{21}$$

Combining (20) and (21), we obtain

$$\log q \leq \log(3H/2) + C(4)(C_0 + \log(E_3 H/\log x)) \log 2 \log 3 \log 5 \log x. \tag{22}$$

Since $E_3 = C_3 \log p \log q(\log\log p + C_6) \leq C_3 \log x \log q(\log\log x + C_6)$ and $H < C_2 C_3 (\log q)^2 (\log\log q + C_5)(\log\log q + C_6)$, combining (13) and (22), we obtain $\log q < 6.0 \times 10^{25}$. Moreover, $\log p = \log x < \log(7H/4) + C(3)(C_0 + \log((e_1 + 2)H)) \log 2 \log 3 \log 5$ gives $\log p < 7.1 \times 10^{12}$.

Now we conclude that in both cases, we have $\log p < 7.1 \times 10^{12}$ and $\log q < 6.0 \times 10^{25}$. Observing that $(e_1 - 1) \log 2 < f_1 \log p + g_1 \log q, (e_2 - 1) \log 3 < f_2 \log p + g_2 \log q$ and $(e_3 - 1) \log 5 < f_3 \log p + g_3 \log q$, we have $e_1 < 1.1 \times 10^{59}, e_2 < 10^{63}$ and $e_3 < 1.5 \times 10^{63}$. □

The last step is to reduce our upper bounds into feasible ones.

Lemma 3.4. $x \leq 1550712$.

Since $x \geq 2^H - 1$, we have

$$|\Lambda| < \frac{7H}{4x} < \frac{7 \times 2^H}{4(2^H - 1)} \exp(\log H - H \log 2). \tag{23}$$

Let M be the matrix defined by $m_{12} = m_{13} = m_{21} = m_{23} = 0$ and $m_{11} = m_{22} = \gamma$ and $m_{3i} = \lfloor C\gamma \log a_i \rfloor$. L be the reduced matrix of M.

Now we know that $H < H_0 = 1.5 \times 10^{126}$ and Lemma 3.7 of de Weger's book[13] with $C = 10^{380}, \gamma = 2$ gives that $X_1 > H_0$ and we see that (23) has no solutions with $X_1 > H > H_1 = 854$. So that $H \leq 854$.

Iterating this argument with $C = 10^{10}, \gamma = 3$ gives that $X_1 > H_1$ and we see that $H \leq H_2 = 30$. Finally, iterating this argument with $C = 150000, \gamma = 3$ gives that $X_1 > H_2$ and we see that $H \leq 19$.

Now we have $|\Lambda| \geq -15\log 2 + 8\log 3 + \log 5$ for $H \leq 19$. Since $\frac{7H}{4x} > -15\log 2 + 8\log 3 + \log 5 = 0.001128\cdots$, we conclude that $x \leq 1550712$.

The final step is checking all possibilities of x.

If $x = 2^{e_1} - 1$, then $e_1 \in \{2, 3, 4, 5, 6, 7, 9, 11, 13, 17, 19\}$. If $x = (3^{e_2} - 1)/2$, then $e_2 \in \{2, 3, 4, 5, 7, 9, 11, 13\}$. Moreover, if $x = (5^{e_3} - 1)/4$, then $e_3 \in \{2, 3, 5, 7\}$.

Here we exhibit only the proof of $x \neq 2^9 - 1$. If $x = 2^9 - 1 = 7 \times 73$, then $(p, q) = (7, 73)$. So that p must divide either $3^{e_2} - 1$ or $5^{e_3} - 1$. If $p \mid (3^{e_2} - 1)$, then $6 \mid e_2$, which is impossible by Lemma 2.4. If $p \mid 5^{e_3} - 1$, then $6 \mid e_3$, which contradicts 2.4 again. Thus x cannot be $2^9 - 1$.

4 Consequences from the abc conjecture

In Aug. 31. 2012, Mochizuki[10] claims to prove the abc conjecture. If Mochizuki's proof is right, Mochizuki's theorem gives that, if $(x^n - 1)/(x - 1) = y^m z^l$ with $n \geq 3, lm \geq 2$ and $y < z$, then for any given $\epsilon > 0$, up to only finitely many counterexamples, we have

1. $(n, m, l) = (3, 1, 2)$,
2. $(n, l) = (3, 2), m \geq 2$ and $\log y < \epsilon \log z$,
3. $(n, m, l) = (3, 1, 3), (4, 1, 2)$ and $\log y < (1 + \epsilon) \log z$, or
4. $l = 1, m \geq 2$ and $\log y < \frac{1+\epsilon}{(n-2)m-(n-1)} \log z$.

Moreover, Mochizuki's theorem implies that for any fixed y, z, $(x^n - 1)/(x-1) = y^m z^l$ has at most two integer solutions. Another consequence of Mochizuki's theorem is that $(x_1^{n_1} - 1)/(x-1) = y^{m_1} z^{l_1}$ and $(x_2^{n_2} - 1)/(x-1) = y^{m_2} z^{l_2}$ have only finitely many solutions in $(x_1, x_2, n_1, n_2, y, z, m_1, m_2, l_1, l_2)$ with $y < z$ and $n_1, n_2 \geq 3, l_1, l_2 \geq 1$.

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Tomohiro Yamada
Center for Japanese language and culture
Osaka University
562-8558
8-1-1, Aomatanihigashi, Minoo, Osaka
Japan
e-mail: `tyamada1093@gmail.com`