TANGENTIAL REPRESENTATIONS ON A SPHERE

Toshio Sumi
Faculty of Arts and Science
Kyushu University

1. INTRODUCTION

Let $G$ be a finite group. The Smith problem is as follows. Let $\Sigma$ be a homotopy sphere with smooth $G$-action such that $\Sigma$ has just two fixed points, say $a$ and $b$. Are tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic as real $G$-modules? Two real $G$-modules $U$ and $V$ are called Smith equivalent if there exists a smooth action of $G$ on a sphere $\Sigma$ such that $S^G = \{a, b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$ as real $G$-modules. We know infinitely many Oliver groups possessing non-isomorphic Smith equivalent real modules. We consider about the subset $\text{Sm}(G)$ of the real representation ring $\text{RO}(G)$ of $G$ consisting of all differences $U - V$ of Smith equivalent real $G$-modules. Recently we have several results corresponding to the Smith set. In this note, we study a sufficient condition for the Smith set to be an additive subgroup of the real representation ring $\text{RO}(G)$. This work is a continuous study from [24].

2. SMITH PROBLEM

The Smith problem asks whether the Smith set $\text{Sm}(G)$ is zero or not. There are many results corresponding to the Smith problem.

Atiyah and Bott [1] or Milnor [7] showed that for a homotopy sphere $\Sigma$ with semi-free smooth compact Lie group with just two fixed points, the tangential representations are isomorphic. Thus, any Smith equivalent real modules over an abelian simple group are isomorphic, that is, $\text{Sm}(C) = 0$ for a prime order cyclic group $C$. Sanchez [18] generalized the result as follows by computing $G$-signature and using Franz-Bass’s theorem. For a cyclic group $P$ of odd prime power order, Smith equivalent real $P$-modules are isomorphic. Therefore $\text{Sm}(P) = 0$ for any group $P$ of odd prime power order by combining the Smith theory.

On the other hand, Cappell and Shaneson [2] showed that there exists non-isomorphic, Smith equivalent real module over a cyclic group $C_{4n}$ of order $4n$ for $n \geq 2$, that is, $\text{Sm}(C_{4n}) \neq \{0\}$. Petrie [17] showed that the Smith set of an abelian group of odd order which has at least four non-cyclic subgroups is nontrivial, eg. $\text{Sm}(C_{pqrs} \times C_{pqrs}) \neq 0$.

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where $p, q, r, s$ are distinct odd primes. And in 1980's, Dovermann, Suh, Masuda, etc.
studied the Smith equivalent real modules.

Oliver [13] showed that $G$ acts smoothly on a disk without fixed points if and only if
there are no subgroups $P$ and $H$ such that $P$ is a $p$-group, $H/P$ is cyclic, $G/H$ is a $q$-group
for some primes $p$ and $q$, possibly $p = q$. A group acting on a disk without fixed points
is called an Oliver group. Laitinen and Morimoto [5] showed that $G$ is an Oliver group if
and only if there exists a one fixed point $G$-action on sphere. Laitinen and Pawafowski [6]
showed that there exists Smith equivalent, non-isomorphic real $G$-modules for a perfect
group $G$ with $r_G \geq 2$ by connecting sum with a sphere with just one fixed point, where $r_G$
is the number of real conjugacy classes of elements of $G$ not of prime power order. After
that, Pawafowski and Solomon [14] extended to that $\text{Sm}(G) \neq \emptyset$ if $G$ is a gap Oliver group
with $r_G \geq 2$ except $\text{Aut}(A_6)$ and $\text{PSL}(2, 27)$. A group $G$ is a gap group if there exists a real
$G$-module $V$ such that

- $\dim V^L = 0$ for any prime power index subgroup $L$ of $G$ and
- for any subgroups $P$ of prime power order and $H$ with $H > P$,

$$\dim V^P \geq 2 \dim V^H.$$  

In particular, a perfect group $G$ with $r_G \geq 2$ is a gap Oliver group. A study for gap groups
is seen in [12, 19, 20, 22, 23].

Now we need some notations. A real conjugacy class $(x)^*$ of an element $x$ of $G$ is the
union of the conjugacy class

$$(x) = \{g^{-1}xg \mid g \in G\}$$

of $x$ and one of its inverse $x^{-1}$. We denote by $\overline{\text{NPP}}(G)$ the set of elements of $G$ not of prime
power order, by $\overline{\text{NPP}}(G)$ the set of elements of the real conjugacy classes of elements of
$\text{NPP}(G)$. Then $r_G$ is the cardinality of the set $\overline{\text{NPP}}(G)$. For a prime $p$, let $N_p(G)$ be
the set of normal subgroups $N$ of $G$ with $[G : N] \leq p$. We denote by $\text{RO}(G)$ the real
representation ring, by $\mathcal{P}(G)$ the set of all subgroups of $G$ of prime power, possibly 1,
order, by $O^p(G)$ the Dress subgroup of type $p$ for a prime $p$ defined as

$$O^p(G) = \bigcap_{L \leq G, [G:L]=p^a} L,$$

and by $\mathcal{L}(G)$ the set of all prime power, possible 1, index subgroups of $G$. Then for
$L \in \mathcal{L}(G)$, $L$ contains $O^p(G)$ for some prime $p$. We put

$$\cap p(G) = \bigcap_{N \in \mathcal{P}(G)} N$$

which quotient is an elementary abelian $p$-group and denote by $G^{\text{nil}}$ the smallest normal
subgroup of $G$ by which quotient is nilpotent:

$$G^{\text{nil}} = \bigcap_p O^p(G).$$
Note that 

\[ G \supseteq \cap p(G) \supseteq O^p(G) \supseteq G^{\nil}. \]

For families \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of subgroups of \( G \) and a subset \( A \) of \( \text{RO}(G) \), we put

\[
A_{\mathcal{F}_1} = \bigcap_{P \in \mathcal{F}_1} \ker(\text{Res}_P^G : \text{RO}(G) \to \text{RO}(P)) \cap A,
\]

\[
A_{\mathcal{F}_2} = \bigcap_{L \in \mathcal{F}_2} \ker(\text{Fix}^L : \text{RO}(G) \to \text{RO}(N_G(L)/L)) \cap A,
\]

and

\[
A_{\mathcal{F}_1} \cap A_{\mathcal{F}_2} = \bigcap_{P \in \mathcal{F}_1} \ker(\text{Res}_P^G) \cap \bigcap_{L \in \mathcal{F}_2} \ker(\text{Fix}^L) \cap A.
\]

The automorphism group \( \text{Aut}(A_6) \) of the alternating group \( A_6 \) is not a gap group, \( r_{\text{Aut}(A_6)} = 2 \), and \( \text{Sm}(G) = 0 \) [8]. Morimoto [8] gave a condition

\[ \text{Sm}(G) \subset \text{RO}(G)_{\mathcal{P}_o(G)}^{N_G(G)} = \text{RO}(G)^{\mathcal{P}_o(G)} \]

for Smith equivalent real modules. The rank of \( \text{RO}(G)^{N_G(G)} \) is equal to

\[ r_G - r_{G,\cap 2(G)}, \]

where \( r_{G,\cap 2(G)} \) is the cardinality of the set \( \pi(\text{NPP}(G)) \) for a canonical projection \( \pi : G \to G/\cap 2(G) \) (cf. [14]). This condition implies that there are Oliver solvable groups \( G \) such that \( r_G \geq 2 \) and \( \text{Sm}(G) = 0 \) [15]. The group \( \text{PSL}(2, 27) \) is an extension of \( \text{PSL}(2, 27) \) by a field automorphism group of order 3 which is a gap non-solvable group, \( r_{\text{PSL}(2, 27)} = 2 \) and \( \text{Sm}(\text{PSL}(2, 27)) \neq 0 \) [9]. Moreover, putting together with [16], for an Oliver non-solvable group \( G \) with \( r_G \geq 2 \), \( \text{Sm}(G) = 0 \) if and only if \( G \) is isomorphic to \( \text{Aut}(A_6) \).

3. SUBSETS OF THE SMITH SET

Sanchez’s criterion and Petrie’s observation says that

\[ \text{Sm}(G) \subset \text{RO}(G)^{[G]}_{\mathcal{P}_o(G)}, \]

where \( \mathcal{P}_o(G) \) is the set of subgroups of \( G \) of order 1, 2, 4, or odd prime power. Thus we have

\[ \text{Sm}(G) \subset \text{RO}(G)^{N_G(G)}_{\mathcal{P}_o(G)}. \]

Note that if \( G \) has no element of order 8 then \( \mathcal{P}_o(G) = \mathcal{P}(G) \). Recall that two real \( G \)-modules \( U \) and \( V \) are Smith equivalent if there exists a smooth action of \( G \) on a sphere \( \Sigma \) such that \( S^G = \{a, b\}, T_a(\Sigma) \cong U \) and \( T_b(\Sigma) \cong V \) as real \( G \)-modules and put

\[ \text{Sm}(G) = \{[U] - [V] \mid U \text{ and } V \text{ are Smith equivalent}\}. \]

Similarly we consider the sets \( \text{PSm}^p(G) \) (resp. \( \text{LSm}(G) \)) of all differences \( [U] - [V] \) such that \( U \) and \( V \) are Smith equivalent and in addition the homotopy sphere \( \Sigma \) satisfies that \( \Sigma^p \) is connected for any prime power order subgroups \( P \) of \( G \) (resp. for any 2-groups of \( G \)).
The set $\text{PSm}^c(G)$ (resp. $\text{LSm}(G)$) is empty if and only if $G$ is of order prime power (resp. 2-power). It holds the inclusions

$$\text{PSm}^c(G) \subset \text{LSm}(G) \subset \text{Sm}(G)$$

and

$$\text{LSm}(G) \subset \text{RO}(G)^\mathcal{P}(G).$$

**Theorem 3.1.** Let $G$ be an Oliver group whose nil-quotient $G/G^{\text{nil}}$ is not a 2-group. Then

$$\text{RO}(G)^{L(G)}_\mathcal{P}(G) \subset \text{PSm}^c(G).$$

Moreover, we have

**Theorem 3.2.** Let $G$ be an Oliver non-gap group with $[G : O^2(G)] = 2$. Suppose that all elements $x$ of $G \setminus O^2(G)$ of order 2 such that $C_G(x)$ is not a 2-group. Then

$$\text{RO}(G)^{L(G)}_\mathcal{P}(G) \subseteq \text{PSm}^c(G).$$

We denote by $\text{SG}(m, n)$ the small group of order $m$ and type $n$ which is obtained as $\text{SmallGroup}(m, n)$ in the software GAP [3]. Morimoto studied (or is studying) the set $\text{Sm}(G)^\mathcal{P}(G) \setminus \text{Sm}(G)^{L(G)}_\mathcal{P}(G)$. He [9] showed that for $G = \text{PEL}(2, 27)$, $\text{SG}(864, 2666)$, $\text{SG}(864, 4666)$, $\text{Sm}(G)^{L(G)}_\mathcal{P}(G) = 0$ but $\text{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \text{Sm}(G) \cong \mathbb{Z}$. Also he and his colleagues [4] showed that if a Sylow 2-subgroup is normal, then

$$\text{Sm}(G) \subset \text{RO}(G)^{N_2(G)}_\mathcal{P}(G)$$

and in particular $\text{Sm}(G) = 0$ holds for $G = \text{SG}(1176, 220), \text{SG}(1176, 221)$.

For an Oliver group, we see $\text{PSm}^c(G) \neq 0$ to show $\text{Sm}(G)^{\mathcal{P}(G)} \neq 0$. We have no rich examples so that $\text{Sm}(G)^{\mathcal{P}(G)} \neq \text{Sm}(G)$, whole $\text{Sm}(G) \setminus \text{Sm}(G)^{\mathcal{P}(G)}$ is a finite set. We do
not have an example for an Oliver group $G$ such that $\text{PSm}^c(G) \neq \text{Sm}(G)_{\mathcal{P}(G)}$. It remains the problem whether $\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)}$ for an Oliver group.

4. Criterion for the Smith set to be a group

We discuss for Oliver groups $G$ such that $\text{PSm}^c(G)$ is a subgroup of $\text{RO}(G)$. We introduce two conditions. One is a part of a sufficient condition to show $\text{Sm}(G)_{\mathcal{P}(G)} \backslash \text{Sm}(G)^{G(G)} \neq 0$ and the other is a sufficient condition so that $\text{Sm}(G)_{\mathcal{P}(G)}$ is a group.

Let $Q = \bigcap_{p \neq 2} O^p(G)$ be a normal subgroup of $G$ with odd index and let $N$ be a normal subgroup of $G$ with $G^{\text{nil}} \leq N \leq \cap 2(G) \cap Q$. Then

$$Q \geq \cap 2(G) \cap Q \geq N \geq G^{\text{nil}} \geq O^2(Q).$$

**Definition 4.1.** We say that $G$ satisfies the quasi-$N$-$\mathcal{P}$-condition if there are real $Q$-modules $U$ and $V$ such that

- $\dim U^{\cap 2(G) \cap Q} = \dim V^{N} = 0$ and
- $[\mathbb{R} \oplus U] - [V] \in \text{RO}(Q)_{\mathcal{P}(Q)}$.

In particular, the quasi-$G^{\text{nil}}$-$\mathcal{P}$-condition is simply called quasi-nil-$\mathcal{P}$-condition.

**Definition 4.2.** We say that $G$ satisfies the weak-nil-$\mathcal{P}$-condition if there are real $G$-modules $U$ and $V$ such that

- $\dim U^{\cap 2(G)} = \dim V^{G^{\text{nil}}} = 0$ and
- $[\mathbb{R} \oplus U] - [V] \in \text{RO}(G)_{\mathcal{P}(G)}$.

**Lemma 4.3.** If $G$ satisfies the quasi-nil-$\mathcal{P}$-condition, then $G$ satisfies the weak Nil-$\mathcal{P}$-condition.

**Proposition 4.4** (cf. [10, Lemma 15]). Let $G$ be a finite group with $O^2(G) = G$. The following statements are equivalent.

1. $G^{\text{nil}}$ has a sub-quotient isomorphic to $D_{2pq}$ for distinct primes $p, q$.
2. $G$ satisfies the quasi-nil-$\mathcal{P}$-condition.

Morimoto and Qi [11] obtained a sufficient condition for an Oliver group $G$ to hold that $\text{Sm}(G)_{\mathcal{P}(G)}$ is not equal to $\text{Sm}(G)^{G(G)}$. This result supplies that $\text{Sm}(G) = \text{Sm}(G)_{\mathcal{P}(G)} \cong \mathbb{Z}$ for $G = \text{SG}(864, 2666)$ or $\text{SG}(864, 4666)$. For $G = \text{SG}(864, 2666)$ or $\text{SG}(864, 4666)$, $G/G^{\text{nil}}$ is a cyclic group of order 3 and $\text{RO}(G)_{\mathcal{P}(G)}$ is generated by two element $\mathbb{R}[G/G^{\text{nil}}] + X_1$ and $3(\mathbb{R}[G/G^{\text{nil}}] - \mathbb{R}) + X_2$ for some elements $X_1, X_2 \in \text{RO}(G)^{G^{\text{nil}}}$ and thus, $G$ satisfies the weak-nil-$\mathcal{P}$-condition since $G/G^{\text{nil}}$ is a cyclic group of order 3. We see it in the next section. Indeed, $G$ has a sub-quotient isomorphic to $D_{12}$ and $G$ satisfies the quasi-nil-$\mathcal{P}$-condition.

**Definition 4.5.** For a normal subgroup $N$ of $G$, we say that $G$ satisfies the $N$-$\mathcal{P}$-condition if there are real $G$-modules $U$ and $V$ such that $U^N = V^N = 0$ and $[\mathbb{R} \oplus U] - [V] \in \text{RO}(G)_{\mathcal{P}(G)}$. If $N = G^{\text{nil}}$ we say that $G$ satisfies the Nil-$\mathcal{P}$-condition.
Lemma 4.6 or Theorem 4.8 in [9] essentially yields us the following two theorems.

**Theorem 4.6.** If a gap Oliver group $G$ satisfies the weak-Nil-$\mathcal{P}$-condition with $NPP(G) \cap G^{ni1} \neq \emptyset$ and has an element of $NPP(G)$ outside $O^p(G)$ for some prime $p$, then

$$\text{PSm}^c(G) \setminus \text{RO}(G)^{l(G)}_{\mathcal{P}(G)} \neq 0.$$ 

Note that under the assumption that $NPP(G) \cap G^{ni1} \neq \emptyset$ the inequality $\text{RO}(G)^{N_2(G)}_{\mathcal{P}(G)} \neq \text{RO}(G)^{l(G)}_{\mathcal{P}(G)}$ if and only if $NPP(G) \setminus O^p(G)$ is not empty for some prime $p$. By using the multiplication of $\text{RO}(G)$, we get the following theorem.

**Theorem 4.7.** Let $G$ be a gap Oliver group satisfying the Nil-$\mathcal{P}$-condition. Then

$$\text{PSm}^c(G) = \text{RO}(G)^{N_2(G)}_{\mathcal{P}(G)} = \text{Sm}(G)_{\mathcal{P}(G)}$$

and in particular $\text{Sm}(G)_{\mathcal{P}(G)}$ is an additive group.

If a Sylow 2-subgroup of $G$ is normal, $G$ does not satisfy the Nil-$\mathcal{P}$-condition. Although the Nil-$\mathcal{P}$-condition is a sufficient one for an Oliver group $G$ such that $\text{Sm}(G)^{\mathcal{P}(G)}$ is an additive group, it is not a necessary condition. For example, $A_5 \times C_4$ does not satisfy the Nil-$\mathcal{P}$-condition but the following result holds.

**Proposition 4.8.** $\text{PSm}^c(A_5 \times C_4) = \text{Sm}(A_5 \times C_4) = \text{RO}(A_5 \times C_4)^{[A_5]}$.

**Problem.** $\text{PSm}^c(A_5 \times (C_4)^n) = \text{Sm}(A_5 \times (C_4)^n)$ holds. Is it true that $\text{PSm}^c(A_5 \times (C_4)^n) = \text{RO}(A_5 \times (C_4)^n)^{[A_5 \times (C_2)^p]}$ for $n \geq 2$?

5. **Quasi-Nil-$\mathcal{P}$-condition**

In this section we study properties for the weak-Nil-$\mathcal{P}$-condition. Remark that there is an Oliver group which satisfies the weak-Nil-$\mathcal{P}$-condition but does not satisfy the Nil-$\mathcal{P}$-condition (eg. SG(864, 2666), SG(864, 4666)).

**Proposition 5.1.** Let $K$ be a subgroup of $G$ such that $\cap 2(G) \cdot K = G$. If $K$ satisfies the weak-$(G^{ni1} \cap K)$-$\mathcal{P}$-condition, then $G$ satisfies the weak-Nil-$\mathcal{P}$-condition.

**Theorem 5.2.** Let $G$ be a gap Oliver group. Suppose that $NPP(G) \cap G^{ni1}$ is not empty and that there is an element $NPP(G)$ outside of $O^p(G)$ for some prime $p$. If an odd index subgroup $K$ of $G$ satisfies the weak-$(G^{ni1} \cap K)$-$\mathcal{P}$-condition, then

$$\text{PSm}^c(G) \setminus \text{RO}(G)^{l(G)}_{\mathcal{P}(G)} \neq 0.$$ 

Morimoto and Qi [10, Lemma 21 and Theorem 22] showed that $\text{Sm}(G)^{\mathcal{P}(G)} \neq \text{Sm}(G)^{l(G)}_{\mathcal{P}(G)}$ for an odd integer $n > 1$, an odd prime $p$, and $G = D_{2n} \int C_p$, the wreath product of the
dihedral group $D_{2n}$ of order $2n$ by a cyclic group $C_p$ of order $p$. The group $G$ satisfies the assumption of Proposition 5.1 as follows. The group $G$ has a presentation

$$a_i^n = b_j^2 = (a_i b_j)^2 = 1, \ (\forall i),$$

$$\langle a_1, b_1, \ldots, a_p, b_p, c \mid a_i a_j = a_j a_i, a_i b_j = b_j a_i, b_i b_j = b_j b_i, \ (i \neq j), \rangle,$$

$$c^p = 1, c^{-1}a_i c = a_{i+1}, c^{-1}b_i c = b_{i+1}, \ (\forall i)$$

where $a_{p+1} = a_1$ and $b_{p+1} = b_1$. The group $G^{\text{nil}}$ is a subgroup of $G$ generated by elements $a_1, \ldots, a_p$ and $b_i b_j$ ($i < j$), and then $G/G^{\text{nil}} \cong C_{2p}$. Thus $G$ is a gap Oliver group. Put $K = O^p(G)$. Let $f: D_{2n}^p \to D_{2n}$ be the first projection and let $\hat{U}$ and $\hat{V}$ be $\mathcal{P}(D_{2n})$-matched real $D_{2n}$-modules such that $\hat{U}^{D_{2n}} = \mathbb{R}$ and $\hat{V}^{D_{2n}} = 0$. The real $K$-modules $f^* \hat{U}$ and $f^* \hat{V}$ implies that $K$ satisfies the assumption of Proposition 5.1 since $f(G^{\text{nil}}) = D_{2n}$. (Or directly, two real $G$-modules $\text{Ind}_K^G f^* \hat{U}$ and $\text{Ind}_K^G f^* \hat{V}$ implies that $G$ satisfies the weak-Nil-$\mathcal{P}$-condition.)

Before closing this section, we should say the strongness of the weak-Nil-$\mathcal{P}$-condition. Let $G$ be a finite group such that $G/G^{\text{nil}}$ is a nilpotent group of odd order and there are an element of $G^{\text{nil}}$ not of prime power order and an element of $G$ outside $G^{\text{nil}}$ not of prime power order. Then

$$\text{RO}(G)^{[G^{\text{nil}}]}_{\mathcal{P}(G)} \neq \text{RO}(G)^G_{\mathcal{P}(G)}.$$  

Note that if a Sylow 2-subgroup of $G$ is normal then $\text{Sm}(G) \subset \text{RO}(G)^{N(G)}_{\mathcal{P}(G)}$ (cf. [4]) and $G$ does not satisfy the weak-Nil-$\mathcal{P}$-condition. Otherwise, if $G$ has a sub-quotient isomorphic to $D_{2qr}$ for some distinct primes $q$ and $r$, there are real $G$-modules $U$ and $V$ such that the equalities $U^{G^{\text{nil}}} = 0 = V^{G^{\text{nil}}}$ hold and that $\mathbb{R}[G/G^{\text{nil}}] \oplus U$ and $V$ are $\mathcal{P}(G)$-matched:

$$\mathbb{R} + [(\mathbb{R}[G/G^{\text{nil}}] - \mathbb{R}) \oplus U] - [V] = \mathbb{R}[G/G^{\text{nil}}] + [U] - [V] \in \text{RO}(G)^{G^{\text{nil}}}_{\mathcal{P}(G)}.$$  

Thus, $G$ satisfies the weak-Nil-$\mathcal{P}$-condition and in addition if $G$ is a gap Oliver group then

$$\text{PSm}^c(G)^{[G^{\text{nil}}]} \neq \text{PSm}^c(G).$$  

6. Nil-$\mathcal{P}$-condition

In this section we study properties for the Nil-$\mathcal{P}$-condition.

**Proposition 6.1.** If $G$ satisfies the Nil-$\mathcal{P}$-condition, then $G$ satisfies the weak-Nil-$\mathcal{P}$-condition.

**Proposition 6.2.** If a quotient group of $G$ satisfies the Nil-$\mathcal{P}$-condition, then $G$ also satisfies the Nil-$\mathcal{P}$-condition.

**Proposition 6.3.** Let $N$ be a normal subgroup of $G$. If there are a subgroup $K$ of $G$ and an epimorphism $f: K \to H$ such that $f(K \cap N) = H$, $KN = G$ and $H$ has sub-quotient isomorphic to $D_{2pq}$, where $p$ and $q$ are distinct primes, then $G$ satisfies the $N$-$\mathcal{P}$-condition.
For a perfect group $G$, the weak-Nil-$\mathcal{P}$-condition and Nil-$\mathcal{P}$-condition are equivalent and moreover equivalent to that $G$ has a sub-quotient isomorphic to a dihedral group $D_{2pq}$ for distinct primes $p$ and $q$.

**Proposition 6.4** (cf. [21]). *Simple groups except the following groups satisfy the Nil-$\mathcal{P}$-condition.*

1. **Cyclic group**
2. **Projective special linear groups**: $\text{PSL}(2, 4) = \text{PSL}(2, 5) = A_5$, $\text{PSL}(2, 7) = \text{PSL}(3, 2)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 9) = A_6$, $\text{PSL}(2, 17)$, $\text{PSL}(3, 4)$, $\text{PSL}(3, 8)$
3. **Suzuki groups** $\text{Sz}(8)$, $\text{Sz}(32)$
4. **Projective unitary groups**: $\text{PSU}(3, 3)$, $\text{PSU}(3, 4)$, $\text{PSU}(3, 8)$

**Theorem 6.5.** *Let $q > 1$ be a prime power. The following groups are gap groups satisfying the Nil-$\mathcal{P}$-condition.*

1. **Symmetric groups** $S_n$, $n \geq 7$
2. **Projective general linear groups** $\text{PGL}(2, q)$, $q \neq 2, 3, 4, 5, 7, 8, 9, 17$
3. **Projective general linear groups** $\text{PGL}(3, q)$, $q \neq 2, 4, 8$
4. **Projective general linear groups** $\text{PGL}(n, q)$, $n \geq 4$
5. **General linear groups** $\text{GL}(2, q)$, $q \neq 2, 3, 4, 5, 7, 8, 9, 17$
6. **General linear groups** $\text{GL}(3, q)$, $q \neq 2, 4, 8$
7. **General linear groups** $\text{GL}(n, q)$, $n \geq 4$
8. **The automorphism group of sporadic groups**

The Smith sets of $\text{PGL}(2, q)$ and $\text{PGL}(3, q)$ have been already obtained in [24]. This can be proved by finding subgroups as in Proposition 6.3. The groups listed up in Theorem 6.5 are non-solvable gap group. Then we have the following theorem.

**Theorem 6.6.** *Let $G$ be a group which has quotient isomorphic to a group in Theorem 6.5. Then*

$$\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)} = \text{RO}(G)_{\mathcal{P}(G)}^{N(G)}.$$  

**Corollary 6.7.** *Let $K$ be a group in Theorem 6.5 and $F$ any finite group. Then for $G = K \times F$,*

$$\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)} = \text{RO}(G)_{\mathcal{P}(G)}^{N(G)}.$$  

**References**


Faculty of Arts and Science, Kyushu University, Motooka 744, Nishi-ku, Fukuoka, 819–0395, Japan
E-mail address: sumi@artsci.kyushu-u.ac.jp