Some aspects of a finite $T_0$-$G$-space

藤田亮介 (Ryousuke Fujita)

Introduction

The purpose of our presentation was to study actions of finite groups on finite $T_0$-spaces, i.e. topological spaces having finitely many points with the $T_0$-separation axioms. The definition of $T_0$-separation axiom is, for each pair of distinct points, there exists an open set containing one but not the other. A remarkable feature of a finite $T_0$-space is that it has the structure of a poset. Conversely, one can give any finite poset the structure of a finite $T_0$-space. The equivariant theory of finite $T_0$-spaces was first made by Stong [11]. After that, Kono and Ushitaki investigated the homeomorphism groups of finite spaces with group actions ([6], [7], [8]). Here a finite space is a topological space having finitely many points. In particular, they studied the homeomorphism groups of fixed point set $X^G$ and $G$-actions on homeomorphism groups induced by given $G$-action on $X$, where $X$ is a finite space with a $G$-action.

First we define a simplicial complex induced from a finite $T_0$-space. Recall that a finite $T_0$-space has a poset structure (see Proposition 2.2). Let $X$ be a finite poset. The order complex $\Delta(X)$ of $X$ is the abstract simplicial complex on the vertex set $X$ whose faces are the chains of $X$, including the empty chain. The dimension of a simplex is defined to be the length of the chain, where the length of a chain is one less than its number of elements. In particular, the length of the empty chain is $-1$. When the dimension of a simplex $\sigma$ is $k$, we write $\dim \sigma = k$. Next we shall define the geometric realization $|\Delta(X)|$ of $\Delta(X)$ by

$$|\Delta(X)| = \{ m : X \to [0,1] | \sum_{x \in X} m(x) = 1, \text{supp}(m) \in \Delta(X) \},$$

where for a map $m : X \to [0,1]$, we mean that $\text{supp}(m) = \{ x \in X | m(x) > 0 \}$. The numbers $(m(x) | x \in X)$ are the barycentric coordinates of $m$. For a simplex $\sigma \in \Delta(X)$, we put

$$|\sigma| = \{ m \in |\Delta(X)| | \text{supp}(m) = \sigma \}.$$

We can define a metric topology on $|\Delta(X)|$. In details, we have a metric $d$ on $|\Delta(X)|$ defined by

$$d(m_1, m_2) = \left( \sum_{x \in X} (m_1(x) - m_2(x))^2 \right)^{\frac{1}{2}}.$$

Then we have $|\sigma| = \{ m \in |\Delta(X)| | \sum_{x \in \sigma} m(x) = 1 \}$, where $|\sigma|$ indicates the closure of $|\sigma|$. Moreover a metric space $|\Delta(X)|$ is equipped with a $CW$-complex structure whose $n$-cell
is a set \( \{ \sigma | \sigma \in \Delta(X), \dim \sigma = n \} \). Let \( (p_x | x \in X) \) be a family of points in euclidean \( n \)-space \( \mathbb{R}^n \). Consider the continuous map
\[
f : |\Delta(X)| \to \mathbb{R}^n, \quad m \mapsto \sum_{x \in X} m(x)p_x.
\]
If \( f \) is an embedding, we call the image of \( f \) a simplicial polyhedron in \( \mathbb{R}^n \) of type \( \Delta(X) \), that is, \( f(|\Delta(X)|) \) is a realization of \( \Delta(X) \) as a polyhedron in \( \mathbb{R}^n \).

Now, we shall introduce McCord’s result [9, Theorem 2], which provides insight into understanding relations between finite \( T_0 \)-spaces and simplicial complexes.

**Proposition 1.1.** There exists a correspondence that assigns to each finite \( T_0 \)-space \( X \) a finite simplicial complex \( \Delta(X) \), whose vertices are the points of \( X \), such that the map \( \mu_X : |\Delta(X)| \to X \) induced from the correspondence above is a weak homotopy equivalence. Moreover, each map \( \varphi : X \to Y \) of finite \( T_0 \)-spaces is also a simplicial map \( \Delta(X) \to \Delta(Y) \), and \( \varphi \mu_X = \mu_Y | \varphi | \) where \( | \varphi | : |\Delta(X)| \to |\Delta(Y)| \) is a continuous map induced by \( \varphi \).

Let \( G \) be a finite group. In this note, we focus on the equivariant order complex \( \Delta(X) \) of a finite \( T_0 \)-\( G \)-space \( X \), that is, a finite \( T_0 \)-space with a \( G \)-action, and then its orbit space \( \Delta(X)/G \). In particular, we are interested in the following questions:

(i) Does \( |\Delta(X)| \) have a \( G \)-\( CW \)-complex structure?

(ii) Is there the orbit space version of Proposition 1.1?

Our results related the above questions are the following.

**Theorem A.** Let \( X \) be a finite \( T_0 \)-\( G \)-space. Then \( |\Delta(X)| \) is a finite \( G \)-\( CW \)-complex.

We will prepare the following technical condition:

(C) If \( g_0, g_1, \ldots, g_k \) are elements of \( G \) and \( (x_0, x_1, \ldots, x_k) \) and \( (g_0x_0, g_1x_1, \ldots, g_kx_k) \) are both simplices of \( K \), then there exists an element \( g \) of \( G \) such that \( gx_i = g_ix_i \) for all \( i \). Here overlaps of some of \( x_i \) are allowed.

**Theorem B.** If \( \Delta(X) \) satisfies property (C), there exists a weak homotopy equivalence \( \tilde{\mu}_X : |\Delta(X)|/G \to X/G \).

The rest of this note is organized as follows. In section 2, we briefly review finite \( (T_0^-) \)-space theory. In section 3, we investigate an equivariant version of finite \( T_0 \)-spaces and prove Theorem A. The last section studies orbit spaces of equivariant complexes and prove Theorem B.

## 2 Finite \( (T_0^-) \)-spaces

In this section, we survey well-known properties about finite \( (T_0^-) \)-spaces. General reference may be found in [2], [6] and [10]. Let \( X \) denote a finite space, i.e. a topological space having finitely many points. Let a set \( U_x \) be the minimal open set which contains a point \( x \) of \( X \), that is, \( U_x \) is the intersection of all open sets containing \( x \). It is easy to see that a set \( \{ U_x | x \in X \} \) constitute a basis for the topology of \( X \). Now we can define a preorder on \( X \) by
\[
x \leq y \quad \text{if} \quad x \in U_y.
\]
In other words, every open set containing \( y \) also contains \( x \) if and only if \( x \leq y \).
Proposition 2.1. Let $x$ and $y$ be elements of a finite space $X$. Then $X$ is $T_0$-space if and only if $U_x = U_y$ implies $x = y$.

Proposition 2.2. A finite $T_0$-space with the above preorder $\leq$ is a poset.

If $X$ is now a finite preordered set, one can define a topology on $X$ given by the basis \( \{ y \in X \mid y \leq x \} \) for each $x \in X$. Note that if $y \leq x$, then $y$ is contained in every basic set containing $x$, and therefore $y \in U_x$. Conversely, if $y \in U_x$, then $y \in \{ z \in X \mid z \leq x \}$. After all, $y \leq x$ if and only if $y \in U_x$. This shows that these two applications, relating topologies and preorders on a finite set, are mutually inverse. Thus we have

Proposition 2.3. A finite $T_0$-space corresponds to a finite poset.

Example 2.4. Let $X = \{a, b, c\}$ be a finite space whose topology is $\{\emptyset, \{a, b, c\}, \{b\}, \{c\}\}$. This space is $T_0$. Immediately, $U_a = \{a, b, c\}$, $U_b = \{b\}$ and $U_c = \{c\}$. Therefore $b \leq a$ and $c \leq a$, but there exists no order relation between $b$ and $c$.

Example 2.5. Let $X = \{a, b, c, d\}$ be a finite space whose topology is $\{\emptyset, \{a, b, c, d\}, \{b, c, d\}, \{b\}, \{b, c\}, \{b, d\}\}$. This space is also $T_0$. Immediately, $U_a = \{a, b, c, d\}$, $U_b = \{b\}$, $U_c = \{b, c\}$ and $U_d = \{b, d\}$. On the order relation, we see the following Hasse diagram:

```
  a  \\
 / \  \\
/   \\
\  \\
 b-\  \\
 \  \\
 \  \\
 \  \\
 d-\  \\
 c
```

Figure 1.

Proposition 2.6. Let $X$ be a preordered set. A set $F_x = \{y \in X \mid x \leq y\}$ is a closed set of $X$. Moreover $F_x$ is the closure of the set $\{x\}$.

Definition 2.7. A subset $U$ of a preordered set $X$ is a down-set if for every $x \in U$ and $y \leq x$, it holds that $y \in U$. Dually, a subset $F$ of a preordered set $X$ is an up-set if for every $x \in F$ and $y \geq x$, it holds that $y \in F$. Open sets of finite spaces correspond to down-sets and closed sets to up-sets.

Proposition 2.8. Let $X$ and $Y$ be finite spaces, and $f$ be a map from $X$ to $Y$. Then $f$ is continuous if and only if $f$ is an order-preserving map.

Proposition 2.9. Let $X$ be a finite space, $f$ a continuous map of $X$ into itself. If $f$ is either one-to-one or onto, then it is a homeomorphism.

Next we state connectivity. First, for each $U_x$, we let $U_x \subset A \cup B$, where $A$ and $B$ are open sets of a finite space $X$. Then $x$ is in one set, say $x \in A$, immediately $U_x \subset A$. Thus any finite space is locally connected.

Proposition 2.10. Let $x, y$ be two comparable points of a finite space $X$ and $x \leq y$. Then there exists a path from $x$ to $y$ in $X$, that is, a map $\alpha$ from the unit interval $I$ to $X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. 
Let $X$ be a finite preordered set. A fence in $X$ is a sequence $x_0,x_1,\ldots,x_n$ of points such that any two consecutive are comparable. $X$ is order-connected if any two points $x,y \in X$ there exists a fence starting in $x$ and ending in $y$.

**Proposition 2.11.** Let $X$ be a finite space. Then the following are equivalent:

(i) $X$ is a connected topological space.

(ii) $X$ is an order-connected preordered set.

(iii) $X$ is a path-connected topological space.

If $X$ and $Y$ are finite spaces, we can consider the finite set $Y^X$ of continuous maps from $X$ to $Y$ with the pointwise order: $f \leq g$ if $f(x) \leq g(x)$ for every $x \in X$.

**Proposition 2.12.** Let $X$ and $Y$ be two finite spaces. Then pointwise order on $Y^X$ corresponds to the compact-open topology.

**Corollary 2.13.** Let $f,g:X \to Y$ be two maps between finite spaces. Then $f \simeq g$ if and only if there is a fence $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$. Moreover, if $A \subset X$, then $f \simeq g \text{ rel } A$ if and only if there exists a fence $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$ such that $f_i|_A = f|_A$ for every $0 \leq i \leq n$.

Any finite space is homotopy equivalent to a finite $T_0$-space.

**Proposition 2.14.** Let $X$ be a finite space. Let $X_0$ be the quotient $X/\sim$ where $x \sim y$ if $x \leq y$ and $y \leq x$. Then $X_0$ is $T_0$ and the quotient map $q:X \to X_0$ is a homotopy equivalence.

Therefore, when studying homotopy types of finite spaces, we can restrict our attention to finite $T_0$-spaces.

**Definition 2.15.** A point $x$ in a finite $T_0$-space $X$ is a down beat point if $x$ cover one and only one element of $X$. This is equivalent to saying that the set $\hat{U}_x = U_x \setminus \{x\}$ has a maximum. Dually, $x \in X$ is an up beat point if $x$ is covered by a unique element or equivalently if $\hat{F}_x = F_x \setminus \{x\}$ has a minimum, where $F_x$ denotes the closure of the set $\{x\}$. In any of these cases, we say that $x$ is a beat point of $X$.

**Proposition 2.16.** Let $X$ be a finite $T_0$-space and let $x \in X$ be a beat point. Then $X \setminus \{x\}$ is a strong deformation retract of $X$.

**Definition 2.17.** A finite $T_0$-space is a minimal finite space if it has no beat points. A core of a finite space $X$ is a strong deformation retract which is a minimal finite space.

**Proposition 2.18.** Let $X$ be a minimal finite space. A map $f:X \to X$ is homotopic to the identity if and only if $f = 1_X$.

Immediately, we have the following corollary.

**Corollary 2.19.** (Classification Theorem) A homotopy equivalence between minimal finite spaces is a homeomorphism. In particular, the core of a finite space is unique up to homeomorphism and two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.

By the Classification Theorem, a finite space is contractible if and only if its core is a point. In fact, a one-point finite space has a core of the one-point. Therefore any contractible finite space has a point which is a strong deformation retract. This property is false in general for non-finite spaces.
3 Finite $T_0$-$G$-spaces

In this section, we treat an equivariant version of finite $T_0$-spaces. Let $G$ be a topological group (a group, for short) and $X$ a finite $T_0$-space. A $G$-invariant subspace $A \subset X$ is an equivariant strong deformation retract if there is an equivariant retraction $r : X \to A$ such that $ir$ is homotopic to $1_X$ via a $G$-homotopy which is stationary at $A$. A finite $T_0$-space which is a $G$-space will be a finite $T_0$-$G$-space.

**Remark** If a topological group $G$ acts on a finite topological space effectively, then it must be a finite topological group [7, Proposition 3.9]. Therefore, from now on, we assume that $G$ is finite.

**Proposition 3.1.** Let $X$ be a finite $T_0$-$G$-space. Then there exists a core of $X$ which is $G$-invariant and an equivariant strong deformation retract of $X$.

**Proposition 3.2.** A contractible finite $T_0$-$G$-space has a point which is fixed by the action of $G$.

This proposition deduces Stong’s result stated in introduction. Note that $A_p(G)$ is a finite $T_0$-$G$-space by conjugation. If $A_p(G)$ is contractible, $A_p(G)$ has exactly one point core which is $G$-invariant. Therefore $A_p(G)$ has a fixed point by the action of $G$. Consequently, $G$ has a non-trivial normal $p$-subgroup.

**Proposition 3.3.** Let $X$ and $Y$ be finite $T_0$-$G$-spaces and let $f : X \to Y$ be a $G$-map which is a homotopy equivalence. Then $f$ is an equivariant homotopy equivalence.

Let $X$ be a finite $T_0$-$G$-space and $x, y$ points of $X$. If $x \in U_y$, then $gx \in gU_y = U_{gy}$. Therefore a $G$-action on a finite $T_0$-space $X$ preserves the order. Thus $\Delta(X)$ is a $G$-simplicial complex (in short, $G$-complex). Let $\mathbb{N}_0$ be the union set of natural numbers $\{1, 2, 3, \cdots \}$ and $\{0\}$.

**Definition 3.4.** Let $G$ be a finite group. A $CW$-complex $Z$ with a $G$-action is called a $G$-$CW$-complex if it satisfies the following conditions:

(i) The $G$-action determines a cellular map, that is, for any $g \in G$, $gZ^i \subset Z^i$ for each $i \in \mathbb{N}_0$, where $Z^i$ denotes the union of cells of dimension $\leq i$ and is called the $i$-skeleton of $Z$.

(ii) If $g(e) = e$, then $g$ is trivial on $\bar{e}$, that is, $Z^i \supset \bar{e}$, where $\bar{e}$ is the closure of $e$.

**Proof of Theorem A.**

**Proof.** For $g \in G$ and $m \in |\Delta(X)|$, we define a map $g(m) : X \to [0, 1]$ by

$$(g(m))(x) := m(g^{-1}(x)) \quad \text{for } x \in X.$$ 

Then we have

$$\sum_{x \in X} (g(m))(x) = \sum_{x \in X} m(g^{-1}(x)) = \sum_{g^{-1}(x) \in X} m(g^{-1}(x)) = 1,$$
on the other hand,

\[
\text{supp}(g(m)) = \{ x \in X \mid (g(m))(x) > 0 \} = \{ x \in X \mid m(g^{-1}(x)) > 0 \} = \{ x \in X \mid g^{-1}(x) \in \text{supp}(m) \} = g(\text{supp}(m)) \in \Delta(X).
\]

Therefore we have that \( g(m) \in |\Delta(X)| \). Thus we can define an isometric map \( g : |\Delta(X)| \to |\Delta(X)| \). For each \( \sigma \in \Delta(X) \), it holds that \( g(|\sigma|) = |g(\sigma)| \). In particular, a map \( g \) is a cellular map.

Let \( g(|\sigma|) = |\sigma| \). Immediately, we have \( g(\sigma) = \sigma \). Since \( g \) is an automorphism between totally ordered sets, it is an identity map. Therefore \( g^{-1} : \sigma \to \sigma \) is also an identity map. Let \( m \) be any element of \( |\sigma| \).

\[
\text{Case } x \in \sigma : \text{ It follows that } (g(m))(x) = m(g^{-1}(x)) = m(x).
\]

\[
\text{Case } x \in X \setminus \sigma : \text{ Since } g^{-1}(x) \in X \setminus g^{-1}(\sigma) = X \setminus \sigma, \text{ we get that } (g(m))(x) = m(g^{-1}(x)) = 0 = m(x).
\]

Therefore \( g(m) = m \). Thus we obtain that \( [\sigma] \subset |\Delta(X)|^g \).

Referring to [5, p.229], we now prepare the following technical properties concerning a \( G \)-complex \( K \):

\begin{enumerate}
  \item [(P_1)] For any \( g \in G \) and simplex \( \sigma \) of \( K \), \( g \) leaves \( \sigma \cap g\sigma \) pointwise fixed.
  \item [(P_2)] If \( g_0, g_1, \cdots, g_k \) are elements of \( G \) and \( (x_0, x_1, \cdots, x_k) \) and \( (g_0x_0, g_1x_1, \cdots, g_kx_k) \) are both simplices of \( K \), then there exists an element \( g \) of \( G \) such that \( gx_i = g_i x_i \) for all \( i \). Here overlaps of some of \( x_i \) are allowed.
  \item [(P_3)] Let \( g \) be an element of \( G \) and \( \sigma \) a simplex of \( K \). If \( g(\sigma) = \sigma \), \( g \) leaves \( \sigma \) pointwise fixed.
\end{enumerate}

**Proposition 3.5.** It holds that \( (P_2) \implies (P_1) \implies (P_3) \).

**Proposition 3.6.** Let \( X \) be a finite \( T_0 \)-\( G \)-space. Then a \( G \)-complex \( \Delta(X) \) holds both property \( (P_1) \) and property \( (P_3) \).

On a \( G \)-complex, we can see a geometric simplex as a cell. One immediate consequence of this observation is the following.

**Proposition 3.7.** Let \( |K| \) be the geometric realization of a \( G \)-complex \( K \) with property \( (P_3) \). Then \( |K| \) is a \( G \)-\( CW \)-complex.

The following result is an equivariant version of Proposition 1.1 in a sense.

**Proposition 3.8.** Let \( X \) be a finite \( T_0 \)-\( G \)-space. For each subgroup \( H \) of \( G \), it holds that \( \Delta(X^H) = \Delta(X)^H \) and the map \( \mu_X^H : |\Delta(X)|^H \to X^H \) is a weak homotopy equivalence.

4 Orbit spaces

Next we will devote the study of the orbit space of a \( G \)-complex.
Proposition 4.1. Let $X$ be a finite $T_0$-G-space. Then the orbit space $X/G$ is a finite $T_0$-space.

Let $X$ and $Y$ be finite sets, and $\mathcal{P}(X)$ the power set of $X$. A map $f : X \to Y$ induces a map $\mathcal{P}(X) \to \mathcal{P}(Y)$, which we denote also by $f$. Let $K$ be a simplicial complex such that $X$ is the set of vertices of $K$. Then it is easy to see that the image $f(K)$ becomes a simplicial complex such that $f(X)$ is the set of vertices of $f(K)$. We apply this observation to our situation.

Let $K$ be a $G$-complex and $X$ be the set of vertices of $K$. Concerning the induced $G$-action on $X$, we consider its orbit space $X/G$ and the orbit map $p : X \to X/G$. As observes above, $p$ induced a map $\mathcal{P}(X) \to \mathcal{P}(X/G)$, which we denote by $p$ as well and $p(K)$ becomes a simplicial complex such that $X/G$ is the set of vertices of $p(K)$. For $s \in K$, we denote $p(s)$ by $\bar{s}$.

Next consider another kind of orbit space. Let $K$ be a $G$-complex. Denote by $K/G$ the orbit space of the $G$-action on $K$ and by $\pi : K \to K/G$ the orbit map. For $s \in K$, we denote $\pi(s)$ by $[s]$. Note that $K/G$ is not a simplicial complex in general and $K/G$ does not coincide with $p(K)$ in general.

Proposition 4.2. [5, Lemma 5.10] Let $K$ be a $G$-complex satisfying property $(P_2)$ and $X$ be the set of vertices of $K$. Then the orbit space $K/G$ becomes a simplicial complex such that the set of vertices $K/G$ is $X/G$ and $K/G$ is naturally isomorphic to $p(K)$. Moreover the orbit map $\pi : K \to K/G$ is a simplicial map preserving dimension of simplexes.

Corollary 4.3. If $K$ is a $G$-complex satisfying property $(P_2)$, $\lvert K \rvert / G$ is homeomorphic to $\lvert K/G \rvert$.

Furthermore, we add simplicial notion for both posets and (finite) cell complexes to investigate the simplicial structure of the orbit spaces in detail.

Definition 4.4. A simplicial poset $P$ is a finite poset with a smallest element $\hat{0}$ such that every interval

$$[\hat{0}, y] = \{ x \in P | \hat{0} \leq x \leq y \}$$

for $y \in P$ is a boolean algebra, i.e., $[\hat{0}, y]$ is isomorphic to the set of all subsets of a finite set, ordered by inclusion. When a boolean algebra is the set of all subsets of a finite set consisting of $n$ elements, we denote the boolean algebra by $B_n$. Let $x$ be an element of $P$ such that $[\hat{0}, x]$ is isomorphic to a boolean algebra $B_n$. Then the dimension of $x$ is said to be $n - 1$, denoted by $\dim x = n - 1$. Remark that $\dim \hat{0} = -1$. Moreover, a simplicial poset $P$ is $n$-dimensional, if it contains at least one point $x$ such that $\dim x = n$ but no $(n + 1)$-dimensional points.

The set of all faces of a (finite) simplicial complex with empty set added forms a simplicial poset ordered by inclusion, where the empty set is the smallest element. Such a simplicial poset is called the face poset of a simplicial complex, and two simplicial complexes are isomorphic if and only if their face posets are isomorphic. Therefore, a simplicial poset can be thought of as a generalization of a simplicial complex. Figure 2 shows that a 2-simplicial complex and its face poset.
A $CW$-complex is said to be regular if all closed cells are homeomorphic to closed disks. Although a simplicial poset is not necessarily the face poset of a simplicial complex, it is always the face poset of a regular $CW$-complex. Let $P$ be a simplicial poset. To each element $y \in P \setminus \{0\} = \overline{P}$, we assign a (geometric) simplex whose face poset is $[0, y]$ and glue those geometric simplices according to the order relation in $P$. Then, we get the $CW$-complex in which the closure of each cell is identified with a simplex, the structure of faces being preserved; moreover, all characteristic mappings are embeddings. This $CW$-complex is called a simplicial cell complex associated to $P$ and is denoted by $|P|$. For instance, if two 2-simplices are identified on their boundaries via the identity map, then it is not a simplicial complex but a $CW$-complex obtained from a simplicial poset (see Figure 3). Clearly, this $CW$-complex is homeomorphic to the 2-sphere $S^2$. The simplicial cell complex $|P|$ has a well-defined barycentric subdivision which is isomorphic to the order complex $\Delta(\overline{P})$ of the poset $\overline{P}$.

By definition, we have the following proposition.

**Proposition 4.5.** Let $S$ be a finite cell complex. Then $S$ is simplicial if and only if for each cell $\sigma \subset S$, the closure $\overline{\sigma}$ of $\sigma$ is isomorphic to a simplex $\Delta$ of the same dimension with $\sigma$ as a cell complex.

In a word, a simplicial cell complex is a cell complex such that each closed cell is a geometric simplex. Obviously, the geometric realization of any finite simplicial complex is a simplicial cell complex.

**Definition 4.6.** Let $S$ be a simplicial cell complex and $V(S)$ the set of all 0-cells of $S$. Let $\sigma$ be a cell of $S$. We put $V(\sigma) = V(S) \cap \overline{\sigma}$. For each cell $\sigma \subset S$, there is an embedding

$$\varphi_{\sigma} : \Delta^{\dim \sigma}(V(\sigma)) \to \overline{\sigma} \subset S,$$
where $\Delta^{\dim \sigma}(V(\sigma))$ is the $\dim \sigma$-simplex whose vertex set is $V(\sigma)$. We say $\varphi_{\sigma}$ a characteristic map of $\sigma$.

**Proposition 4.7.** A simplicial poset corresponds to a simplicial cell complex.

Let $P$ be a simplicial poset and $x \in P$. A half-open interval $[\hat{0}, x]$ is a subset $\{y \in P | \hat{0} \leq y \leq x\}$ of $P$.

**Definition 4.8.** Let $P$ and $Q$ be simplicial posets. A simplicial poset map $f : P \rightarrow Q$ is a map such that for any $x \in P$, $\dim f(x) \leq \dim x$ and $f([\hat{0}, x]) = ([\hat{0}, f(x)]).

For a simplicial poset $P$, we put $V(P) := \{x \in P | \dim x = 0\}$, which is called the vertex set of $P$. Similarly, for each $x \in P$, $V(x) := V([\hat{0}, x]) = [\hat{0}, x] \cap V(P)$, which is also called the vertex set of $x$. A simplicial poset map $f$ is order-preserving and satisfies $f(V(x)) = V(f(x))$ for $x \in P$. Note that $V(P) = \bigcup_{x \in P} V(x)$. Moreover we put

$$K_P := \{V(x) | x \in P\},$$

which is a simplicial complex whose vertex set is $V(P)$. Here we see $K_P$ as a simplicial poset, so that a surjection $\varphi_P : P \twoheadrightarrow K_P$ defined by $\varphi_P(x) = V(x)$ is a simplicial poset map.

**Definition 4.9.** Let $X$ and $Y$ be simplicial cell complexes. A simplicial cell complex map $f : X \rightarrow Y$ is a cellular map such that for any cell $\sigma \in X$, $f(\sigma)$ is a cell of $Y$ and $f|_{\sigma} : \sigma \rightarrow f(\sigma) \subset Y$ extends linearly the map $f|_{V(\sigma)} : V(\sigma) \rightarrow V(f(\sigma)) \subset Y$. Note that $f(\sigma)$ is the compact set of a Hausdorff space $Y$.

Let $X$ and $Y$ be simplicial cell complexes. Let $\mathcal{F}(X)$ (respectively, $\mathcal{F}(Y)$) be a simplicial poset corresponding to $X$ (respectively, $Y$). A simplicial cell complex map $f : X \rightarrow Y$ defines a simplicial poset map $\mathcal{F}(f) : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ by $\sigma \mapsto f(\sigma)$ for each cell $\sigma \in X$. Conversely, we have the following.

**Proposition 4.10.** For any simplicial poset map $\alpha : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$, there exists uniquely a simplicial cell complex map $f : X \rightarrow Y$ such that $\mathcal{F}(f) = \alpha$. In particular, if a simplicial poset map $\alpha : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ is bijective, then $f$ is an isomorphism from $X$ to $Y$.

**Proposition 4.11.** For any simplicial poset $P$, there exists some simplicial cell complex $X$ with $\mathcal{F}(X) \cong P$.

From the above two propositions, there is uniquely an isomorphism class $[X]$ such that $\mathcal{F}(X) \cong P$. Then a simplicial cell complex $X$ is said to be a realization of $P$, denoted by $|P|$ as well. Under this notation, we have a simplicial cell complex map $|\varphi_P| : |P| \rightarrow |K_P|$.

Let $K$ be a $G$-complex. Now, we shall investigate the structure of the orbit space $K/G$. Let $\sigma$ and $\tau$ be simplices of $K$. We define a partial ordering on $K/G$ as follows:

$$\pi(\tau) \leq \pi(\sigma) \text{ if and only if there exists an element } g \in G \text{ such that } g(\tau) \subset \sigma,$$

where the map $\pi : K \twoheadrightarrow K/G$ is the orbit map. Note that the orbit space $K/G$ has the minimum $\hat{0} = \pi(\emptyset)$. Moreover we denote the orbit map from $|K|$ to $|K|/G$ by $\pi$ as well.
Proposition 4.12. If a $G$-complex $K$ has property $(P_1)$, $K/G$ is a simplicial poset. Moreover $|K|/G$ is a simplicial cell complex such that $\{\pi(\sigma)\mid \sigma \in K\backslash \{\emptyset\}\}$ is the set of all cells of $|K|/G$.

Proposition 4.13. If a $G$-complex $K$ has property $(P_1)$, it holds that $|K|/G \cong |K/G|$ as a simplicial cell complex.

Corollary 4.14. Let $X$ be a finite $T_0$-$G$-space. The orbit space $|\Delta(X)|/G$ is a finite simplicial cell complex associated to a simplicial poset $\Delta(X)/G$. Moreover we have $|\Delta(X)|/G \cong |\Delta(X)/G|$.

Let $X$ be a finite $T_0$-$G$-space. Since the orbit map $p : X \to X/G$ is continuous, it is an order-preserving map. It determines a simplicial map

$$\Delta(p) : \Delta(X) \to \Delta(X/G),$$

and also a continuous map $|\Delta(p)| : |\Delta(X)| \to |\Delta(X/G)|$. Noting $|\Delta(X/G)|$ is a $G$-space with a trivial $G$-action, we have a continuous map $\tilde{p} : |\Delta(X)|/G \to |\Delta(X/G)|$ such that the following diagram commutes

$$\begin{array}{ccc}
|\Delta(X)| & \xrightarrow{q} & |\Delta(X)/G| \\
\downarrow & & \downarrow \tilde{p} \\
|\Delta(X)|/G & \xrightarrow{p} & |\Delta(X/G)|
\end{array}$$

where $q$ is the orbit map from $|\Delta(X)|$ to $|\Delta(X)|/G$.

Proposition 4.15. Let $X$ be a finite $T_0$-$G$-space. A simplicial complex $K_{\Delta(X)/G}$ concides with $\Delta(X/G)$.

In consequence we have the following commutative diagram:

$$\begin{array}{ccc}
|\Delta(X)|/G & \xrightarrow{\cong} & |\Delta(X)/G| \\
\downarrow \tilde{p} & & \downarrow |\varphi_{\Delta(X)/G}| \\
|\Delta(X/G)| & \xrightarrow{id} & |\Delta(X/G)|
\end{array}$$

A simplicial action of $G$ on a simplicial complex $K$ is called regular in the sense of Bredon if $K$ possesses property $(P_2)$ for the action of each subgroups of $G$. Now, we shall present an interesting example.

Example 4.16. Let $n$ be an integer larger than one. Let $X_{2n+2}$ be a set consisting of $2n+2$ elements as follows:

$$X_{2n+2} = \bigcup_{i=1}^{n+1}\{x_i, x_{-i}\}.$$
We set
\[
\begin{align*}
U(x_i) := \{x_i\} \cup \{x_j, x_{-j}\}, \quad \text{and} \\
U(x_{-i}) := \{x_{-i}\} \cup \{x_j, x_{-j}\},
\end{align*}
\]
for \(i = 1, 2, \ldots, n + 1\). First note that each point \(x_i\) determines the smallest open set \(U(x_i)\) on \(X_{2n+2}\), that is, \(U_{x_i} = U(x_i)\). Therefore we define a \(T_0\)-topology on \(X_{2n+2}\). Let \(g\) be a map from \(X_{2n+2}\) to itself by \(g(x_i) = x_{-i}\). We set \(G := (g)(\text{that is, a group is generated by } g)\). Evidently, \(G\) is a cyclic group whose order is two. Since \(|\Delta(X_{2n+2})|\) is homeomorphic to the \(n\)-sphere \(S^n\), it holds that \(|\Delta(X_{2n+2})|/G \cong \mathbb{R}P^n\), where \(\mathbb{R}P^n\) is the \(n\)-dimensional real projective space. Note that \(|\Delta(X_{2n+2})|/G\) is a simplicial cell complex by Proposition 4.12. On the other hand, \(X_{2n+2}/G\) is a totally ordered set with \(n + 1\) elements. Therefore \(|\Delta(X_{2n+2}/G)|\) is homeomorphic to a \(n\)-simplex \(\Delta^n(X_{2n+2}/G)\).

Since the map \(\tilde{p} : |\Delta(X_{2n+2})|/G \to |\Delta(X_{2n+2}/G)|\) is not a weak homotopy equivalence, \(\tilde{p}\) is not an isomorphism between simplicial cell complexes. If \(\Delta(X_{2n+2}/G)\) is a simplicial complex, the map \(|\varphi_{\Delta(X_{2n+2}/G)}|\) is an isomorphism, and \(\tilde{p}\) is also an isomorphism. This is a contradiction. Hence \(\Delta(X_{2n+2}/G)\) is not a simplicial complex, thereby \(G\)-action on \(\Delta(X_{2n+2})\) is not regular in the sense of Bredon.

**Proof of Theorem B.**

Let \(X\) be a finite \(T_0\)-\(G\)-space. By Proposition 1.1, there is a weak homotopy equivalence \(\mu_X : |\Delta(X)| \to X\). Then \(\mu_X\) determines a continuous map \(\tilde{\mu}_X : |\Delta(X)|/G \to X/G\) such that the following diagram commutes.

\[
\begin{array}{ccc}
|\Delta(X)|/G & \xrightarrow{\tilde{p}} & |\Delta(X)/G| \\
\downarrow{\tilde{\mu}_X} & & \downarrow{\mu_{X/G}} \\
X/G & & 
\end{array}
\]

Therefore \(\tilde{p}\) is a weak homotopy equivalence if and only if \(\tilde{\mu}_X\) is so. In general, \(\tilde{\mu}_X\) is not a weak homotopy equivalence (see Example 4.16).

Remark that both \(|\Delta(X)|/G\) and \(|\Delta(X/G)|\) are CW-complexes. Therefore, we have

**Claim 1.** \(\tilde{\mu}_X\) is a weak homotopy equivalence if and only if \(\tilde{p}\) is a homotopy equivalence.

We consider the case where \(\tilde{p}\) is a homeomorphism.

**Claim 2.** Let \(X\) be a finite \(T_0\)-\(G\)-space. Then the following conditions are equivalent:
(1) \(\tilde{p}\) is a homeomorphism.
(2) \(\Delta(X)/G\) is a simplicial complex.
(3) \(\Delta(X)\) has property (P2).

**Proof.** (1) \(\Rightarrow\) (2) Since \(\tilde{p}\) is a homeomorphism, \(\varphi_{\Delta(X)/G}\) is injective. Let \(U\) be a subset of \(X/G\). Then there exists only one element \(s\) of \(\Delta(X)/G\) at most with \(V(s) = U\). Therefore \(\Delta(X)/G\) is a simplicial complex. (2) \(\Rightarrow\) (1) Since \(\Delta(X)/G\) is a simplicial complex, it holds that \(|\Delta(X)/G| = |\Delta(X/G)|\). Noting that \(\varphi_{\Delta(X)/G}\) is surjective, \(\tilde{p}\) is also surjective.
By Proposition 2.9, $\tilde{p}$ is a homeomorphism. (2) $\Rightarrow$ (3) Let $\sigma = \{x_i \mid i = 0, \cdots, k\}$ and $\tau = \{g_ix_i \mid g_i \in G, \ i = 0, \cdots, k\}$ be simplices of $\Delta(X)$. If $x_i = x_j$, then

$$g_jx_j = (g_jg_i^{-1})(g_ix_i) \in \tau \cap (g_jg_i^{-1})\tau.$$ 

Since a $G$-complex $\Delta(X)$ has property $(P_1)$, we have $g_jx_j = (g_jg_i^{-1})^{-1}(g_jx_j) = g_ix_i$, so that $g_ix_i = g_jx_j = g_jx_i$. Hence we assume that each $x_i (i = 0, \cdots, k)$ is distinct, then both $\sigma$ and $\tau$ are $k$-simplices of $\Delta(X)$. Therefore both $\pi(\sigma)$ and $\pi(\tau)$ are elements of $\Delta(X)/G$ such that $V(\pi(\sigma)) = V(\pi(\tau)) = \{\pi(x_i) \mid i = 0, \cdots, k\}$. By assumption, $\pi(\sigma) = \pi(\tau)$. In consequence there is some $g \in G$ such that $\tau = g(\sigma)$ and $g_ix_i = g_jx_i (i = 0, \cdots, k)$.

(3) $\Rightarrow$ (2) It follows from Proposition 4.2. $\square$

Combining Claim 1 and Claim 2, we obtain Theorem B. $\square$

References


