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ON SOME DIFFERENTIAL SUBORDINATIONS

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ABSTRACT. The purpose of this work is to present a new geometric approach to some problems in differential subordination theory. We also discuss the new results closely related to the generalized Briot-Bouquet differential subordination.

1. INTRODUCTION

Let $\mathcal{H}$ denote the class of all analytic functions in the unit disc $\mathbb{D} = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. A univalent function f maps $\mathbb{D}$ onto a convex domain E if and only if [6]

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{for all } z \in \mathbb{D} \tag{1.1}$$

and then f is said to be convex in $\mathbb{D}$ (or briefly convex). Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{D}$ we denote by $\mathcal{K}$. We say that $f \in \mathcal{A}$ is convex of order α, $0 \le \alpha < 1$ when

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad \text{for all } z \in \mathbb{D}. \tag{1.2}$$

Functions that are convex of order α introduced Robertson in [5]. For two analytic functions f, g, we say that f is subordinate to g, written as $f \prec g$, if and only if there exists an analytic function ω with property $|\omega(z)| \le |z|$ in $\mathbb{D}$ such that $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}). \tag{1.3}$$

The idea of subordination was used for defining many of classes of functions studied in geometric function theory. For obtaining the main result, we shall use the methods of differential subordinations. The main results in the theory of differential subordinations was introduced by Miller and Mocanu in [1],[3]. A function p, analytic in $\mathbb{D}$, is said to satisfy a first order differential subordination if

$$\phi\left(p(z), zp'(z)\right) \prec h(z), \tag{1.4}$$

where $(p(z), zp'(z)) \in D \subset \mathbb{C}^2$, $\phi : \mathbb{C}^2 \to \mathbb{C}$ is analytic in $\mathbb{D}$, h is analytic and univalent in $\mathbb{D}$. The function q is said to be a *dominant* of the differential subordination (1.4) if $p \prec q$ for all p satisfying (1.4). If $\widetilde{q}$ is a *dominant* of (1.4) and $\widetilde{q} \prec q$ for all *dominants* q of (1.4), then we say that $\widetilde{q}$ is the *best dominant* of the differential subordination (1.4).

The following lemma will be required in our present investigation.

Lemma 1.1. [1], [3, p.24] *Assume that $\mathcal{Q}$ is the set of functions $f \in \mathcal{H}$ that are injective on $\overline{\mathbb{D}} \setminus E(f)$, where*

$$E(f) := \{\zeta : \zeta \in \partial\mathbb{D} \ \text{ and } \ \lim_{z \to \zeta} f(z) = \infty\},$$

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and are such that

$$f'(\zeta) \neq 0 \qquad (\zeta \in \partial(\mathbb{D}) \setminus E(f)).$$

Let $\psi \in \mathcal{Q}$ with $q(0) = a$ and let

$$\varphi(z) = a + a_m z^m + \cdots$$

be analytic in $\mathbb{D}$ with

$$\varphi(z) \not\equiv a \quad \textbf{and} \quad m \in \mathbb{N}.$$

If $\varphi \not\prec \psi$ in $\mathbb{D}$, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{D} \quad \textbf{and} \quad \zeta_0 \in \partial\mathbb{D} \setminus E(q),$$

for which

$$\varphi(|z| < r_0) \subset \psi(\mathbb{D}),$$
$$\varphi(z_0) = \psi(\zeta_0)$$

and

(1.5) $$z_0\varphi'(z_0) = s\zeta_0\psi'(\zeta_0),$$

for some $s \geq m$.

2. Main results

Theorem 2.1. *Let $h \in \mathcal{Q}$ be convex univalent in $\mathbb{D}$, and let $p(z)$ be analytic in $\mathbb{D}$ such that*

(2.1) $$p(0) = h(0), \quad \mathfrak{Re}\,\{\phi(h(z))\} > 0 \quad z \in \mathbb{D},$$

and

(2.2) $$p(z) + zp'(z)\phi(p(z)) \prec h(z), \quad z \in \mathbb{D},$$

where $\phi(z)$ is analytic in a domain D containing $h(\mathbb{D})$. Then we have

$$p(z) \prec h(z) \quad z \in \mathbb{D}.$$

Proof. Let us put

(2.3) $$p(z) + zp'(z)\phi(p(z)) = h(z), \quad p(0) = h(0).$$

Then, there is an unique solution $p(z)$ which satisfies the equation (2.3). If $p(z) \not\prec h(z)$, then there exists a point z_0, $|z_0| < 1$ for which

$$p(z_0) = h(\zeta_0), \quad p(|z| < |z_0|) \subset h(\mathbb{D}), \quad |\zeta_0| = 1,$$

see Fig. 1 below. Then it follows that from (2.3) we have

(2.4) $$p(z_0) + z_0p'(z_0)\phi(p(z_0)) = h(\zeta_0) + z_0p'(z_0)\phi(h(\zeta_0)).$$

Let l be the tangential line at the point $w = p(z_0) = h(\zeta_0)$ and let m be the perpendicular line at this point. Further let α be the point of intersection with m and the real axis. The case when m is parallel to the real axis, and α doesn't exist, we shall consider afterwards. If α exists, then we have

(2.5) $$p(z_0) + z_0p'(z_0)\phi(p(z_0)) = h(\zeta_0) + \frac{z_0p'(z_0)}{p(z_0) - \alpha}\{p(z_0) - \alpha\}\,\phi(h(\zeta_0)).$$

Because $z_0p'(z_0)$ is outdoor normal vector to the boundary of $h(|z| \leq 1)$ and $p(|z| \leq |z_0|)$ at the point $w = p(z_0) = h(\zeta_0)$ thus both $z_0p'(z_0)$ and $p(z_0) - \alpha$ lie on the line m and are of the same argument. This geometric observation yields to that

$$\frac{z_0p'(z_0)}{p(z_0) - \alpha} \geq 0.$$

Therefore, we have

(2.6) $$\arg\left\{\frac{z_0p'(z_0)}{p(z_0) - \alpha}\{p(z_0) - \alpha\}\,\phi(h(\zeta_0))\right\} = \arg\{p(z_0) - \alpha\} + \arg\{\phi(h(\zeta_0))\}$$

and

$$|\arg\{\phi(h(\zeta_0))\}| < \frac{\pi}{2}. \tag{2.7}$$

Using together (2.5), (2.6) and (2.7) we observe that $p(z_0) + z_0 p'(z_0)\phi(p(z_0))$ lies outside the set $h(\mathbb{D})$ because $h(\mathbb{D})$ is convex, and this contradicts (2.2).

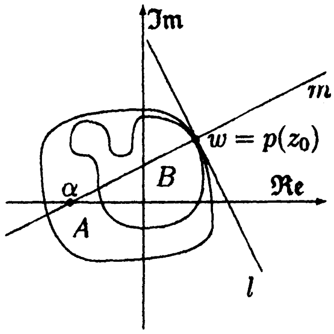

Fig. 1. $A = h(|z| \le 1)$, $B = p(|z| \le |z_0|)$

If m is parallel to the real axis, then $p(z_0)$ and $h(\zeta_0)$ are real. Moreover, we have $z_0 p'(z_0)/p(z_0) \ge 0$. Therefore, and by (2.4) we obtain that $p(z_0) + z_0 p'(z_0)\phi(p(z_0))$ lies on the boundary of $h(\mathbb{D})$ but this contradicts (2.2), too. It completes the proof. □

The above proof of Theorem 2.1 is something else than that of Miller and Mocanu in [2, p.186], see also [3, p.120].

Theorem 2.2. *Let $f(z) = z + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$, and convex of order α, $0 \le \alpha < 1$. Then we have*

$$\frac{zf'(z)}{f(z)} \prec q(z), \quad z \in \mathbb{D}, \tag{2.8}$$

where $q(z)$ satisfies the differential equation

$$q(z) + \frac{zq'(z)}{q(z)} = \frac{1 + (1-2\alpha)z}{1-z}, \quad q(0) = 1. \tag{2.9}$$

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \; p(0) = 1, \quad z \in \mathbb{D}, \tag{2.10}$$

Because $f(z)$ is convex of order α, from (2.10), it follows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + (1-2\alpha)z}{1-z} = h(z). \tag{2.11}$$

On the other hand, from Theorem 2.1 we have $p(z) \prec h(z)$, where $h(z)$ is a convex function. If $p(z) \not\prec q(z)$, then there exists a point $z_0 \in \mathbb{D}$ such that

$$p(z_0) = q(\zeta_1), \quad p(|z| < |z_0|) = h(\mathbb{D}), \quad |\zeta_1| = 1.$$

Then, from Lemma 1.1, we have

$$
\begin{aligned}
p(z_0)+\frac{z_0p'(z_0)}{p(z_0)} &= q(\zeta_1)+\frac{s\zeta_1q'(\zeta_1)}{q(\zeta_1)}\\
&= s\left(q(\zeta_1)+\frac{\zeta_1q'(\zeta_1)}{q(\zeta_1)}\right)+(1-s)q(\zeta_1)\\
&= s\left\{\frac{1+(1-2\alpha)\zeta_1}{1-\zeta_1}\right\}+(1-s)p(z_0),\quad (1\leq s).
\end{aligned}\tag{2.12}
$$

On the other hand, it follows that

$$
\Re\mathfrak{e}\left\{\frac{1+(1-2\alpha)e^{i\theta}}{1-c^{i\theta}}\right\}=\alpha,
$$

where $0\leq\theta<2\pi$.

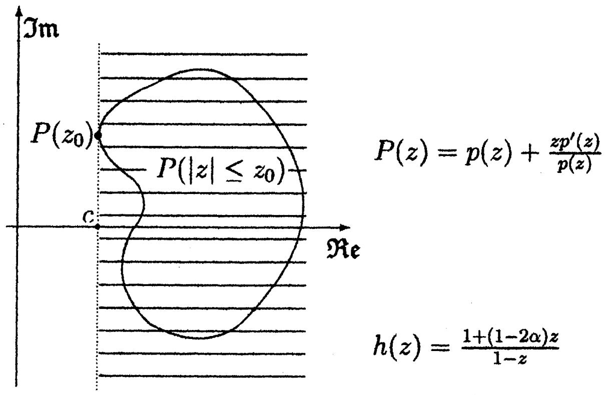

Fig.2. $h(\mathbb{D})$, $c=\alpha$.

Therefore, from (2.12), we have

$$
\begin{aligned}
\Re\mathfrak{e}\left\{p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right\} &= m\Re\mathfrak{e}\left\{\frac{1+(1-2\alpha)\zeta_1}{1-\zeta_1}\right\}+(1-m)\Re\mathfrak{e}\{p(z_0)\}\\
&= m\alpha+\Re\mathfrak{e}\{p(z_0)\}\\
&\leq m\alpha+(1-m)\alpha\\
&= \alpha.
\end{aligned}\tag{2.13}
$$

This contradicts (2.11) and therefore, it completes the proof.

□

The above proof of Theorem 2.2 is different from the earlier one presented by Miller and Mocanu in [1, p.165].

Theorem 2.3. *Let $h\in\mathcal{Q}$ be convex univalent in $\mathbb{D}$, and let $\gamma\neq 0$ with $\Re\mathfrak{e}\gamma\geq 0$. If $p(z)$ is analytic in $\mathbb{D}$ such that*

$$
p(z)+zp'(z)/\gamma\prec h(z),\quad z\in\mathbb{D},\tag{2.14}
$$

then we have

$$
p(z)\prec h(z)\quad z\in\mathbb{D}.\tag{2.15}
$$

Proof. From the hypothesis, we have

$$
\Re\mathfrak{e}\left\{\frac{1}{\gamma}\right\}=\Re\mathfrak{e}\left\{\frac{\overline{\gamma}}{|\gamma|^2}\right\}\geq 0.
$$

If we put

$$\phi(z) = \frac{1}{\gamma} \quad z \in \mathbf{C},$$

then

$$\Re\{\phi(h(z))\} > 0 \quad z \in \mathbb{D}.$$

Applying Theorem 2.1 we get (2.15). □

The above proof of Theorem 2.3 is different from the earlier one presented by Miller and Mocanu in [1, p.167].

Theorem 2.4. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$, and suppose that*

$$p(z) - \frac{zp'(z)}{p(z)} \prec q(z), \quad z \in \mathbb{D}, \tag{2.16}$$

where $q(z)$ satisfies the differential equation

$$q(z) - \frac{zq'(z)}{q(z)} = \frac{1+z^2}{1-z^2}, \quad q(0) = 1. \tag{2.17}$$

Then we have

$$p(z) \prec q(z) \quad z \in \mathbb{D}. \tag{2.18}$$

Proof. From the hypothesis (2.17), we have

$$q(z) = \frac{1+z}{1-z}. \tag{2.19}$$

If $p(z) \not\prec q(z)$, then there exists a point $z_0 \in \mathbb{D}$ such that

$$p(z_0) = q(\zeta_0), \quad p(|z| < |z_0|) = h(\mathbb{D}), \quad |\zeta_0| = 1.$$

It follows that from (1.5) we have

$$\begin{aligned} p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)} &= q(\zeta_0) - \frac{s\zeta_0 q'(\zeta_0)}{q'(\zeta_0)} \\ &= s\left(q(\zeta_0) - \frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right) + (1-s)q(\zeta_0), \quad s \geq 1. \end{aligned}$$

From (2.17) and (2.19) we have

$$\begin{aligned} \Re\left\{p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right\} &= s\Re\left\{q(\zeta_0) - \frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right\} + (1-s)\Re\{q(\zeta_0)\} \\ &= 0 + 0 = 0. \end{aligned}$$

because $|\zeta_0| = 1$. Therefore, $p(z_0) - (z_0 p'(z_0))/p(z_0) \notin q(\mathbb{D}) = \{w : \Re\{w\} > 0\}$. By subordination principle, this contradicts (2.16) and therefore, it completes the proof. □

Theorem 2.5. *Let β and γ be complex with $\beta \neq 0$ and let $p(z)$ and $q(z)$ be analytic in $\mathbb{D}$ with $p(0) = q(0)$. If $\Re\{\beta q(z) + \gamma\} > 0$ and $\beta q(z) + \gamma$ is convex univalent in $\mathbb{D}$ and if*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec q(z), \quad z \in \mathbb{D}, \tag{2.20}$$

then we have

$$p(z) \prec q(z) \quad z \in \mathbb{D}. \tag{2.21}$$

Proof. If $p(z) \not\prec q(z)$, then there exists a point $z_0 \in \mathbb{D}$ such that

$$p(z_0) = q(\zeta_0), \quad p(|z| < |z_0|) = h(\mathbb{D}), \quad |\zeta_0| = 1.$$

Then, from Lemma 1.1, we have

$$p(z_0) + \frac{z_0 p'(z_0)}{\beta p(z_0) + \gamma} = q(\zeta_0) + \frac{s\zeta_0 q'(\zeta_0)}{\beta q(\zeta_0) + \gamma}, \tag{2.22}$$

where $1 \leq s$. With the notation as as in the figure **Fig.** 1 and in the same way as in the proof of Theorem 2.1 we obtain

$$\begin{aligned} \arg \frac{s\zeta_0 q'(\zeta_0)}{\beta q(\zeta_0) + \gamma} &= \arg \left\{ \frac{s\zeta_0 q'(\zeta_0)}{q(\zeta_0) - \alpha} \frac{q(\zeta_0) - \alpha}{\beta q(\zeta_0) + \gamma} \right\} \\ &= \arg \left\{ \frac{q(\zeta_0) - \alpha}{\beta q(\zeta_0) + \gamma} \right\} \\ &= \arg \{q(\zeta_0) - \alpha\} - \arg \{\beta q(\zeta_0) + \gamma\}, \end{aligned} \tag{2.23}$$

where α is the point of intersection with m and the real axis. From the hypothesis, we have

$$|\arg \{\beta q(\zeta_0) + \gamma)\}| < \frac{\pi}{2}. \tag{2.24}$$

Using together (2.22), (2.23) and (2.24) we observe that $p(z_0) + z_0 p'(z_0)/(\beta p(z_0) + \gamma)$ lies outside the set $q(\mathbb{D})$ because $q(\mathbb{D})$ is convex, and this contradicts (2.20). In the case when m is parallel to the real axis, then as in the proof of Theorem 2.1 we obtain that $p(z_0) + z_0 p'(z_0)/(\beta p(z_0) + \gamma)$ lies on the boundary of $Q(\mathbb{D})$, $Q(z) = q(z) + zq'(z)/(\beta q(z) + \gamma)$ but this contradicts (2.20), too. It completes the proof. □

The Theorem 2.5 above was proved earlier differently as Corollary 1.1 in [2, p.167].

Theorem 2.6. *Let λ be a complex number with $|\lambda| \leq 1$, and let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 0$. If $p(z)$ satisfies*

$$p(z) + \frac{zp'(z)}{\lambda p(z) + 1} \prec \frac{z(2 + \lambda z)}{1 + \lambda z}, \quad z \in \mathbb{D}, \tag{2.25}$$

then we have

$$p(z) \prec z \quad z \in \mathbb{D}. \tag{2.26}$$

Proof. If $p(z) \not\prec z$, then there exists a point $z_0 \in \mathbb{D}$ such that

$$p(z_0) = \zeta_0, \quad p(z < |z_0|) = h(\mathbb{D}), \quad |\zeta_0| = 1.$$

Then, from Lemma 1.1, we have

$$p(z_0) + \frac{z_0 p'(z_0)}{\lambda p(z_0) + 1} = \zeta_0 + \frac{s\zeta_0 q'(\zeta_0)}{\lambda \zeta_0) + 1}, \tag{2.27}$$

where $1 \leq s$. A simple calculation shows that either

$$\zeta_0 + \frac{s\zeta_0 q'(\zeta_0)}{\lambda \zeta_0) + 1} \in h(|z| = 1) \quad \text{when} \quad s = 1 \tag{2.28}$$

or

$$\zeta_0 + \frac{s\zeta_0 q'(\zeta_0)}{\lambda \zeta_0) + 1} \notin h(|z| \leq 1) \quad \text{when} \quad s \geq 1. \tag{2.29}$$

This contradicts (2.25) and therefore it finishes the proof. □

For other results on the subordination in geometric function theory we refer also to the recent papers [4] and [7].

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