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The Stokes semigroup on spaces of bounded functions

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Abstract

In this brief note, we review recent results on the Stokes semigroup on spaces of bounded functions especially for bounded domains based on the papers [1], [3] (and also [2]). The Stokes semigroup on a bounded domain is an analytic semigroup on spaces of bounded functions as was recently proved in [1] based on an a priori L^{∞}-estimate for solutions to the linear Stokes equations. The proof for the a priori L^{∞}-estimate is a blow-up argument. Very recently, a direct approach for the analyticity of the semigroup is found in [3], where a necessary resolvent estimate is established by so called Masuda-Stewart technique for elliptic operators. In this note, we sketch the proofs for the analyticity of the semigroup on L^{∞} both in indirect and direct ways.

1 Introduction

We consider the initial-boundary problem for the Stokes equations in the domain $\Omega\subset\mathbf{R}^{n}$, $n\geq 2$:

$$\begin{aligned} v_{t}-\Delta v+\nabla q=0 \quad &\text{in} \quad \Omega\times(0,T), && (1.1)\\ \operatorname{div} v=0 \quad &\text{in} \quad \Omega\times(0,T), && (1.2)\\ v=0 \quad &\text{on} \quad \partial\Omega\times(0,T), && (1.3)\\ v=v_{0} \quad &\text{on} \quad \Omega\times\{t=0\}. && (1.4)\end{aligned}$$

It is well known that the solution operator of the linear Stokes equations $S(t):v_{0}\mapsto v(\cdot,t)$, called the Stokes semigroup, is an analytic semigroup on L^{r}-solenoidal space, $r\in(1,\infty)$, for various kinds of domains including bounded domains with smooth boundaries [27], [9]. However, it had been a long-standing open problem whether or not the Stokes semigroup is an analytic semigroup on L^{∞}-type spaces even if the domain Ω is bounded. For a half space the Stokes semigroup is an analytic semigroup on L^{∞}-type spaces since explicit solution formulas are available [6], [28], [19]. In this note, we review recent results on the analyticity of the semigroup on L^{∞} especially for bounded domains based on works [1], [3] (and also [2]).

To state a result, let $C_{0,\sigma}(\Omega)$ denote the L^{∞}-closure all smooth solenoidal vector fields with compact support in Ω. When Ω is bounded, $C_{0,\sigma}(\Omega)$ agrees with the space of all continuous solenoidal vector fields vanishing on $\partial\Omega$ [18], [1]. Our typical result is the following:

Theorem 1.1 ([1]). *Let Ω be a bounded domain in $\mathbf{R}^{n}$, $n\geq 2$, with C^{3}-boundary. Then the Stokes semigroup $S(t):v_{0}\mapsto v(\cdot,t)$ is a C_{0}-analytic semigroup on $C_{0,\sigma}(\Omega)$.*

For the Laplace operator or general elliptic operators it is well known that the corresponding semigroup is analytic on L^∞-type spaces. K. Masuda was the first to prove the analyticity of the semigroup associated to general elliptic operators on $C_0(\mathbf{R}^n)$ including the case of higher orders [20], [21], [22]. This result was then extended by H. B. Stewart to the case for the Dirichlet problem [31] and more general boundary condition [32]. We refer to a book by A. Lunardi [16, Chapter 3] for this Masuda-Stewart method which applies to many other situations. However, it seems that their localization argument does not directly apply to the Stokes equations because of the presence of pressure.

In the sequel, we review two approaches in proving the analyticity of the Stokes semigroup on L^∞. The analyticity of the Stokes semigroup on L^∞ was first proved by a contradiction argument called a blow-up argument [1]. We sketch the proof for an a priori L^∞-estimate for solutions to the non-stationary Stokes equations (1.1)-(1.4). Recently, a direct proof is found in [3], where a necessary resolvent estimate is established by the Masuda-Stewart technique for elliptic operators. The former is the original proof based on a heuristic observation. The latter is rather involved, but we are able to prove the maximum angle of the analytic semigroup on L^∞ which does not follow from a contradiction argument.

2 A blow-up argument

A blow-up argument is a typical indirect argument to obtain an a priori upper bound for solutions; see [11], [23], [24] for semilinear heat equations and [14], [12] for the Navier-Stokes equations. Let us give a heuristic idea of our argument. Our goal is to establish the a priori L^∞-estimate for solutions (v, q) of the form,

$$\sup_{0<t\leq T_0} \left\|N(v,q)\right\|_{L^\infty(\Omega)}(t) \leq C\|v_0\|_{L^\infty(\Omega)} \tag{2.1}$$

for some T_0 and the constant C, where $N(v,q)(x,t)$ denotes the norm for solutions up to second orders,

$$N(v,q)(x,t) = \left|v(x,t)\right| + t^{1/2}\left|\nabla v(x,t)\right| + t\left|\nabla^2 v(x,t)\right| + t\left|v_t(x,t)\right| + t\left|\nabla q(x,t)\right|. \tag{2.2}$$

The a priori estimate (2.1) in particular implies that the Stokes semigroup is (a positive angle of) a C_0-analytic semigroup on $C_{0,\sigma}(\Omega)$. We define analytic semigroups for semigroups. For the Banach space X and the semigroup $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ we call $T(t)$ an analytic semigroup if $t\|\mathrm{d}T(t)/\mathrm{d}t\|_{\mathcal{L}}$ is bounded in $(0,1]$, where $\mathcal{L}(X)$ denotes the space of all bounded linear operators from X onto itself and is equipped with the norm $\|\cdot\|_{\mathcal{L}}$. Although the angle of the analytic semigroup depends on the constant in (2.1), the estimate (2.1) is stronger than that of the resolvent estimate discussed later in Section 3. The following statement is a special case of general analyticity results proved in [1].

Theorem 2.1 ([1]). *Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary. Then there exist constants T_0 and C such that the a priori L^∞-estimate (2.1) holds for all solutions (v,q) for $v_0 \in C^\infty_{c,\sigma}(\Omega)$. In particular, the Stokes semigroup $S(t) : v_0 \mapsto v(\cdot,t)$ is a C_0-analytic semigroup on $C_{0,\sigma}(\Omega)$.*

To argue by contradiction, suppose that the estimate (2.1) were false for any choice of constants T_0 and C. Then, there are a sequence of solutions $\{(v_m, q_m)\}_{m=1}^{\infty}$ and a sequence of points $t_m \downarrow 0$ such that

$$\sup_{0<t\leq t_m} \left\|N(v_m, q_m)\right\|_{L^\infty(\Omega)}(t) \leq 1, \tag{2.3}$$

$$\|v_{0,m}\|_{L^\infty(\Omega)} \leq \frac{1}{m}, \tag{2.4}$$

$$\left\|N(v_m, q_m)\right\|_{L^\infty(\Omega)}(t_m) \geq \frac{1}{2}. \tag{2.5}$$

We take the point $x_m \in \Omega$ such that $N(v_m, q_m)(x_m, t_m) \geq 1/4$ and resale (v_m, q_m) around the point (x_m, t_m) to get the blow-up sequence,

$$u_m(x,t) = v_m(x_m + t_m^{1/2}x, t_m t), \quad p_m(x,t) = t_m^{1/2} q_m(x_m + t_m^{1/2}x, t_m t).$$

Then, the blow-up sequence (u_m, p_m) solves the Stokes equations in the domain $\Omega_m \times (0,1]$, where $\Omega_m = \Omega_{x_m}/t_m^{1/2}$ is the rescaled domain which expands to either the whole space or a half space depending on whether $d_m/t_m^{1/2}$, $d_m = d_\Omega(x_m)$, converges or not. Here, $d_\Omega(x)$ denotes the distance from $x \in \Omega$ to the boundary $\partial\Omega$. The estimates (2.3)-(2.5) are inherited to the estimates

$$\sup_{0<t\leq 1} \left\|N(u_m, p_m)\right\|_{L^\infty(\Omega_m)}(t) \leq 1, \tag{2.6}$$

$$\|u_{0,m}\|_{L^\infty(\Omega_m)} \leq \frac{1}{m}, \tag{2.7}$$

$$N(u_m, p_m)(0,1) \geq \frac{1}{4}. \tag{2.8}$$

The basic strategy is to show the compactness of the blow-up sequence (u_m, p_m) and the uniqueness of its limit. If (u_m, p_m) (subsequently) converges to a limit (u, p) strongly enough, (2.8) implies $N(u,p)(0,1) \geq 1/4$. If the limit (u,p) is unique, it is natal to expect $u \equiv 0$ and $\nabla p \equiv 0$. This yields a contradiction. The first part is "compactness" of a blow-up sequence and the second part is "uniqueness" for the limit problem. If the problem is the heat equation, it is easy to realize this argument. However, for the Stokes equations this strategy is highly non-trivial because of the presence of pressure.

To solve both compactness of the blow-up sequence and uniqueness of its limit, a key is *the harmonic-pressure gradient estimate* in terms of velocity,

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x,t)\right| \leq C_\Omega \left\|W(v)\right\|_{L^\infty(\partial\Omega)}(t) \tag{2.9}$$

for $W(v) = -(\nabla v - \nabla^T v)n_\Omega$. When $n = 3$, the tangential vector field $W(v)$ agrees with the tangential component of vorticity, i.e., $-\mathrm{curl}\, v \times n_\Omega$. Here, n_Ω denotes the unit outward normal vector field on $\partial\Omega$. The estimate (2.9) is a special case of an estimate for solutions of the homogeneous Neumann problem. We invoke that the pressure q is harmonic in Ω. A key observation is that the Neumann data of the pressure q is transformed into the surface divergence of the tangential component of vorticity, i.e., $\Delta v \cdot n_\Omega = \mathrm{div}_{\partial\Omega}\, W(v)$ as $\mathrm{div}\, v = 0$ in Ω. Then, the estimate (2.9) is reduced to investigating an a priori estimate for solutions of the homogeneous Neumann problem:

$$\Delta q = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial n_\Omega} = \mathrm{div}_{\partial\Omega} W \text{ on } \partial\Omega. \tag{2.10}$$

The question is for what kind of domains the estimate (2.9) holds. Since the estimate (2.9) may not hold for general domains, we call Ω *strictly admissible* if the a priori estimate (2.9) holds for all solutions of the Neumann problem (2.9). Of course, a half space is strictly admissible. It is proved in [1], [2] by a blow-up argument that bounded and exterior domains with C^3-boundaries are strictly admissible.

Lemma 2.2 ([1]). *Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary. Then there exists a constant C such that the a priori estimate*

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x)\right| \leq C\|W\|_{L^\infty(\partial\Omega)} \tag{2.11}$$

holds for all solutions of the Neumann problem (2.10) for tangential vector fields $W \in L^\infty(\partial\Omega)$.

Recently, it turned out that the estimate (2.11) was also found by C. E. Kenig, F. Lin, and Z. Shen [13], independently of the works [1], [2]. In [13] they proved the estimate (2.11) for $C^{1,\gamma}$-bounded domains directly by estimating the Green function. Note that for layer-type domains the estimate (2.11) does not hold. In fact, $q = x^1$ does not satisfy the estimate (2.4) in a layer $\Omega = \{a < x_n < b\}$. Thus, layer-type domains are not strictly admissible. We conjecture that quasi-cylindrical domains, i.e., $\overline{\lim}_{|x|\to\infty} d_\Omega(x) < \infty$, are not strictly admissible (see [4, 4, 6.32]).

We apply the harmonic-pressure gradient estimate (2.9) in order to solve both compactness of the blow-up sequence (u_m, p_m) and uniqueness of a limit problem. The estimate (2.9) is scale invariant so (2.9) for (v_m, q_m) is inherited to the blow-up sequence (u_m, p_m) with the scale invariant constant C_Ω, i.e.,

$$\sup_{x\in\Omega_m} d_{\Omega_m}(x)\left|\nabla p_m(x,t)\right| \leq C_\Omega \left\|W(u_m)\right\|_{L^\infty(\partial\Omega_m)}(t). \tag{2.12}$$

Now, we observe the compactness of the blow-up sequence. When Ω_m expands to the whole space, we apply the parabolic regularity theory [15] to get a uniform local Hölder bound for the blow-up sequence in the interior of $\Omega_m \times (0, 1]$, which implies that $N(u_m, p_m)(x, t)$ subsequently converges to $N(u, p)(x, t)$ locally uniformly near the point $(0, 1) \in \mathbf{R}^n \times (0, 1]$. Up to boundary is more involved. When Ω_m expands to a half space, we apply the Hölder estimate for the Stokes equations [27], [29], [30] and obtain a uniform local Hölder bound for the blow-up sequence up to the boundary of Ω_m. Note that, without using (2.12), we can not obtain a uniform local Hölder bound for the blow-up sequence even in the interior of Ω_m. In fact, $v = g(t)$ and $q = -g'(t) \cdot x$ solves (1.1) and (1.2), and $N(v, q)$ is bounded in $\Omega \times (0, T]$ for any $g \in C^1[0, T]$, but v_t and ∇q may not be Hölder continuous in time variables.

The estimate (2.12) plays an important role also for the uniqueness of a limit problem. When Ω_m expands to the whole space, the problem is reduced to the heat equation. In fact, the estimate (2,12) implies that $\nabla p_m \to 0$ locally uniformly in $\mathbf{R}^n \times (0, 1]$. When Ω_m expands to a half space, the bound (2.12) is inherited to the limit, i.e., $\sup \{t^{1/2} x_n |\nabla p(x, t)| \mid x \in \mathbf{R}^n_+,\ 0 < t \leq 1\} < \infty$, which implies a necessary pressure decay condition for the uniqueness, i.e., $\nabla p \to 0$ as $x_n \to \infty$. We apply the L^∞-type uniqueness result due to V. A. Solonnikov [28] to get $u \equiv 0$ and $\nabla p \equiv 0$. For the detailed proof see [1].

Remarks 2.3. (i) The statement of Theorem 2.1 is valid for general strictly admissible domains with uniformly regular boundaries [1].
(ii) It is natural to extend the result for

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f \cdot \nabla\varphi dx = 0 \quad \text{for } \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ denotes the homogeneous Sobolev space of the form $\hat{W}^{1,1}(\Omega) = \{\varphi \in L^1_{\mathrm{loc}}(\Omega) \mid \nabla\varphi \in L^1(\Omega)\}$. In fact, for bounded domains, the Stokes semigroup is a non-C_0-analytic semigroup on $L^\infty_\sigma(\Omega)$ [1]. For unbounded domains, the space $L^\infty_\sigma(\Omega)$ includes non-decaying functions. It is proved also for exterior domains that the Stokes semigroup is uniquely extendable to a non-C_0-analytic semigroup on $L^\infty_\sigma(\Omega)$ [2].
(iii) In general, it is unknown whether or not $S(t)$ is a bounded analytic semigroup on L^∞-type spaces in the sense that both $\|S(t)\|_{\mathcal{L}}$ and $\|\mathrm{d}S(t)/\mathrm{d}t\|_{\mathcal{L}}$ are bounded in $(0,\infty)$ for $X = C_{0,\sigma}(\Omega)$ or L^∞_σ. For bounded domains, we are able to prove that $S(t)$ is a bounded analytic semigroup on $C_{0,\sigma}(\Omega)$ (and also on $L^\infty_\sigma(\Omega)$) via the energy inequality [1]. Recently, P. Maremonti [18] proved that $S(t)$ is a bounded semigroup on $L^\infty_\sigma(\Omega)$ for exterior domains based on the a priori L^∞-estimate (2.1). Note that it is unknown whether $\|\mathrm{d}S(t)/\mathrm{d}t\|_{\mathcal{L}}$ is bounded in $(0,\infty)$ for $X = L^\infty_\sigma(\Omega)$.

3 Resolvent approach

As we have seen a contradiction argument in the preceding section, the harmonic-pressure gradient estimate (2.9) plays a key role in proving the analyticity of the Stokes semigroup on L^∞. It is interesting to discuss the resolvent problem corresponding to (1,1)-(1.4):

$$\lambda v - \Delta v + \nabla q = f \quad \text{in } \Omega, \tag{3.1}$$
$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{3.2}$$
$$v = 0 \quad \text{on } \partial\Omega. \tag{3.3}$$

We establish the a priori estimate for

$$M_p(v,q)(x,\lambda) = |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| + |\lambda|^{n/2p}\|\nabla^2 v\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p}\|\nabla q\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})},$$

and $p > n$ of the form,

$$\sup_{\lambda\in\Sigma_{\vartheta,\delta}} \left\|M_p(v,q)\right\|_{L^\infty(\Omega)}(\lambda) \leq C\|f\|_{L^\infty(\Omega)} \tag{3.4}$$

for some constant $C > 0$ independent of f. Here $\Omega_{x,r}$ denotes the intersection of Ω with an open ball $B_x(r)$ centered at $x \in \Omega$ with radius $r > 0$, i.e., $\Omega_{x,r} = B_x(r) \cap \Omega$ and $\Sigma_{\vartheta,\delta}$ denotes the sectorial region in the complex plane given by $\Sigma_{\vartheta,\delta} = \{\lambda \in \mathbf{C}\backslash\{0\} \mid |\arg\lambda| < \vartheta,\ |\lambda| > \delta\}$ for $\vartheta \in (\pi/2,\pi)$ and $\delta > 0$. The approach is inspired by the Masuda-Stewart technique for elliptic operators (see, e.g., [16]). The estimate (3.4) in particular implies that the Stokes semigroup is an analytic semigroup of angle $\pi/2$ on L^∞-type spaces. Furthermore, as noted in Remarks 3.2 (ii) the method applies also to different type of boundary conditions.

In order to prove the estimate (3.4) directly, we use the harmonic-pressure gradient estimate (2.9) which is available also for the resolvent Stokes equations (3.1)-(3.3), i.e.,

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x)\right| \leq C_\Omega\left\|W(v)\right\|_{L^\infty(\partial\Omega)} \tag{3.5}$$

holds for $W(v) = -(\nabla v - \nabla^T v)n_\Omega$. We estimate the sup-norm for $M_p(v,q)(x,\lambda)$ by using the estimate (3.5) and the L^p-estimate for the resolvent Stokes equations with inhomogeneous divergence-free condition [7], [8].

From the estimate (3.4), we define the Stokes operator in L^∞ and observe that the operator generates an analytic semigroup on L^∞-type spaces. Let us observe the generation of an analytic semigroup on $C_{0,\sigma}(\Omega)$. By the L^p-theory, the solutions (v,q) exist for $f \in C^\infty_{c,\sigma}(\Omega)$ and satisfy the estimates (3.4) and (3.5). We extend the solution operator $R(\lambda) : f \mapsto v_\lambda$ by the estimate (3.4) and a uniform approximation for $f \in C_{0,\sigma}(\Omega)$. (The solution operator to the pressure gradient $f \mapsto \nabla q_\lambda$ is also uniquely extended for $f \in C_{0,\sigma}$). We observe that $R(\lambda)$ is injective on $C_{0,\sigma}$ since the estimate (3.5) immediately implies that $f = 0$ for $v_\lambda = R(\lambda)f = 0$ and $f \in C_{0,\sigma}$. The operator $R(\lambda)$ may be regarded as a surjective operator from $C_{0,\sigma}$ to the range of $R(\lambda)$. The open mapping theorem then implies the existence of a closed operator A such that $R(\lambda) = (\lambda - A)^{-1}$; see [5, Proposition B.6]. We call *A the Stokes operator* in $C_{0,\sigma}(\Omega)$. The estimate (3.4) says that the Stokes operator A is a sectorial operator in $C_{0,\sigma}$. Although the following statement has a general form as well as Theorem 2.1, here, we restrict the statement for bounded domains.

Theorem 3.1 ([3]). *Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary. Let $p > n$. For $\vartheta \in (\pi/2, \pi)$ there exist constants δ and C such that the a priori estimate (3.4) holds for all solutions $(v, \nabla q)$ of (3.1)-(3.3) for $f \in C_{0,\sigma}(\Omega)$ and $\lambda \in \Sigma_{\vartheta,\delta}$. In particular, the Stokes operator A generates a C_0-analytic semigroup on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$.*

Remarks 3.2. (i) The direct resolvent approach clarifies the angle of the analytic semigroup e^{tA} on $C_{0,\sigma}$. Theorem 3.1 asserts that e^{tA} is angle $\pi/2$ on $C_{0,\sigma}$ which does not follow from the a priori L^∞-estimates for the non-stationary Stokes equations (2.1).
(ii) We observe that our argument applies to other boundary conditions, for example, to the Robin boundary condition, i.e., $B(v) = 0$ and $v \cdot n_\Omega = 0$ on $\partial\Omega$ where

$$B(v) = \alpha v_{\tan} + (D(v)n_\Omega)_{\tan} \quad \text{for } \alpha \geq 0. \tag{3.6}$$

Here $D(v) = (\nabla v + \nabla^T v)/2$ denotes the deformation tensor and $f_{\tan}$ the tangential component of the vector field f on $\partial\Omega$. Note that the case $\alpha = \infty$ corresponds to the Dirichlet boundary condition (1.3); see [25] for generation results subject to the Robin boundary conditions on L^∞ for $\mathbf{R}^n_+$. The L^p-resolvent estimates for the Robin boundary condition was established in [10] for concerning analyticity and was later strengthened in [26] to non-divergence free vector fields. We use the generalized resolvent estimate in [26] to extend our result in spaces of bounded functions to the Robin boundary condition.

In the sequel, we sketch the proof for the a priori estimate (3.4). Our argument can be divided into the following three steps:

(i) (Localization) We first localize a solution (v, q) of the resolvent Stokes equations (3.1)-(3.3) in a domain $\Omega' = B_{x_0}((\eta+1)r) \cap \Omega$ for $x_0 \in \Omega, r > 0$ and parameters $\eta \geq 1$ by setting $u = v\theta_0$ and $p = (q - q_c)\theta_0$ with a constant q_c and the smooth cutoff function θ_0 around $\Omega_{x_0,r}$ satisfying $\theta_0 \equiv 1$ in $B_{x_0}(r)$ and $\theta_0 \equiv 0$ in $B_{x_0}((\eta+1)r)^c$. We then observe that (u, p) solves the resolvent Stokes equations with inhomogeneous divergence-free condition in the localized domain Ω'. Applying the L^p-estimates for the localized Stokes equations we have

$$\begin{aligned}&|\lambda| \|u\|_{L^p(\Omega')} + |\lambda|^{1/2} \|\nabla u\|_{L^p(\Omega')} + \|\nabla^2 u\|_{L^p(\Omega')} + \|\nabla p\|_{L^p(\Omega')} \\ &\leq C_p \left(\|h\|_{L^p(\Omega')} + \|\nabla g\|_{L^p(\Omega')} + |\lambda| \|g\|_{W_0^{-1,p}(\Omega')} \right),\end{aligned} \tag{3.7}$$

where $W_0^{-1,p}(\Omega')$ denotes the dual space of the Sobolev space $W^{1,p'}(\Omega')$ with $1/p + 1/p' = 1$. The external forces h and g contain error terms appearing in the cut-off procedure and are explicitly given by

$$h = f\theta_0 - 2\nabla v \nabla \theta_0 - v\Delta\theta_0 + (q - q_c)\nabla\theta_0, \quad g = v \cdot \nabla\theta_0. \tag{3.8}$$

(ii) (Error estimates) A key step is to estimate the error terms of the pressure such as $(q-q_c)\nabla\theta_0$. We here simplify the description by disregarding the terms related to g in order to describe the essence of the proof. Now, the error terms related to h are estimated in the form

$$\|h\|_{L^p(\Omega')} \leq Cr^{n/p}\left((\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)} + (\eta+1)^{-(1-n/p)}\left(r^{-2}\|v\|_{L^\infty(\Omega)} + r^{-1}\|\nabla v\|_{L^\infty(\Omega)}\right)\right). \tag{3.9}$$

If we disregard the term $(q-q_c)\nabla\theta_0$ in h, the estimates (3.9) easily follows from the estimates of the cutoff function θ_0, i.e. $\|\theta_0\|_\infty+(\eta+1)r\|\nabla\theta_0\|_\infty+(\eta+1)^2r^2\|\nabla^2\theta_0\|_\infty \leq K$ with some constant K. We invoke the harmonic-pressure gradient estimate (3.5) in order to handle the pressure term in terms of velocity through the *Poincaré-Sobolev-type inequality*:

$$\|\varphi-(\varphi)\|_{L^p(\Omega_{x_0,s})} \leq Cs^{n/p}\|\nabla\varphi\|_{L^\infty_d(\Omega)} \quad \text{for all } \varphi \in \hat{W}^{1,\infty}_d(\Omega), \tag{3.10}$$

with some constant C independent of $s>0$, where (φ) denotes the mean value of φ in $\Omega_{x_0,s}$ and $\hat{W}^{1,\infty}_d(\Omega) = \{\varphi \in L^1_{\mathrm{loc}}(\bar{\Omega}) \mid \nabla\varphi \in L^\infty_d(\Omega)\}$. By taking $q_c=(q)$ and applying (3.10) for $\varphi=q$ and $s=(\eta+1)r$, we obtain the estimate (3.9) via (3.5).
(iii) (Interpolation) Once we establish the error estimates for h and g, it is easy to obtain the estimate (3.4) by applying the interpolation inequality,

$$\|\varphi\|_{L^\infty(\Omega_{x_0,r})} \leq C_I r^{-n/p}\left(\|\varphi\|_{L^p(\Omega_{x_0,r})} + r\|\nabla\varphi\|_{L^p(\Omega_{x_0,r})}\right) \quad \text{for } \varphi \in W^{1,p}_{\mathrm{loc}}(\bar{\Omega}), \tag{3.11}$$

for $\varphi=u$ and ∇u. Now taking $r=|\lambda|^{-1/2}$ we obtain the estimate for $M_p(v,q)(x_0,\lambda)$ with the parameters η of the form,

$$M_p(v,q)(x_0,\lambda) \leq C\left((\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)} + (\eta+1)^{-(1-n/p)}\|M_p(v,q)\|_{L^\infty(\Omega)}(\lambda)\right) \tag{3.12}$$

for some constant C independent of η. The second term in the right-hand side is absorbed into the left-hand side by letting η sufficiently large provided $p>n$.

Actually, in the procedure (ii) we take q_c by the mean value of q in $\Omega_{x_0,(\eta+2)r}$ since we estimate $|\lambda|\|g\|_{W^{-1,p}_0}$. By using the equation (3.1) we reduce the estimate of $|\lambda|\|g\|_{W^{-1,p}_0}$ to the L^∞-estimate for the boundary value of $q-q_c$ on $\partial\Omega'$. In order to estimate $\|q-q_c\|_{L^\infty(\Omega')}$ we use a uniformly local L^p-norm bound for ∇q besides the sup-bound for ∇v. This is the reason why we need the norm $\|M_p(v,q)\|_{L^\infty(\Omega)}(\lambda)$ in the right-hand side of (3.12). For general elliptic operators, the estimate (3.12) is valid without invoking the uniformly local L^p-norm bound for second derivatives of a solution. See [3] for the detailed proof.

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