HOMOTOPY COMMUTATIVITY IN LOCALIZED GAUGE GROUPS

DAI SUKE KISHIMOTO

1. INTRODUCTION AND STATEMENT OF THE RESULT

This is a survey the paper [KKTh] written with Akira Kono and Stephen Theriault.

Throughout the paper, we only consider the Lie group $G = \text{SU}(n)$ for simplicity, while most results hold for other simply connected, simple Lie groups. Let us recall $p$-local properties of $G.$

Theorem 1.1 (Mimura, Nishida and Toda [MNT]). There exist $p$-local spaces $B_1, \ldots, B_{p-1}$ satisfying

$$G_{(p)} \simeq B_1 \times \cdots \times B_{p-1},$$

where the mod $p$ cohomology of $B_i$ is given by

$$H^*(B_i; \mathbb{Z}/p) = \Lambda(x_{2i+1+2k(p-1)} | 0 \leq k < \frac{n-i-1}{p-1}), \quad |x_j| = j.$$

This is called the mod $p$ decomposition of $G.$ Observe that if $p \geq n,$ each $B_i$ has the homotopy type of $S^{2i+1}_{(p)}$ or a point. Then we can say that the $p$-local homotopy type of $G$ degenerates as $p$ gets larger. So it is natural to consider degeneration of the $H$-structure of $G_{(p)}$ as $p$ gets larger.

As for homotopy commutativity, the complete answer was given by McGibbon [M] as:

Theorem 1.2 (McGibbon [M]). $G_{(p)}$ is homotopy commutative if and only if $p > 2n.$

Later, this result was generalized by Kaji and Kishimoto [KaKi] and Kishimoto [Ki] to homotopy nilpotency.

Our object to study is a gauge group which is the topological group of all automorphisms of a principal bundle, i.e. self-maps of the total space which are compatible with the action of the fiber and cover the identity map of the base space. Recall that principal $G$-bundles over $S^4$ are classified by $\pi_4(BG) \cong \mathbb{Z}$. We write the gauge group of the principal $G$-bundle over $S^4$ corresponding to the integer $k \in \mathbb{Z} \cong \pi_4(BG)$ by $\mathcal{G}_k$. The homotopy theory of gauge groups has been studied in many directions (cf. [CS, Ko, KiKo]). In each work, we have seen that $\mathcal{G}_k$ has a close relation with $G$ as is expected from definition. So we may expect that $\mathcal{G}_k$ possesses $p$-local properties analogous to $G$. As for the mod $p$ decomposition, our expectation has been proved to be true.

---

*The second author is partially supported by the Grant-in-Aid for Scientific Research (C)(No.25400087) from the Japan Society for Promotion of Sciences.*
Theorem 1.3 (Kishimoto, Kono and Tsutaya [KKTs]). There exist $p$-local spaces $B_1, \ldots, B_{p-1}$ satisfying
\[ G_{k(p)} \simeq B_1 \times \cdots \times B_{p-1} \]
and homotopy fibrations
\[ \Omega \Omega_0^3 B_i \rightarrow B_i \rightarrow B_{i-2}, \]
where we regard the spaces $B_i$ of Theorem 1.1 are indexed by $\mathbb{Z}/(p-1)$. Moreover, the homotopy fibrations are trivial if $p \geq n+2$.

In particular, we can say that the $p$-local homotopy type of $G_k$ degenerates as $p$ gets larger, analogously to $G$. Now we naturally ask whether there is a gauge group version of Theorem 1.2. Let us state our main result.

Theorem 1.4. Suppose $n \geq 4$.

1. For $p < 2n+1$, $G_{k(p)}$ is not homotopy commutative.
2. For $p > 2n+1$, $G_{k(p)}$ is homotopy commutative.
3. For $p = 2n+1$, $G_{k(p)}$ is homotopy commutative if and only if $p$ divides $k$.

Remark 1.5. Note that the integer $k$ only appears in the border case $p = 2n+1$.

2. Noncommutativity

In this section, we give a sketch of the proof of the noncommutativity result on $G_{k(p)}$. We first recall basic facts of gauge groups briefly. Let $\epsilon_i$ be a generator of $\pi_{2i-1}(G) \cong \mathbb{Z}$ for $i = 2, \ldots, n$. Recall that there is a natural homotopy equivalence
\[ B G_k \simeq \text{map}(S^4, BG; k\overline{\epsilon}_2), \]
where map$(X, Y; f)$ stands for the connected component of the space of maps from $X$ to $Y$ containing a map $f : X \rightarrow Y$ and $\overline{\epsilon}_2 : S^4 \rightarrow BG$ is the adjoint of $\epsilon_2$. See [AB]. Then the evaluation map map$(S^4, BG; k\overline{\epsilon}_2) \rightarrow BG$ induces a homotopy fibration
\[ (2.1) \quad G_k \xrightarrow{\pi} G \xrightarrow{\delta} \Omega_0^3 G, \]
where $\pi$ is a loop map. The map $\delta$ is identified as:

Lemma 2.1 (Whitehead [W]). The map $\delta$ is the adjoint of the Samelson product $\langle \epsilon_2, 1_G \rangle$.

Hereafter, everything will be localized at the prime $p$.

We now sketch the proof of noncommutativity of $G_k$. Suppose that there are $2 \leq i, j \leq n$ such that
\[ (2.2) \quad \langle \epsilon_2, \epsilon_i \rangle = 0, \quad \langle \epsilon_2, \epsilon_j \rangle = 0, \quad \langle \epsilon_i, \epsilon_j \rangle \neq 0. \]
Since $\delta \circ \epsilon_\ell$ is the adjoint of $\langle \epsilon_2, \epsilon_\ell \rangle$ by Lemma 2.1, $\delta \circ \epsilon_\ell$ is null homotopic for $\ell = i, j$. Then for $\ell = i, j$, $\epsilon_\ell$ lifts to $\tilde{\epsilon}_\ell : S^{2\ell - 1} \to \mathcal{G}_k$ through $\pi : \mathcal{G}_k \to G$. Consider the Samelson product $\langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle$.

Since $\pi$ is an H-map, we have

$$\pi \circ \langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle = \langle \pi \circ \tilde{\epsilon}_i, \pi \circ \tilde{\epsilon}_j \rangle = \langle \epsilon_i, \epsilon_j \rangle$$

which is nontrivial by assumption. Then in particular, we obtain that $\mathcal{G}_k$ is not homotopy commutative. So our task is to find $2 \leq i, j \leq n$ satisfying (2.2), which is easily done by the following classical result if $n \geq 4$.

**Theorem 2.2 (Bott [B]).** If $2 \leq i, j \leq n$ and $i + j > n$, the order of the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is a nonzero multiple of

$$\frac{(i + j - 1)!}{(i - 1)!(j - 1)!}.$$

3. **COMMUTATIVITY**

In this section, we give a brief sketch of the proof of the commutativity result on $\mathcal{G}_k$. If the map $\pi$ in the homotopy fibration (2.1) has a homotopy section, we have a decomposition

$$\mathcal{G}_k \simeq G \times \Omega(\Omega_0^3 G)$$

as spaces. If this decomposition is as H-spaces and $G$ is homotopy commutative (i.e. $p > 2n$ by Theorem 1.2), we obtain that $\mathcal{G}_k$ is homotopy commutative as desired. Then we give a criterion for the decomposition being as H-spaces, where we omit the proof.

**Lemma 3.1 (cf. [KiKo]).** If there is an H-map $\hat{s} : G \to \mathcal{G}_k$ such that $\pi \circ \hat{s}$ is a homotopy equivalence, then there is a homotopy equivalence as H-spaces

$$\mathcal{G}_k \simeq G \times \Omega(\Omega_0^3 G).$$

In particular, if moreover $p > 2n$, $\mathcal{G}_k$ is homotopy commutative.

For the rest of this section, we assume $p > 2n$. Then in particular, $G \simeq S^3 \times S^5 \times \cdots \times S^{2n-1}$.

Since $G$ is homotopy commutative, it follows from Lemma 2.1 that $\pi$ has a homotopy section $s : G \to \mathcal{G}_k$, not necessarily an H-map. We replace this homotopy section with an H-map. To this end, we employ the loop-suspension technique.

**Theorem 3.2 (James [J]).** Consider a map $f : X \to Y$ where $Y$ is a homotopy associative H-space. There is a unique (up to homotopy) H-map $\tilde{f} : \Sigma X \to Y$ satisfying $\tilde{f} \circ E \simeq f$ for the suspension map $E : X \to \Sigma X$, where $\tilde{f}$ is called the extension of $f$.

We put $A = S^3 \vee S^5 \vee \cdots \vee S^{2n-1}$ and let $i : A \to G$ be the inclusion of a wedge into a product. Let $F$ be the homotopy fiber of the extension $\tilde{i} : \Sigma A \to G$, and let $\lambda : F \to \Sigma$ be the fiber inclusion. By an easy diagram chasing, we can prove:
Lemma 3.3. Consider a map \( f : G \to Z \) where \( Z \) is a homotopy associative \( H \)-space. If the composite \( F \overset{\lambda}{\to} \Omega \Sigma A \overset{\overline{i}}{\to} \mathcal{G}_k \) is null homotopic, there is an \( H \)-map \( \hat{f} : G \to Z \) satisfying the homotopy commutative square

\[
\begin{array}{ccc}
\Omega \Sigma A & \overset{i}{\to} & G \\
\downarrow \overline{\overline{i}} & & \downarrow \overline{f} \\
Z & \underset{Z}{\cong} & \mathcal{G}_k.
\end{array}
\]

Suppose now that the composite \( F \overset{\lambda}{\to} \Omega \Sigma A \overset{s \circ i}{\to} \mathcal{G}_k \) is null homotopic. Then it follows from Lemma 3.3 that there is an \( H \)-map \( \hat{s} : G \to \mathcal{G}_k \) satisfying the homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega \Sigma A & \overset{i}{\to} & G \\
\downarrow s \circ i & & \downarrow \hat{s} \\
\mathcal{G}_k & \underset{\mathcal{G}_k}{\cong} & \mathcal{G}_k.
\end{array}
\]

In particular, there is a chain of homotopies

\[
\pi \circ \hat{s} \circ i \simeq \pi \circ \hat{s} \circ \overline{i} \circ E \simeq \pi \circ \overline{s \circ i} \circ E \simeq \pi \circ s \circ i \simeq i.
\]

In the mod \( p \) homology, the map \( i : A \to G \) induces the inclusion of ring generators. Then \( \pi \circ \hat{s} \) turns out to be the identity map on ring generators in the mod \( p \) homology, hence since \( \pi \circ \hat{s} \) is an \( H \)-map, it is an isomorphism in the mod \( p \) homology. So we obtain that \( \pi \circ \hat{s} \) is a \( p \)-local homotopy equivalence. Then all we have to do is prove that the composite \( F \overset{\lambda}{\to} \Omega \Sigma A \overset{\overline{i}}{\to} \mathcal{G}_k \) is null homotopic. To this end, we analyze the fiber inclusion \( \lambda \).

Let \( F' \) be the homotopy fiber of the adjoint \( \Sigma A \to BG \) of the inclusion \( i : A \to G \). Since the extension \( \overline{i} : \Omega \Sigma A \to \Omega \Sigma A \) is the loop of the above adjoint, we get:

Lemma 3.4. \( F \simeq \Omega F' \) and the fiber inclusion \( \lambda : \Omega F' \to \Omega \Sigma A \) is a loop map.

Let \( L \) be the free Lie algebra generated by \( \tilde{H}_*(A; \mathbb{Z}/p) \). Then as in [CN], the induced map \( \tilde{i}_* : H_*(\Omega \Sigma A; \mathbb{Z}/p) \to H_*(G; \mathbb{Z}/p) \) is identified with the map between universal envelopes

\[
U(L) \to U(L/[L, L])
\]

induced from the abelianization \( L \to L/[L, L] \). Moreover, there is a splitting

\[
U(L) \cong U([L, L]) \otimes U(L/[L, L]),
\]

hence the image of \( \lambda_* : H_*(F; \mathbb{Z}/p) \to H_*(\Omega \Sigma A; \mathbb{Z}/p) \) is identified with \( U([L, L]) \subset U(L) \). A little more consideration shows that the Lie algebra generators of \([L, L]\) are spherical and lift to \( F \). So we obtain:

Theorem 3.5. There is a wedge of spheres \( R \) such that \( F' \simeq \Sigma R \), and the composite \( R \overset{E}{\to} \Omega \Sigma R \overset{\lambda}{\to} \Omega \Sigma A \) is a wedge of iterated Samelson products of \( \mu_j : S^{2j-1} \overset{\text{incl}}{\to} A \overset{E}{\to} \Omega \Sigma A \).
Corollary 3.6. If $p > 2n + 1$, the composite $F \xrightarrow{\lambda} \Omega \Sigma A \xrightarrow{\pi} G_k$ is null homotopic.

Proof. Put $\tilde{\mu}_j = (s \circ i) \circ \mu_j$. We consider the Samelson product $\langle \tilde{\mu}_i, \tilde{\mu}_j \rangle$. Since $\pi$ is an H-map and $G$ is homotopy commutative, we have

$$\pi \circ \langle \tilde{\mu}_i, \tilde{\mu}_j \rangle = \langle \pi \circ \tilde{\mu}_i, \pi \circ \tilde{\mu}_j \rangle = 0.$$  

Then $\langle \tilde{\mu}_i, \tilde{\mu}_j \rangle$ lifts to a map $S^{2i+2j-1} \to \Omega(\Omega_k^2 G)$ by the homotopy fibration $\Omega(\Omega_k^2 G) \to G$. Since $p > 2n + 1$, we have $\pi_{2m}(\Omega(\Omega_k^2 G)) = 0$ for $m \leq 2n - 1$ by [To], implying that the above lift is null homotopic. Then we obtain $\langle \tilde{\mu}_i, \tilde{\mu}_j \rangle = 0$, hence

$$0 = \langle \tilde{\mu}_j, \langle \cdots \langle \tilde{\mu}_{j_{m-1}}, \tilde{\mu}_{j_m} \cdots \rangle \rangle = (s \circ i) \circ \langle \mu_j, \langle \cdots \langle \mu_{j_{m-1}}, \mu_{j_m} \rangle \cdots \rangle \rangle$$

since $s \circ i$ is an H-map. Thus by Theorem 3.5, the composite $R \xrightarrow{E} \Omega \Sigma R \xrightarrow{\lambda} \Omega \Sigma A \xrightarrow{\pi} G_k$ is null homotopic. Therefore we obtain the desired result by the uniqueness of the extension and Lemma 3.4.

4. The case $p = 2n + 1$

Throughout this section, we assume $p = 2n + 1$.

As in the previous section, it is sufficient for proving the commutativity result to show that the homotopy section $s : G \to G_k$ is an H-map. This is equivalent to show that the adjoint

$$s : \Sigma G \to BG_k \simeq \text{map}(S^4, BG : k\epsilon_2)$$

extends to the projective plane $P^2G$. By the exponential law, this is equivalent to existence of a map $\mu : S^4 \times P^2G \to BG$ satisfying a homotopy commutative diagram

$$
\begin{array}{ccc}
S^4 \vee \Sigma G & \xrightarrow{\mu} & BG \\
\downarrow \text{incl} & & \\
S^4 \times P^2G & \xrightarrow{\mu} & BG.
\end{array}
$$

Since $P^2G$ is the cofiber of the Hopf construction $\Sigma G \wedge G \to \Sigma G$ and $\Sigma G \wedge G$ has the homotopy type of a wedge of spheres of dimension $\leq 2n^2 - 1 = \frac{(p-1)^2}{2} - 1$, we see that the obstruction for existence of $\mu$ lies in $\pi_2(BG)$ for $* \leq \frac{(p-1)^2}{2} + 3$. Since the obstruction is torsion in $\pi_2(BG)$, we see from [To] that it is of order at most $p$. Moreover, we also see that the obstruction is linear in $k$. Then we get:

Proposition 4.1. If $p$ divides $k$, the homotopy section $s$ is an H-map, hence $G_k$ is homotopy commutative.

When $p$ does not divide $k$, we can prove that the obstruction is nontrivial by looking at the Steenrod operation on the mod $p$ cohomology of $BG$. Then we have:

Proposition 4.2. If $p$ does not divide $k$, the homotopy section $s$ cannot be an H-map.
Corollary 4.3. If $p$ does not divide $k$, $G_k$ is not homotopy commutative.

Proof. Suppose that $G_k$ is homotopy commutative. Then the argument in the previous section ensures that there is an H-map $\hat{s} : G \to G_k$ such that the composite $e = \pi \circ \hat{s}$ is a homotopy equivalence. If we put $s = \hat{s} \circ e^{-1}$, $s$ is a homotopy section of $\pi$ and is an H-map, which contradicts to Proposition 4.2.

REFERENCES


Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan
E-mail address: kishi@math.kyoto-u.ac.jp