数理解析研究所講究録
第 1880 巻 2014 年 133-179

Discrete surfaces of constant mean curvature

Fran Burstall, Udo Hertrich-Jeromin, Wayne Rossman, Susana Santos

Abstract. We propose a unified definition for discrete analogues of constant mean curvature surfaces in spaces of constant curvature as a special case of discrete special isothermic nets. Bäcklund transformations and Lawson's correspondence are discussed. It is shown that the definition generalizes previous definitions and a construction for discrete cmc surfaces of revolution in space forms is provided.

MSC 2000. *53A10*, *53C42*, 53A30, 52C26, 37K35, 37K25

1. Introduction

Discrete surfaces of constant mean curvature (discrete "cmc surfaces") have been studied in recent years from a variety of different points of view. Two essentially antithetic approaches, one from variational principles and the other from integrable systems, lead to substantially different definitions: normally, for example, discrete soap films and bubbles, i.e., "discrete variationally cmc surfaces", are triangulated whereas the definition of "integrable discrete cmc surfaces" makes use of special coordinates and, therefore, leads to "discrete cmc *nets*", i.e., quadrilateral surfaces. Even in cases where it is sensible to compare the two approaches, such as that of a discrete catenoid in Euclidean space, it turns out that different notions are obtained: each approach leads to a different class of discrete surfaces that can be viewed as analogues of the smooth catenoid.

The present paper is concerned with the integrable systems approach to discrete cmc surfaces.

A key feature of this approach is its compatibility with the transformation theory of the (smooth) surface class under consideration: for a given class of surface, not only is a similar transformation theory sought for the discrete case but a discrete surface in the class should be created by repeated (Bäcklund-Darboux-)transformations of the smooth class; or, otherwise said, every 2-dimensional subnet of a multidimensional net created by repeated transformation of a discrete surface in the class should itself be a discrete net of the class. This is what has recently been coined "multidimensional consistency", see the very clear and essential description of integrable discretization in [6].

A key idea in the definition of integrable discrete cmc surfaces has been to consider them as special discrete isothermic nets: that is, to discretize a (conformal) curvature line net on a smooth cmc surface — recall that smooth surfaces of constant mean curvature (in any space form) are isothermic, i.e., allow a parametrization by conformal curvature line parameters. This has been the pioneering idea in [3], where the authors introduced the notion of *discrete minimal surfaces in Euclidean space* alongside the notion of discrete isothermic surfaces and their Christoffel transformation[1]. Subsequently, the notion of *discrete surfaces of constant mean curvature in Euclidean space* has been introduced alongside a notion of a Darboux transformation for discrete

[1]Note the parallel with Christoffel's original paper [13], where his transformation is introduced — motivated by an observation about minimal surfaces.

isothermic nets in [17], see also [4], and the notion of *discrete horospherical surfaces in hyperbolic space* — as an analogue of smooth cmc 1 surfaces — has been introduced alongside a notion of a Calapso transformation for discrete isothermic nets in [18]. In all three cases the constant mean curvature surfaces can be characterized as isothermic surfaces with a special behaviour of their transformations, as we will discuss below.

Note that for all three classes of surfaces $H^2 + \kappa \geq 0$, where H is the mean curvature of the surface and κ is the ambient curvature. It is straightforward to use the Calapso transformation for discrete isothermic nets to extend this family of definitions to discrete analogues of any constant mean curvature surfaces with

$$H^2 + \kappa \geq 0.$$

Here, the key observation is that, for smooth constant mean curvature surfaces in space forms, the Calapso transformation becomes a conformal variant of the Lawson correspondence and that Bianchi permutability can then be used to carry over the characterization of cmc surfaces in Euclidean space to other space forms. However, these ideas turn out to be useless for surfaces with

$$H^2 + \kappa < 0$$

as, for example, for minimal surfaces in hyperbolic space.

For discrete isothermic surfaces in Euclidean space, a construction of a mean curvature function or rather a "mean curvature sphere congruence" was given in [4] and shown to be constant for discrete minimal or constant mean curvature surfaces[2]. Note that this mean curvature function is defined at the *vertices* of a discrete isothermic net. Very recently, new ideas from [22] have led to substantial progress in this direction: a new definition of discrete cmc surfaces in Euclidean space relies on the requirement that a mean curvature function — defined via Steiner's formula on the *faces* of a discrete (isothermic) net — be constant, see also [8]. This definition is equivalent to the one via isothermic transformations, see [9].

Our mission in the present paper will be to add another definition of discrete cmc surfaces to the list. However, the aim is not just to promote mathematical pluralism: our definition provides a uniform definition of discrete cmc nets in all space forms alike — in particular, we also capture the previously inaccessible case of

$$H^2 + \kappa < 0.$$

In fact, we define the much wider class[3] of "discrete special isothermic surfaces" based on [10] and [12, Def. 2.18], see Definition 3.12; these come equipped with a "type number" $N \in \mathbb{N}$ — discrete cmc nets in space forms will be the $N = 1$ case. Hence our definition does not only provide a generalization in allowing any ambient space form and value of the mean curvature, but also in discussing a wider class of discrete isothermic nets — and we expect it to inaugurate a new direction of research in the field.

We shall start our investigation with a short discussion of discrete isothermic surfaces and their transformations — not only to remind the reader of some facts and to fix notations but also to introduce our perspective on discrete isothermic nets

[2]In the minimal case the reverse is in fact also true as shown in [3].

[3]In contrast to the generic terminology of "special discrete isothermic nets" used earlier our "discrete special isothermic surface" will be a technical term.

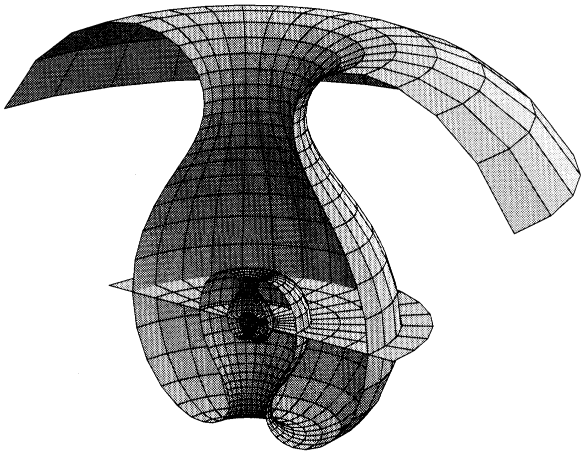

FIGURE 1.1. Discrete minimal net in H^3

via loops of flat connections, which will be central to all that follows, see Lemma 2.5. This will set the scene for the central section of this paper: we shall investigate the properties of polynomial loops of parallel sections, called "polynomial conserved quantities", and relations to the geometry of the underlying isothermic net. Excluding some degenerate cases we will arrive at the notion of "discrete special isothermic nets of type N" in a natural way. It turns out that the Darboux transformation for discrete isothermic nets behaves nicely on these special isothermic nets, which gives rise to a "Bäcklund transformation" for special isothermic nets; in particular, we will prove a Bianchi permutability theorem that establishes "3D-consistency" for special isothermic nets. Hence our discrete special isothermic nets satisfy the two fundamental discretization principles of the "discrete Erlanger programme" of [6]:

- *Transformation group principle* — this is built into our construction as we are working in conformal geometry which is the natural symmetry group for special isothermic (smooth) surfaces and (discrete) nets alike, see [12, Sect. 2.2.3];
- *Consistency principle* — which is established by our Bianchi permutability theorem for the Bäcklund transformation, see Theorem 4.7.

Certain (very) special Bäcklund transforms of a special isothermic net, its "complementary nets", will provide the basis for establishing the relation of our approach with the previous approaches to discrete cmc surfaces via their transformations as discrete isothermic surfaces in [3], [17], [4] and [18], as discussed above. Moreover, we obtain a characterization for discrete cmc surfaces in space forms, i.e., special isothermic nets of type 1, with

$$H^2 + \kappa \geq 0$$

via complementary nets — as one may have expected; and the lack of their existence when

$$H^2 + \kappa < 0$$

provides one possible explanation why the aforementioned approach to define discrete cmc nets in space forms was doomed to failure in this case. In the same context we also obtain characterizations of type 2 special isothermic nets, which discretize the classical "special isothermic surfaces" of Darboux [14] and Bianchi [2], see also [15, §§84–86].

In the final section we arrive at the main subject of this paper: we give a definition of discrete cmc surfaces in space forms as special isothermic nets of type 1. Clearly, the rich theory developed in the more general case of special isothermic nets descends to a similarly rich theory for discrete cmc nets in space forms — in particular, we have the discrete analogues of the Lawson correspondence and the Bäcklund transformation and our discretization satisfies the two discretization principles above for an "integrable discretization". Note that we consider Möbius geometry as the natural ambient geometry for constant mean curvature surfaces in space forms as these arise in "Lawson families" of cmc surfaces with different ambient curvatures: the confinement to a space form subgeometry appears as a symmetry breaking phenomenon initiated by part of the geometric data attached to a special isothermic net and, in particular, a discrete cmc net.

Despite the obvious merits of our definition we make a great effort to convince the reader of its value by providing detailed analysis of how the aforementioned previous approaches tie in with our definition. However, we do not conceal its problems: we provide an example of a single discrete isothermic net which is a cmc net in a whole family of different space forms[4]; even though this seems to be a rather singular example we can, at the moment, only speculate about how to obviate this anomaly. On the positive side, we provide a method to construct discrete cmc nets of revolution for any prescribed mean curvature H and ambient curvature κ. In particular, we show how to explicitely construct discrete analogues of smooth constant mean curvature surfaces that were previously unavailable: for example, we construct (see Figure 1.1) the discrete analogue of a "hyperbolic catenoid", that is, a minimal surface of revolution in hyperbolic space, see [1].

Acknowledgements. It is our pleasure to thank our colleagues A. Bobenko, T. Hoffmann, U. Pinkall, W. Schief and Y. Suris for many interesting and helpful discussions about the subject.

We also gratefully acknowledge financial support for exchange visits of the second and third authors to Japan and the UK, respectively, from the *Daiwa Anglo-Japanese Foundation* and the *Japanese Ministry of Education.*

The surface graphics in Figures 1.1 & 5.1 were produced using *Mathematica.*

2. Discrete isothermic nets

We consider discrete nets $f : \mathbb{Z}^2 \supset M \to S^3$ in the (conformal) 3-sphere, where

$$M = \{(m, n) \in \mathbb{Z}^2 \,|\, m_1 \leq m \leq m_2, n_1 \leq n \leq n_2\}$$

is a rectangular grid[5]:

[4]This net is not spherical but it "looks" close to a "wrinkled" sphere — note that, in a smooth world, spheres are the only surfaces that have constant mean curvature in different space forms.

[5]It should be straightforward to generalize our results to discrete nets defined on quad-graphs, making it possible to consider discrete isothermic nets with "umbilics", cf. [20].

Definition 2.1. *Such a net will be called a* discrete isothermic net *if there is a (real) function a on the edges of M, that is, a map $(ij) \mapsto a_{ij} \in \mathbb{R}$ with $a_{ji} = a_{ij}$ for all edges (ij), so that*

- (i) *a has equal values on opposite edges of elementary quadrilaterals*
$$(ijkl) = ((m,n)(m+1,n)(m+1,n+1)(m,n+1)),$$
i.e., $a_{(m,n)(m+1,n)} = a_{(m,n+1),(m+1,n+1)}$ and correspondingly for "vertical" edges;
- (ii) *the cross ratios[6] $q_{ijkl} = [f_i; f_j; f_k; f_l]$ on faces factorize as*
$$q_{ijkl} = \frac{a_{ij}}{a_{il}}$$
into two functions of one variable.

Thus we employ the "wide definition" of discrete isothermic nets [4], see also [19, §5.7.2], which discretizes isothermic nets parametrized by curvature line coordinates (not necessarily conformally): as all cross ratios are real, the four vertices of any face of the net are concircular, so that a discrete isothermic net qualifies as a discrete curvature line net or *discrete principal net.* Note that the smallest domain of a discrete net where "discrete isothermic" imposes a condition is a 3×3-grid, $m_2-m_1 = n_2-n_1 = 2$; there the definition can be reformulated as a cross ratio 1 condition on four cross ratios:
$$\frac{q_{(m,n-1)(m+1,n-1)(m+1,n)(m,n)}\, q_{(m-1,n)(m,n)(m,n+1)(m-1,n+1)}}{q_{(m,n)(m+1,n)(m+1,n+1)(m,n+1)}\, q_{(m-1,n-1)(m,n-1)(m,n)(m-1,n)}} = 1;$$
a cross ratio function, satisfying this condition on all 3×3-grids in M, determines the function a uniquely up to a non-zero factor.

As a mild regularity assumption, discretizing the notion of an immersed surface parametrized by curvature lines, we will usually add the requirement that any three of the four vertices of a face uniquely determine the circle of the four vertices, i.e., that any three vertices are in "general position".

Throughout the paper we will use the following notations: if g is a map defined on the vertices of a rectangular grid M, then we let

$$dg_{ij} := g_j - g_i \quad \text{and} \quad g_{ij} := \frac{1}{2}(g_i + g_j); \tag{2.1}$$

note that $(ij) \mapsto g_{ij}$ defines a function on the edges of M whereas $(ij) \mapsto dg_{ij}$ defines a 1-form, that is, $dg_{ij} + dg_{ji} = 0$. With these notations a Leibniz rule holds:

$$d(g \cdot h)_{ij} = g_{ij} \cdot dh_{ij} + dg_{ij} \cdot h_{ij}, \tag{2.2}$$

where "$\cdot$" denotes any product on the target space of g and h.

2.1. **The projective approach and Moutard lifts.** As we are considering nets in the *conformal* 3-sphere it will be helpful to consider
$$S^3 \cong L^4/\mathbb{R} \subset \mathbb{R}P^4, \quad \text{where} \quad L^4 = \{Y \in \mathbb{R}^{4,1} \mid |Y|^2 = 0\},$$
as a quadric in projective 4-space. Recall (from [19] for example) that 2-spheres are, in this model, described by Minkowski 4-spaces in $\mathbb{R}^{4,1}$ and circles by Minkowski 3-spaces or, equivalently, by their (spacelike) orthogonal complements, and that incidence translates into a subspace relation or as orthogonality, respectively. For example,

[6]Note that the cross ratio of four points in S^3 is (up to complex conjugation) a conformal invariant. For a detailed discussion see [19, Sects. 4.9, 6.5 and §7.5.14].

four points $p_n \in S^3$, $n = 1, \dots, 4$, generically lie on a unique 2-sphere $S \subset S^3$ which can be described as

$$S \cong \mathrm{span}\{P_1, \dots, P_4\} \subset \mathbb{R}^{4,1},$$

where $P_n \in L^4$ with $\mathbb{R}P_n = p_n \in S^3 \cong L^4/\mathbb{R}$ — their (complex) cross ratio[7] is given by

$$q = [p_1; p_2; p_3; p_4] = \tag{2.3}$$
$$\frac{\langle P_1, P_2\rangle\langle P_3, P_4\rangle - \langle P_1, P_3\rangle\langle P_2, P_4\rangle + \langle P_1, P_4\rangle\langle P_2, P_3\rangle \pm \sqrt{\det(\langle P_i, P_j\rangle)_{i,j=1,\dots 4}}}{2\langle P_1, P_4\rangle\langle P_2, P_3\rangle};$$

the cross ratio becomes real exactly when the P_n become linearly dependent, i.e., when they span a 3-dimensional Minkowski subspace and the four points p_n are concircular. In this case the (real) cross ratio uniquely determines the relative position of the four points on the circle: given three of the points, say p_1, p_2 and p_4, and a real cross ratio q the fourth point $p_3 = \Gamma^q_{p_2,p_4}(p_1)$, where

$$\Gamma^q_{p,p'}(X) := X + \frac{1}{\langle P, P'\rangle}\{(q-1)\,\langle X, P'\rangle P + (\frac{1}{q} - 1)\,\langle X, P\rangle P'\}, \tag{2.4}$$

as is easily verified from (2.3); note that $\Gamma^q_{p,p'} \in O(4,1)$ descends to a Möbius transformation of S^3 and does not depend on the choice of representatives $P, P' \in L^4$ of $p, p' \in S^3$. Also note that

$$\mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\} \ni q \mapsto \Gamma^q_{p,p'}(p'') \in S^3$$

yields a 1-to-1 parametrization of the circle through three distinct points $p, p', p'' \in S^3$ in terms of the cross ratio, so that $\Gamma^0_{p,p'}(p'') = p'$, $\Gamma^1_{p,p'}(p'') = p''$ and $\Gamma^\infty_{p,p'}(p'') = p$.

Now suppose that $f : M \to S^3$ is an isothermic net and fix a cross ratio factorizing function a. We wish to show that there is a lift F of f with

$$\langle F_i, F_j\rangle = a_{ij} \tag{2.5}$$

on every edge (ij) of M. To this end we have to show that this scaling is compatible on any quadrilateral: thus let $(ijkl)$ denote an elementary quadrilateral and choose a light cone lift $F_i \in L^4$ of f_i; then we normalize lifts $F_j, F_l \in L^4$ of f_j and f_l so that $\langle F_i, F_j\rangle = a_{ij}$ and $\langle F_i, F_l\rangle = a_{il}$. Now we choose the lift

$$F_k := \Gamma^{\frac{a_{ij}}{a_{il}}}_{f_j,f_l}(F_i) = F_i + \frac{a_{ij} - a_{il}}{\langle F_j, F_l\rangle}(F_j - F_l) \tag{2.6}$$

of f_k and readily verify that $\langle F_j, F_k\rangle = a_{il} = a_{jk}$ and $\langle F_l, F_k\rangle = a_{ij} = a_{kl}$.

Note that this lift F of f satisfies the discrete version (2.6) of a Moutard equation.

Conversely, if a light cone lift F of a discrete surface satisfies a Moutard equation, $F_k - F_i \parallel F_j - F_l$ on all faces, then it is isothermic, see [7, Def. 9]. Namely, taking scalar products we learn that

$$\left.\begin{matrix} F_k + F_i \perp F_j - F_l \\ F_k - F_i \perp F_j + F_l \end{matrix}\right\} \quad \Rightarrow \quad \left\{\begin{matrix} \langle F_j, F_k\rangle = \langle F_i, F_l\rangle \\ \langle F_k, F_l\rangle = \langle F_i, F_j\rangle \end{matrix}\right.$$

[7]Note that $\det(\langle P_i, P_j\rangle) < 0$ so that $\sqrt{\det(\langle P_i, P_j\rangle)} \in i\mathbb{R}$. Using the Clifford algebra of $\mathbb{R}^{4,1}$, a Clifford algebra valued cross ratio can be defined whose "imaginary" part encodes the 2-sphere of the four points [19, Sect. 6.5].

and hence

$$\langle F_i, F_k\rangle\langle F_j, F_l\rangle = \langle F_i, F_j - F_l\rangle\langle F_k - F_i, F_l\rangle = (\langle F_i, F_j\rangle - \langle F_i, F_l\rangle)^2$$

so that (2.3) gives

$$[f_i; f_j; f_k; f_l] = \frac{\langle F_i, F_j\rangle}{\langle F_i, F_l\rangle}.$$

From (2.6) it is also straightforward to see that any diagonal vertex star of a discrete isothermic net is cospherical: if $i_{(m,n)}$, $m, n \in \{-1, 0, 1\}$, denote the vertices of a 3×3-grid then the discrete Moutard equation (2.6) shows that the four diagonals $F_{i_{(m,n)}} - F_{p_{(0,0)}}$, $m, n \in \{\pm 1\}$, are linearly dependent so that

$$\dim \operatorname{span}\{F_{i_{(0,0)}}, F_{i_{(1,1)}}, F_{i_{(-1,1)}}, F_{i_{(-1,-1)}}, F_{i_{(1,-1)}}\} \leq 4$$

and the five points lie on a 2-sphere.

Assuming that the vertex star $\{F_{i_{(0,0)}}, F_{i_{(1,0)}}, F_{i_{(0,1)}}, F_{i_{(-1,0)}}, F_{i_{(0,-1)}}\}$ is not cospherical the converse can also be shown, leading to two characterizations of discrete isothermic nets, see [7, Sect. 3]:

Lemma 2.2. . *A discrete net $f : \mathbb{Z}^2 \supset M \to S^3$ in the conformal 3-sphere is isothermic iff*

- (i) *there is a lift $F : M \to L^4$ of f satisfying a discrete Moutard equation $F_k - F_i \parallel F_j - F_l$ on every face $(ijkl)$ iff*
- (ii) *any diagonal vertex star is cospherical.*

The sphere containing a diagonal vertex star of an isothermic net is referred to as the *central sphere* of the net at the center of the star, see [7, Thm. 10].

As an example we investigate discrete surfaces of revolution: consider the discrete net

$$(m, n) \mapsto f_{(m,n)} := (\eta_m, \varrho_m \cos\varphi_n, \varrho_m \sin\varphi_n) \in \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} \cong S^3$$

where η, ϱ and φ are real functions of a discrete parameter. A straightforward cross ratio computation would reveal that

$$q_{(m,n)(m+1,n)(m+1,n+1)(m,n+1)} = -\frac{(d\eta_{m,m+1})^2 + (d\varrho_{m,m+1})^2}{4\varrho_m\varrho_{m+1}\sin^2\frac{d\varphi_{n,n+1}}{2}}$$

identifying the net as a discrete isothermic net. However, we shall proceed differently to show that f is an isothermic net and to find a cross ratio factorizing function a on the edges.

Consider

$$|(x_0, \dots, x_4)|^2 = -x_0^2 + \sum_{i=1}^{4} x_i^2$$

as the quadratic form of the Minkowski scalar product of $\mathbb{R}^{4,1}$ and let

$$F^e := \left(\frac{1+|f|^2}{2}, f, \frac{1-|f|^2}{2}\right) \tag{2.7}$$

denote the *Euclidean lift*[8] of f into the light cone $L^4 \subset \mathbb{R}^{4,1}$. Now observe that

$$F_{(m,n)} := \frac{(-1)^m}{\varrho_m} F^e_{(m,n)} =$$

$$(-1)^m\{\underbrace{(\tfrac{1+\eta_m^2+\varrho_m^2}{2\varrho_m}, \tfrac{\eta_m}{\varrho_m}, 0, 0, \tfrac{1-\eta_m^2-\varrho_m^2}{2\varrho_m})}_{=:\, M_m \in \mathbb{R}^{2,1}} + \underbrace{(0, 0, \cos\varphi_n, \sin\varphi_n, 0)}_{=:\, \Phi_n C \in \mathbb{R}^2}\},$$

that is, there is an orthogonal decomposition $\mathbb{R}^{4,1} = \mathbb{R}^{2,1} \oplus \mathbb{R}^2$ so that

$$F_{(m,n)} = (-1)^m(M_m + \Phi_n C) = (-1)^m \Phi_n(M_m + C), \tag{2.8}$$

where Φ_n are rotations of $\mathbb{R}^2$, $C \in S^1 \subset \mathbb{R}^2$ and M takes values in the hyperbolic plane[9]

$$H^2 = \{Y \in \mathbb{R}^{2,1} \,|\, |Y|^2 = -1, Y_0 > 0\} \subset \mathbb{R}^{2,1}.$$

In particular, $M_m \perp \Phi_n C$ for all (m,n).

Note that $\mathbb{R}^2 = \mathrm{span}\{\Phi_n C \,|\, n \in \mathbb{Z}\}$ defines an elliptic sphere pencil, hence (cf. [19, Sect. 1.2]) a circle

$$L^4 \cap \mathbb{R}^{2,1} = L^4 \cap \{\Phi_n C \,|\, n \in \mathbb{Z}\}^\perp,$$

which is the axis of our discrete surface of revolution. At the same time, it is the infinity boundary of the hyperbolic 2-plane H^2 of the meridian curve.

Clearly, F satisfies the discrete Moutard equation

$$F_{(m+1,n+1)} - F_{(m,n)} = (-1)^{m+1}\{M_{m+1} + \Phi_{n+1}C + M_m + \Phi_n C\} = F_{(m+1,n)} - F_{(m,n+1)}$$

and is therefore a discrete isothermic net with cross ratio factorizing function

$$a_{ij} := \langle F_i, F_j \rangle =$$

$$= \begin{cases} -1 - \langle M_m, M_{m+1}\rangle = \frac{(d\eta_{m,m+1})^2 + (d\varrho_{m,m+1})^2}{2\varrho_m \varrho_{m+1}} & \text{for } (ij) = ((m,n)(m+1,n)) \\ -1 + \langle \Phi_n C, \Phi_{n+1} C\rangle = -2\sin^2 \frac{d\varphi_{n,n+1}}{2} & \text{for } (ij) = ((m,n)(m,n+1)) \end{cases}$$

as soon as $M_{m+1} \neq M_m$ and $\Phi_{n+1}C \neq \Phi_n C$.

2.2. **Quaternions and the Calapso transformation.** The Calapso transformation, or T-transformation, of (discrete) isothermic nets will be central to our investigations — it was introduced in [18] (see also [19, §5.7.16]) using a quaternionic setup for Möbius geometry. Hence we will first briefly discuss the quaternionic approach in order to make contact with earlier work; however, we will provide an independent definition in the following section so that a reader unfamiliar with previous approaches may just skip this section.

Thus, we consider $S^3 \cong \mathrm{Im}\,\mathbb{H} \cup \{\infty\} \subset \mathbb{HP}^1$ and

$$\mathbb{R}^{4,1} \cong \{X \in End(\mathbb{H}^2) \,|\, X = \begin{pmatrix} x & x_\infty \\ x_0 & -x \end{pmatrix}, x \in \mathrm{Im}\,\mathbb{H}, x_0, x_\infty \in \mathbb{R}\} \subset End(\mathbb{H}^2)$$

[8]The Euclidean lift into the (flat) quadric of constant curvature (see [19, Sect. 1.4])

$$\mathcal{Q} = \{Y \in L^4 \,|\, \langle Y, Q\rangle = -1\}, \quad \text{where} \quad Q := (1, 0, 0, 0, -1).$$

[9]Secretly we are using a conformal map $\mathbb{R}^3 \setminus \{\text{axis}\} \to H^2 \times S^1 \subset \mathbb{R}^{2,1} \oplus \mathbb{R}^2 = \mathbb{R}^{4,1}$ adapted to the rotational symmetry of the map f, cf. [19, §1.4.16].

equipped with $|X|^2 = -X^2 = x^2 + x_0x_\infty$ as the quadratic form of the Minkowski product[10]. In particular, we obtain an isometry

$$(2.9)\qquad \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H} \ni x \to X = \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} \in L^4 \subset \mathbb{R}^{4,1};$$

note that, for two such "Euclidean" light cone lifts $X, Y \in L^4$,

$$-2\langle X, Y\rangle = XY + YX = -(y-x)^2 = |y-x|^2.$$

In this setup the Möbius group

$$M\ddot{o}b(3) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in End(\mathbb{H}^2) \,|\, \bar{a}c+\bar{c}a = \bar{b}d+\bar{d}b = 0, \bar{a}d+\bar{c}b \in \mathbb{R}\backslash\{0\}\}/\mathbb{R} \subset \mathbb{P}Gl(2,\mathbb{H})$$

of S^3 acts isometrically on $\mathbb{R}^{4,1}$ via

$$(2.10)\qquad M\ddot{o}b(3) \times \mathbb{R}^{4,1} \ni (A, X) \mapsto A \cdot X := AXA^{-1} \in \mathbb{R}^{4,1}.$$

Now let $f : M \to S^3$ be a discrete conformal net, with cross ratio factorizing function a on the edges, and define

$$(2.11)\qquad \tau_{ij} := \frac{a_{ij}}{2\langle F_i, F_j\rangle} F_iF_j,$$

where F is any light cone lift of f; assuming that $f : M \to \operatorname{Im}\mathbb{H} \subset S^3$ and using the lift (2.9) we find

$$\tau_{ij} = \begin{pmatrix} f_i df_{ij}^* & -f_i df_{ij}^* f_j \\ df_{ij}^* & -df_{ij}^* f_j \end{pmatrix}, \quad \text{where} \quad df_{ij}^* = a_{ij}(df_{ij})^{-1}$$

is the "derivative" of the Christoffel transform f^* of f in $\mathbb{R}^3$, see [4, Thm. 14] or [19, §5.7.7]. Then, for $\lambda \in \mathbb{R}$,

$$(1 + \lambda\tau_{ij})(1 + \lambda\tau_{ji}) = 1 - \lambda a_{ij} \in \mathbb{R}$$

and

$$(1 + \lambda\tau_{ij})(1 + \lambda\tau_{jk}) = (1 + \lambda\tau_{il})(1 + \lambda\tau_{lk})$$

on every elementary quadrilateral $(ijkl)$ so that

$$(2.12)\qquad (ij) \mapsto 1 + \lambda\tau_{ij}, \quad 1 + \lambda\tau_{ij} : \{j\} \times S^3 \to \{i\} \times S^3,$$

defines a flat $M\ddot{o}b(3)$-connection[11] on $M \times S^3$, as long as

$$1 - \lambda a_{ij} \neq 0 \quad \Leftrightarrow \quad 1 + \lambda\tau_{ij} \in M\ddot{o}b(3)$$

for all edges (ij). Hence there is a gauge transformation

$$(2.13)\qquad T^\lambda : M \to M\ddot{o}b(3), \quad T_j^\lambda = T_i^\lambda(1 + \lambda\tau_{ij}),$$

which identifies the $(1 + \lambda\tau)$-connection on $M \times S^3$ with the trivial connection.

The T^λ play a key role in the transformation theory of (discrete) isothermic nets; in particular, it turns out that every $T^\lambda f$ defines[12] a discrete isothermic net: $T^\lambda : M \to M\ddot{o}b(3)$ are the *Calapso transformations* of f and the discrete isothermic nets $T^\lambda f$ are its *Calapso transforms*, see [19, §5.7.16].

---

[10]This is analogous to the Vahlen matrix approach to Möbius geometry (see [19, Sect. 7.1]) using the Clifford algebra of $\mathbb{R}^{4,1}$.

[11]We shall make the notion of a flat (discrete) connection precise in the following section.

[12]Here $M\ddot{o}b(3)$ acts on $S^3 \cong \operatorname{Im}\mathbb{H} \cup \{\infty\}$ by Möbius transformations, i.e., by fractional linear transformations.

The connections $(ij) \mapsto 1 + \lambda\tau_{ij}$ lift to flat $O(4,1)$-connections $(ij) \mapsto \Gamma^\lambda_{ij}$ on the (discrete) vector bundle $M \times \mathbb{R}^{4,1}$ via (2.10) to give

$$X \mapsto \Gamma^\lambda_{ij} \cdot X := \frac{1}{1-\lambda a_{ij}}(1+\lambda\tau_{ij})\, X\, (1+\lambda\tau_{ji}) = \Gamma^{1-\lambda a_{ij}}_{f_i,f_j}(X). \tag{2.14}$$

2.3. **The vector bundle approach.** Clearly, the flat connections Γ^λ on $M \times \mathbb{R}^{4,1}$ in (2.14) can be defined without reference to the quaternionic approach. First we define our setup:

Definition 2.3. *A* connection *on a (discrete) fibre bundle $F \to M$, where the base M is a rectangular grid as before, is a map that assigns to each directed edge (ij) in M an isomorphism*

$$\Gamma_{ij} : F_j \to F_i \quad \textit{so that} \quad \Gamma_{ij}\Gamma_{ji} = 1;$$

it will be said to be a flat connection *if its holonomies around all elementary quadilaterals $(ijkl)$ are trivial,*

$$\Gamma_{ij}\Gamma_{jk}\Gamma_{kl}\Gamma_{li} = 1.$$

With these notions we can now formulate the key definition:

Definition 2.4. *Let $f : M \to S^3$ be a discrete isothermic net with cross ratio factorizing function a. We say that*

$$(\lambda, ij) \mapsto \Gamma^\lambda_{ij} := \Gamma^{1-\lambda a_{ij}}_{f_i,f_j} \in Hom(\{j\} \times \mathbb{R}^{4,1}, \{i\} \times \mathbb{R}^{4,1}) \tag{2.15}$$

defines the isothermic family of connections[13] *of f, where $\lambda \in \mathbb{R}$ so that $1 \neq \lambda a_{ij}$ for all edges (ij).*

Note that the Γ^λ are *metric* connections on $M \times \mathbb{R}^{4,1}$, as all Γ^λ_{ij} are isometries and hence descend to connections on $M \times S^3$.

We already know that, if f is an isothermic net, then the isothermic family of connections (2.15) is flat, see [19, §5.7.5]. Here we shall give an independent proof, not relying on the quaternionic setup, as well as a certain converse of this fact (cf. [18, Thm. 3.14]):

Lemma 2.5. *Let $f : M \to S^3$ be a regular discrete net, i.e., any three vertices of a face are in general position, and let a be a function on the edges. Then the connection given by* (2.15) *is flat if and only if f is isothermic with cross ratio factorizing function a.*

Proof. First note that $\Gamma^{1-\lambda a_{ij}}_{f_i,f_j}(X) = X \bmod f_i$ when $X \perp f_i$, so that $\Gamma^{1-\lambda a_{ij}}_{f_i,f_j}$ projects to the identity on $f_i^\perp/f_i$ and similarly for f_j. Consequently,

$$\Gamma^{1-\lambda a_{ij}}_{f_i,f_j}\Gamma^{1-\lambda a_{jk}}_{f_j,f_k}(X) = X \bmod f_j \quad \text{if} \quad X \perp f_j.$$

The same is true for the product $\Gamma^{1-\lambda a_{il}}_{f_i,f_l}\Gamma^{1-\lambda a_{lk}}_{f_l,f_k}$ so that, if we now assume flatness of the connection,

$$\Gamma^{1-\lambda a_{ij}}_{f_i,f_j}\Gamma^{1-\lambda a_{jk}}_{f_j,f_k} = \Gamma^{1-\lambda a_{il}}_{f_i,f_l}\Gamma^{1-\lambda a_{lk}}_{f_l,f_k} =: \Gamma^\lambda,$$

[13]We have $\Gamma^\lambda_{ij}\Gamma^\lambda_{ji} = 1$ on all edges (ij), so that the Γ^λ qualify as discrete (linear) connections.

we learn that $\Gamma^\lambda = id$ on $(f_j \oplus f_l)^\perp$ since $f_j \neq f_l$. Moreover, f_j and f_l are eigendirections of Γ^λ and

$$\Gamma^\lambda(X) = \begin{cases} \frac{1-\lambda a_{jk}}{1-\lambda a_{ij}} X & \text{if } X \in f_j, \\ X & \text{if } X \perp f_j, f_l, \\ \frac{1-\lambda a_{lk}}{1-\lambda a_{il}} X & \text{if } X \in f_l. \end{cases}$$

As Γ^λ is, along with $\Gamma^{1-\lambda a_{ij}}_{f_i,f_j}$ and $\Gamma^{1-\lambda a_{jk}}_{f_j,f_k}$, an orientation preserving orthogonal transformation we infer that, for all λ,

$$\frac{1-\lambda a_{jk}}{1-\lambda a_{ij}} \frac{1-\lambda a_{lk}}{1-\lambda a_{il}} = 1 \quad \text{and} \quad \Gamma^\lambda = \Gamma^{1/q(\lambda)}_{f_j,f_l}, \quad q(\lambda) := \frac{1-\lambda a_{ij}}{1-\lambda a_{jk}}.$$

Hence $a_{ij} = a_{jk}$ and $a_{il} = a_{lk}$ (in which case $\Gamma^\lambda = id$ for all λ) or $a_{ij} = a_{lk}$ and $a_{jk} = a_{il}$.

Now decompose $F_k \in f_k \setminus \{0\}$ as $F_k = F_i + F_j + F_k^\perp \in f_i \oplus f_j \oplus (f_i \oplus f_j)^\perp$ and observe that

$$\Gamma^\lambda(F_k) = \frac{1}{1-\lambda a_{jk}} \Gamma^{1-\lambda a_{ij}}_{f_i,f_j}(F_k) = \frac{1-\lambda a_{ij}}{1-\lambda a_{jk}} F_i + \frac{1}{1-\lambda a_{jk}} F_k^\perp + \frac{1}{(1-\lambda a_{ij})(1-\lambda a_{jk})} F_j$$

$$\longrightarrow \frac{a_{ij}}{a_{jk}} F_i$$

as $\lambda \to \infty$. This shows that, since $f_k \neq f_i$, we cannot have $a_{ij} = a_{jk}$; hence the function a has equal values on opposite edges of an elementary quadrilateral. Moreover, we learn that

$$f_k = \Gamma^{q(\infty)}_{f_j,f_l}(f_i),$$

showing that the four vertices f_i, f_j, f_k and f_l are concircular and the edge function a factorizes their cross ratio,

$$[f_i; f_j; f_k; f_l] = q(\infty) = \frac{a_{ij}}{a_{jk}}.$$

Hence f is a discrete isothermic net.

Conversely, suppose that f is discrete isothermic with cross ratio factorizing function a; we wish to show that

$$\Gamma^\lambda_{ij}\Gamma^\lambda_{jk} = \Gamma^{1/q(\lambda)}_{f_j,f_l} = \Gamma^{q(\lambda)}_{f_l,f_j} = \Gamma^\lambda_{il}\Gamma^\lambda_{lk},$$

where $q(\lambda) = \frac{1-\lambda a_{ij}}{1-\lambda a_{jk}}$. As the second equation holds and the third is obtained from the first by exchanging the roles of j and l it suffices to prove the first of these equations. Also,

$$\Gamma^{q(\lambda)}_{f_j,f_l}\Gamma^\lambda_{ij}\Gamma^\lambda_{jk}(X) = \begin{cases} X & \text{if } X \in f_j \\ X \bmod f_j & \text{if } X \perp f_j \end{cases}$$

so that flatness of the family of isothermic connections of f follows as soon as $\Gamma^{q(\lambda)}_{f_j,f_l}\Gamma^\lambda_{ij}\Gamma^\lambda_{jk}$ has another isotropic eigendirection — we shall show that f_k serves this purpose: consider

$$\mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\} \ni \lambda \mapsto \Gamma^\lambda_{ij}\Gamma^\lambda_{jk}(f_k) = \Gamma^\lambda_{ij}(f_k), \Gamma^{1/q(\lambda)}_{f_j,f_l}(f_k) \in S^3.$$

Both maps parametrize the same circle in terms of a certain cross ratio, given as a linear fractional transformation of λ; in particular,

$$
\begin{aligned}
\lambda = 0 \quad &\Rightarrow \quad \Gamma^{\lambda}_{ij}(f_k) = \Gamma^{1}_{f_i,f_j}(f_k) = f_k = \Gamma^{1}_{f_j,f_l}(f_k) = \Gamma^{1/q(\lambda)}_{f_j,f_l}(f_k),\\
\lambda = \infty \quad &\Rightarrow \quad \Gamma^{\lambda}_{ij}(f_k) = \Gamma^{\infty}_{f_i,f_j}(f_k) = f_i \,= \Gamma^{\frac{a_{jk}}{a_{ij}}}_{f_j,f_l}(f_k) = \Gamma^{1/q(\lambda)}_{f_j,f_l}(f_k),\\
\lambda = \tfrac{1}{a_{ij}} \quad &\Rightarrow \quad \Gamma^{\lambda}_{ij}(f_k) = \Gamma^{0}_{f_i,f_j}(f_k) = f_j \,= \Gamma^{\infty}_{f_j,f_l}(f_k) = \Gamma^{1/q(\lambda)}_{f_j,f_l}(f_k).
\end{aligned}
$$

As two Möbius transformations of a circle coincide as soon as they coincide at three points we conclude that $\Gamma^{\lambda}_{ij}(f_k) = \Gamma^{1/q(\lambda)}_{f_j,f_l}(f_k)$ for all λ. □

Note that, freeing the second part of the proof from the specific notations of the situation, we have proved the following:

Lemma 2.6. *Write $[p_1; p_2; p_3; p_4] = \frac{a}{b}$ with $a, b \in \mathbb{R}$ for the cross ratio of four concircular points $p_i \in S^3$, $i = 1, \ldots, 4$; then, for all $\lambda \in \mathbb{R}$,*

$$\Gamma^{1-a\lambda}_{p_1,p_2}\Gamma^{1-b\lambda}_{p_2,p_3} = \Gamma^{(1-b\lambda)/(1-a\lambda)}_{p_2,p_4} = \Gamma^{(1-a\lambda)/(1-b\lambda)}_{p_4,p_2} = \Gamma^{1-b\lambda}_{p_1,p_4}\Gamma^{1-a\lambda}_{p_4,p_3}.$$

Thus, for a discrete isothermic net f, there are gauge transformations $T^{\lambda} : M \to O(4,1)$ identifying the connections Γ^{λ} on $M \times \mathbb{R}^{4,1}$ with the trivial connection:

Lemma 2.7. *(Lemma and definition) Let $f : M \to S^3$ be a discrete isothermic net with its isothermic family of connections Γ^{λ}. Then the gauge transformations*

$$T^{\lambda} : M \to O(4,1) \quad \text{with} \quad T^{\lambda}_j = T^{\lambda}_i \Gamma^{\lambda}_{ij}$$

are the Calapso transformations *of f; the isothermic nets $f^{\lambda} := T^{\lambda} f$ are its* Calapso transforms.

Note that $a^{\mu} = \frac{a}{1-\mu a}$ is a cross ratio factorizing function for the Calapso transform f^{μ} of f with cross ratio factorizing function a, see [19, §5.7.16]; the isothermic family of connections of f^{μ} is given by

$$\Gamma^{\mu,\lambda}_{ij} = T^{\mu}_i \Gamma^{\mu+\lambda}_{ij} (T^{\mu}_j)^{-1}, \tag{2.16}$$

which shows that the Calapso transformations of a discrete isothermic net satisfy a 1-parameter group property, see [18] or [19, §5.7.30]:

$$T^{\mu,\lambda} T^{\mu} = T^{\mu+\lambda}. \tag{2.17}$$

3. Polynomial conserved quantities

The second key notion in our definition of discrete cmc nets in space forms will be that of polynomial conserved quantities:

Definition 3.1. *Let $f : M \to S^3$ be an isothermic net. A* polynomial conserved quantity *of f is a map*

$$\mathbb{R} \times M \ni (\lambda, i) \mapsto P_i(\lambda) = \sum_{k=0}^{N} P_i^{(k)} \lambda^k \in \mathbb{R}^{4,1}[\lambda]$$

so that, for every fixed λ,

$$T^{\lambda} P(\lambda) \equiv const.$$

Hence, a polynomial conserved quantity of an isothermic net can be thought of as a polynomial family of parallel sections of the vector bundle $M \times \mathbb{R}^{4,1}$ equipped with the isothermic family of connections:

$$T^\lambda P(\lambda) \equiv const. \quad \Leftrightarrow \quad P_i(\lambda) = \Gamma^\lambda_{ij} P_j(\lambda) \tag{3.1}$$

on all edges (ij) of M.

3.1. **Basic properties.** Clearly, as T^λ acts linearly on $\mathbb{R}^{4,1}$, the polynomial conserved quantities of a given discrete isothermic net f can be superposed:

Lemma 3.2. *The space of polynomial conserved quantities of f is a vector space.*

As a consequence, we can construct new polynomial conserved quantities from a given one by multiplying with real polynomials $p(\lambda)$, thereby raising the degree; for example, if P is a polynomial conserved quantity of f then $(1+\lambda)P(\lambda)$ will be a new polynomial conserved quantity of higher degree. Thus we will be interested in (non-vanishing) polynomial conserved quantities of lowest possible degree. The following lemma provides a criterion:

Lemma 3.3. *Let $P(\mu) = 0$ for a polynomial conserved quantity $P(\lambda) : M \to \mathbb{R}^{4,1}[\lambda]$ of f; then*

$$\tilde{P}(\lambda) := \frac{1}{\lambda - \mu} P(\lambda)$$

is a polynomial conserved quantity of f of lower degree.

Note that, if $P_i(\mu) = 0$ for some $i \in M$ then $P(\mu) \equiv 0$ on M since $T^\mu P(\mu) \equiv const.$

Proof. Writing $P_i(\lambda) \in \mathbb{R}^{4,1}[\lambda]$ in terms of a basis of $\mathbb{R}^{4,1}$ shows that μ is a common zero for all (real) component polynomials, which are therefore divisible by $(\lambda - \mu)$. Hence $\tilde{P}_i(\lambda)$ is polynomial at any $i \in M$.

Clearly

$$T^\lambda \tilde{P}(\lambda) = \frac{1}{\lambda - \mu} T^\lambda P(\lambda) \equiv const$$

for any fixed λ, showing that $\tilde{P}(\lambda)$ is a polynomial conserved quantity of f. □

As a direct consequence of the previous two lemmas we learn that, if two distinct polynomial conserved quantities $P(\lambda)$ and $\tilde{P}(\lambda)$ of degree $N \in \mathbb{N}$ of an isothermic net have the same value at some point $(\mu, i) \in \mathbb{R} \times M$, then there is a polynomial conserved quantity of degree $\leq N - 1$. In particular:

Corollary 3.4. *A non-zero polynomial conserved quantity of lowest possible degree N,*

$$P(\lambda) = \lambda^N Z + \cdots + \lambda^0 Q : M \to \mathbb{R}^{4,1}[\lambda],$$

is uniquely determined by either its top or bottom coefficient Z or Q, respectively.

Since the T^λ are orthogonal transformations, there is another obvious property of a polynomial conserved quantity which will become important later:

Lemma 3.5. *If $P(\lambda)$ is a polynomial conserved quantity of f, then $|P(\lambda)|^2$ depends only on λ; in particular, $|Z|^2$ and $|Q|^2$ are constants. If $P(\lambda) = \lambda Z + Q$ is a linear conserved quantity, then also $\langle Z, Q\rangle \equiv const.$*

Finally note that the equation (3.1) for a polynomial conserved quantity depends crucially on the choice of a cross ratio factorizing function a: however, if $\tilde{a} := \alpha a$ is a new cross ratio factorizing function, then

$$\tilde{\Gamma}^\lambda = \Gamma^{\alpha\lambda}$$

by (2.15); hence $\tilde{P}(\lambda) = P(\alpha\lambda)$ is a new polynomial conserved quantity satisfying (3.1) with the new isothermic family of connections. Consequently:

Lemma 3.6. *If $P(\lambda)$ is a polynomial conserved quantity of f with respect to a as a cross ratio factorizing function then*

$$\tilde{P}(\lambda) := P(\alpha\lambda)$$

is a polynomial conserved quantity of f with respect to $\tilde{a} := \alpha a$ as a new cross ratio factorizing function.

3.2. **Geometric properties.** We now turn to a more detailed analysis of (3.1) and its geometric consequences: using (2.4) the fact that a polynomial conserved quantity $P(\lambda)$ is a family of parallel sections of the isothermic family of connections (2.15) of f reads

$$P_i(\lambda) \;=\; \Gamma^\lambda_{ij} P_j(\lambda) \;=\; P_j(\lambda) + \tfrac{\lambda a_{ij}}{\langle F_i, F_j\rangle}\{\tfrac{1}{1-\lambda a_{ij}}\langle P_j(\lambda), F_i\rangle F_j - \langle P_j(\lambda), F_j\rangle F_i\}$$

on any edge (ij) of M; exchanging the roles of the endpoints i and j of the edge we obtain a similar equation which, as $f_i \neq f_j$, yields two equations

$$\begin{aligned} dP_{ij}(\lambda) &= \tfrac{\lambda a_{ij}}{\langle F_i, F_j\rangle}\{\langle P_j(\lambda), F_j\rangle F_i - \langle P_i(\lambda), F_i\rangle F_j\} \\ &= \tfrac{\lambda a_{ij}}{1-\lambda a_{ij}} \tfrac{1}{\langle F_i, F_j\rangle}\{\langle P_i(\lambda), F_j\rangle F_i - \langle P_j(\lambda), F_i\rangle F_j\} \end{aligned}$$

for some light cone lift F of f. Note that the second equality follows from the first by taking scalar products with F_i and F_j, respectively. Hence, we also obtain the converse:

Lemma 3.7. *$P(\lambda)$ is a polynomial conserved quantity of f if and only if, for all edges (ij) in M,*

$$dP_{ij}(\lambda) = \frac{\lambda a_{ij}}{\langle F_i, F_j\rangle}\{\langle P_j(\lambda), F_j\rangle F_i - \langle P_i(\lambda), F_i\rangle F_j\}. \tag{3.2}$$

Note that, in case F is a Moutard lift of f satisfying (2.5), then (3.2) simplifies to

$$dP_{ij}(\lambda) = \lambda\,\{p_j(\lambda)F_i - p_i(\lambda)F_j\}, \quad \text{where} \quad p(\lambda) := \langle P(\lambda), F\rangle.$$

The integrability $d^2P(\lambda) = 0$ of this equation then yields

$$(p_k(\lambda) - p_i(\lambda))(F_j - F_l) = (p_j(\lambda) - p_l(\lambda))(F_k - F_i),$$

and hence $p(\lambda)$ satisfies the very same Moutard equation (2.6) as F does.

Now the key observation from (3.2) is that this equates a polynomial of degree N and a polynomial of degree $N+1$ with vanishing constant coefficient. Hence, looking at the degree 0 and degree N and $N+1$ terms we obtain the following two corollaries:

Corollary 3.8. *If $P(\lambda) = \lambda^N Z + \cdots + Q$ is a polynomial conserved quantity of f, then $Q \equiv const.$*

Thus a polynomial conserved quantity naturally provides an ambient quadric $\mathcal{Q}$ of constant curvature $\kappa = -|Q|^2$ for the isothermic net, see [19, Sect. 1.4]:

$$\mathcal{Q} = \{Y \in L^4 \,|\, \langle Y, Q\rangle = -1\}. \tag{3.3}$$

Corollary 3.9. *If $P(\lambda) = \lambda^N Z + \lambda^{N-1} Y + \cdots + Q$ is a polynomial conserved quantity of f then:*
(i) $Z_i \perp f_i$ at all points i in M;
(ii) $Z_i + a_{ij}\frac{\langle Y_j, F_j\rangle}{\langle F_i, F_j\rangle} F_i = Z_j + a_{ij}\frac{\langle Y_i, F_i\rangle}{\langle F_i, F_j\rangle} F_j$ for all edges (ij) in M;
(iii) $|Z|^2 \geq 0$ and $|Z|^2 = 0$ if and only if $Z \in f$.

Here, (iii) follows from (i) since $f^\perp$ carries a positive semi-definite metric with only f as a null direction. Also note that, since $|Z|^2$ is constant, either $Z \parallel F$ at all points or, without loss of generality, $|Z|^2 \equiv 1$ as the space of polynomial conserved quantities is linear.

We will be mostly interested in the latter case. To interpret this situation geometrically first note that, if $|Z|^2 \equiv 1$, then $i \mapsto Z_i$ defines a discrete sphere congruence so that every sphere Z_i contains the point f_i, by (i); moreover, the sphere

$$S_{ij} := Z_i + a_{ij}\frac{\langle Y_j, F_j\rangle}{\langle F_i, F_j\rangle} F_i = Z_j + a_{ij}\frac{\langle Y_i, F_i\rangle}{\langle F_i, F_j\rangle} F_j \tag{3.4}$$

belongs to both contact elements[14] defined by Z at the endpoints of an edge so that

$$M \ni i \mapsto Z_i + f_i := \{Z_i + \alpha F_i \,|\, \alpha \in \mathbb{R}\}$$

defines a discrete principal net in Lie geometry, see [6, Sect. 4.1], with curvature spheres S_{ij}. On the other hand, the existence of the curvature spheres S_{ij}, which touch both spheres Z_i and Z_j, can be interpreted as a discrete version of the enveloping condition for a sphere congruence defined at the vertices of a discrete net. Thinking of f as a net in $\mathbb{R}^3$, the spheres Z define a unit normal field n at the vertices of f, which satisfies the trapezoid property of [22, Sect. 3].

We summarize these observations in the following[15]

Corollary 3.10. *(Corollary and definition) If $P(\lambda) = \lambda^N Z + \cdots + Q$ is a polynomial conserved quantity of f with $|Z|^2 = 1$, then f envelops the discrete sphere congruence Z: we say that $f : M \to S^3$* envelops *a discrete sphere congruence $S : M \to S^{3,1}$ if*

- *(i) $S_i \in f_i^\perp$ for all $i \in M$ (incidence) and*
- *(ii) $S_j = S_i \bmod f_i \oplus f_j$ for each edge (ij) of M (touching);*

the common sphere S_{ij} of the two contact elements $S_i + f_i$ and $S_j + f_j$ given by S_i and S_j at the endpoints of an edge will be called a curvature sphere.

Note that the condition for a sphere congruence to be enveloped by a net is a condition on the congruence of contact elements defined by the sphere congruence — hence, if f envelops S and F is any light cone lift of f, then any sphere congruence $S + hF$, $h : M \to \mathbb{R}$, is also enveloped by f.

[14]Recall that, in contrast to Möbius geometry, Lie geometry considers oriented spheres; thus $\pm Z + f$ will define two contact elements which differ by the orientation of their spheres.

[15]We insist on a consistent orientation of the spheres of an enveloped sphere congruence.

Before focusing on this geometrically interesting configuration we shall, for the rest of this section, investigate the degenerate case where we can, without loss of generality, take $Z = F$ as a canonical light cone lift[16] of f.

Lemma 3.11. *If $P(\lambda) = \lambda^N F + \cdots + Q$ is a polynomial conserved quantity of f, then F is a Moutard lift of f.*

Proof. From (ii) of Corollary 3.9 we learn that

$$\langle F_i, F_j\rangle + a_{ij}\langle Y_j, F_j\rangle = \langle F_i, F_j\rangle + a_{ij}\langle Y_i, F_i\rangle = 0$$

since F_i and F_j are linearly independent; thus[17] $\langle Y, F\rangle \equiv const =: c$ and F satisfies (2.5) with the cross ratio factorizing function $\tilde{a} := -c\,a$. □

As an example we seek an isothermic net with a degenerate degree 1 polynomial conserved quantity: this is the lowest degree possible since the top term of a degree 0 polynomial conserved quantity is constant and can therefore not be a (Moutard) lift of an isothermic net.

Now consider, as before,

$$|(x_0, \ldots, x_4)|^2 = -x_0^2 + \sum_{i=1}^{4} x_i^2$$

as the quadratic form of the Minkowski scalar product of $\mathbb{R}^{4,1}$ and the isothermic net

$$\{-1,0,1\}^2 \ni (m,n) \mapsto f_{(m,n)} := (\eta m, \frac{1+\alpha}{2} + (-1)^n \frac{1-\alpha}{2}, \beta n) \in \mathbb{R}^3, \tag{3.5}$$

where $\alpha \in (0,1)$ and $\beta, \eta > 0$, and let

$$Z_{(m,n)} := (-1)^n F_{(m,n)} \quad \text{and} \quad Q := -\frac{4}{1-\alpha}(\frac{1+\alpha}{2}, 0, 1, 0, -\frac{1+\alpha}{2}), \tag{3.6}$$

where $F = (\frac{1+|f|^2}{2}, f, \frac{1-|f|^2}{2})$ is a Euclidean lift[18]. Note that $|Q|^2 > 0$ so that Q describes a sphere[19] and f appears to be a perturbation of a net on that sphere.

Since

$$\langle F_i, F_j\rangle = -\frac{1}{2}|f_j - f_i|^2$$

and f has rectangular faces, so that the cross ratio (2.3) on an elementary quadrilateral becomes

$$[f_i; f_j; f_k; f_l] = -\frac{|f_i - f_j|^2}{|f_i - f_l|^2},$$

---

[16] We neglect the case where Z may have zeroes, i.e., where the degree of $P(\lambda)$ may not be constant.

[17] This we already knew from Lemma 3.5, because $\langle Y, F\rangle$ is the λ^{2N-1}-coefficient of $|P(\lambda)|^2$.

[18] Cf. (2.7) and (2.9): this is the Euclidean lift with respect to $Q_0 = (1,0,0,0,-1)$, defining a flat quadric of constant curvature via (3.3).

[19] Suppose that $|Q|^2 \leq 0$ for a degenerate linear conserved quantity $P(\lambda) = \lambda F + Q$ of an isothermic net f; then

$$a_{ij} = \langle F_i, F_j\rangle = -\tfrac{1}{2}|dF_{ij}|^2 < 0$$

for any edge (ij) since $dF_{ij} \perp Q$ by Lemma 3.5 and f is regular. Hence the cross ratio of any face becomes positive; thus, if we seek an isothermic net with embedded faces, then Q necessarily describes a sphere.

$a_{ij} := \frac{1}{2}\langle Z_i, Z_j\rangle$ provides a cross ratio factorizing function, i.e., Z is a Moutard lift of f. Moreover

$$\langle \lambda Z + Q, Z\rangle = \langle Q, Z\rangle \equiv -2,$$

so that

$$d(\lambda Z + Q)_{ij} - \frac{\lambda a_{ij}}{\langle Z_i, Z_j\rangle}\{\langle \lambda Z + Q, Z\rangle_j Z_i - \langle \lambda Z + Q, Z\rangle_i Z_j\} = 0$$

and $P(\lambda) := \lambda Z + Q$ is a linear conserved quantity for f by Lemma 3.7.

Note that $Z = P(\infty)$ is a lift of f and that

$$P(\frac{4}{(1-\alpha)^2}) = \frac{4}{(1-\alpha)^2}\{Z - 2\frac{\langle Z, Q\rangle}{|Q|^2}Q\}$$

is a lift of a Möbius equivalent net or, more precisely, of the "antipodal" net in the quadric of constant curvature given by Q, see [19, Section 1.4]. We shall come back to this observation later.

Finally observe that our restriction of the domain to $\{-1, 0, 1\}^2$ was not necessary: f, as given by (3.5), can be extended to all of $\mathbb{Z}^2$ while (3.6) keeps defining a linear conserved quantity for f. However, this restriction will be convenient later when we shall recycle this example to demonstrate another aspect of our theory.

3.3. **Special isothermic nets.** As in the smooth case, see [12, Sect. 2.2], we can now use the existence of a polynomial conserved quantity to define a special class of discrete isothermic nets, ordered by the (minimal) degree of an associated polynomial conserved quantity. However, in contrast to the smooth case, where a polynomial conserved quantity is essentially unique because its top degree coefficient encodes the conformal Gauss map of the underlying isothermic surface, the space of polynomial conserved quantities of minimal degree may be higher dimensional[20] and may contain elements with null top degree coefficient, as in the above example, even though we require the existence of a normalized polynomial conserved quantity of minimal degree:

Definition 3.12. *A polynomial conserved quantity* $P(\lambda) = \lambda^N Z + \cdots + Q$ *of an isothermic net* f *will be called* normalized *if* $|Z|^2 \equiv 1$*; we say that* f *is a* special isothermic net of type N *if it has a normalized polynomial conserved quantity of degree* N*, but not of any lower degree.*

As a direct consequence of this definition and the 1-parameter group property (2.17) of the Calapso transformations we obtain stability of the class of special isothermic nets of a fixed type N under the Calapso transformation:

Theorem 3.13. *If* f *is special isothermic of type* N *then so are its Calapso transforms* $f^\mu = T^\mu f$.

Proof. $T^{\mu,\lambda} = T^{\mu+\lambda}(T^\mu)^{-1}$ are the Calapso transformations of f^μ by (2.17), see also [19, §5.7.30]. Hence, if $P(\lambda) = \lambda^N Z + \cdots + Q$ is a polynomial conserved quantity of f, then

$$P^\mu(\lambda) := T^\mu P(\mu + \lambda)$$

[20] However, we do expect uniqueness of a (normalized) polynomial conserved quantity of minimal degree for generic isothermic nets of a sufficient size.

defines a polynomial conserved quantity of f^μ of the same degree; moreover,

$$|P^\mu(\lambda)|^2 = |P(\lambda+\mu)|^2 = \lambda^{2N}|Z|^2 + \lambda^{2N-1}\cdots + \dots$$

showing that $P^\mu(\lambda)$ is normalized as soon as $P(\lambda)$ is. Thus f^μ is special isothermic of type $\leq N$.

On the other hand $f = T^{\mu,-\mu} f^\mu$ is a Calapso transform of f^μ, by (2.17) again, so that the same argument shows that f^μ is special isothermic of type $\geq N$. □

So far it is rather unclear how restrictive the definition of special isothermic of a given type is: as we shall see later (and as one might expect after an equation count), the condition on a discrete isothermic net of being special isothermic of type N is not a local condition — in particular, we will see that every isothermic 3×3-net has plenty of linear conserved quantities and is, therefore, special isothermic of type 1. On the other hand, the condition of being special isothermic of type 0 does already impose a condition on a 3×3-grid, and the corresponding constant conserved quantity is generically unique:

Theorem 3.14. *f is special isothermic of type 0 if and only if f takes values in a 2-sphere.*

Proof. First suppose that f is special isothermic of type 0 with (necessarily[21] normalized) constant conserved quantity $P(\lambda) = Q$. Then Q defines a fixed 2-sphere, since $|Q|^2 = 1$, that contains the points of the net by (i) of Corollary 3.9. This also shows that a (normalized) constant conserved quantity is unique as soon as the isothermic net does not take values in a circle.

Conversely, if f takes values in a fixed 2-sphere $S \subset S^3$ then the Calapso transformations T^λ of f take (up to a constant Möbius transformation) values in the Möbius group $M\ddot{o}b(S)$ of this 2-sphere (see [18, Thm. 3.14] or [19, §5.7.22]). That is, the Calapso transformations T^λ of f can be chosen to fix a unit vector Q defining the 2-sphere S: this yields a normalized constant conserved quantity so that f is special of type 0. □

As a more involved example we take up surfaces of revolution and investigate the symmetry of a corresponding polynomial conserved quantity: suppose $F_{(m,n)} = (-1)^m(M_m + \Phi_n C)$ is the Moutard lift of a discrete surface of revolution, see (2.8), with a rotationally symmetric polynomial conserved quantity, i.e., we assume that $\Phi_n^{-1}P_{(m,n)}(\lambda) =: \hat{P}_m(\lambda)$ does not depend on n. Then, clearly,

$$p_{(m,n)}(\lambda) := \langle P_{(m,n)}(\lambda), F_{(m,n)}\rangle = (-1)^m\langle \hat{P}_m(\lambda), M_m + C\rangle$$

is independent of n; we shall see that the converse also holds:

Lemma 3.15. *A polynomial conserved quantity $P(\lambda)$ of a discrete net f of revolution with canonical lift $F_{(m,n)} = (-1)^m\Phi_n(M_m + C)$ is rotationally symmetric, $P_{(m,n)}(\lambda) = \Phi_n\hat{P}_m(\lambda)$, if and only if*

$$p(\lambda) := \langle P(\lambda), F\rangle$$

does not depend on n.

[21]Remember that a constant conserved quantity cannot be degenerate by (i) of Corollary 3.9, since f is not constant.

Proof. We already know that $p(\lambda)$ depends only on m if $P(\lambda)$ is rotationally symmetric; to prove the converse we assume that $p(\lambda)$ does not depend on n and write

$$P_{(m,n)} = P^{\perp}_{(m,n)} + \Phi_n(\alpha_{(m,n)}C + \beta_{(m,n)}C^{\perp}) \in \mathbb{R}^{2,1} \oplus \mathbb{R}^2, \tag{3.7}$$

where $C^{\perp}$ complements C to form an orthonormal basis of $\mathbb{R}^2$ and α and β are suitable real valued functions. First note that

$$(-1)^m p_m = \langle P^{\perp}_{(m,n)}, M_m\rangle + \alpha_{(m,n)}.$$

Now we fix m and, in order to simplify notation, consider all functions as functions of n only. From (3.2) we get

$$0 = dP_{n,n+1}(\lambda) + \lambda\, p(\lambda)\, dF_{n,n+1};$$

hence, in particular,

$$dP^{\perp}_{n,n+1} = 0 \quad \text{and} \quad d\alpha_{n,n+1} = -\langle dP^{\perp}_{n,n+1}, M\rangle = 0.$$

For a function g on the vertices let $g_{n,n+1} := \frac{g_n+g_{n+1}}{2}$ denote the associated function on the edges, and let $d\varphi_{n,n+1}$ be the rotation angle along the edge, i.e., the rotation angle of $\Phi_{n+1}\Phi_n^{-1}$; observe that

$$\begin{array}{rcll} d(\Phi C^{\perp})_{n,n+1} & = & -2\tan\frac{d\varphi_{n,n+1}}{2}\,(\Phi C)_{n,n+1} & \perp\ dF_{n,n+1}, \\ 2(\Phi C^{\perp})_{n,n+1} & = & \cot\frac{d\varphi_{n,n+1}}{2}\, d(\Phi C)_{n,n+1} & \|\ dF_{n,n+1}. \end{array}$$

Hence the $\mathbb{R}^2$-part of $0 = dP_{n,n+1}(\lambda) + \lambda\, p(\lambda)\, dF_{n,n+1}$ yields

$$0 = \beta_{n,n+1}(\lambda) \quad \text{and} \quad 0 = \alpha(\lambda) + (-1)^m\lambda p(\lambda) + \frac{1}{2}d\beta_{n,n+1}(\lambda)\cot\frac{d\varphi_{n,n+1}}{2}.$$

Thus, considering two consecutive edges, $0 = \beta_n \sin\frac{d\varphi_{n-1,n}+d\varphi_{n,n+1}}{2}$ so that β vanishes and

$$P_{(m,n)}(\lambda) = P^{\perp}_m(\lambda) + (-1)^{m+1}\lambda p_m(\lambda)\,\Phi_n C \tag{3.8}$$

is clearly rotationally symmetric. □

As a simple consequence we see that a linear conserved quantity of a discrete surface of revolution f is rotationally symmetric as soon as f is rotationally symmetric in the space form defined by the constant term:

Corollary 3.16. *A linear conserved quantity $P(\lambda) = \lambda Z + Q$ of a net f of revolution is rotationally symmetric if and only if Q is.*

Proof. By (3.8), the constant term of a rotationally symmetric polynomial conserved quantity has no $\mathbb{R}^2$-component in the decomposition (3.7); for the converse observe that, in the case of a linear conserved quantity, $p_{(m,n)} = \langle Q, F_{(m,n)}\rangle$. □

4. The Bäcklund transformation

Bianchi's Bäcklund transformation of smooth constant mean curvature surfaces in Euclidean space turned out to be a special case of the Darboux transformation of isothermic surfaces: considering the Darboux transformation as an initial value problem depending on a real (spectral) parameter, the Darboux transforms of a constant mean curvature surface turn out to have constant mean curvature as soon as a certain relation between the initial value and the parameter is satisfied, see [19, §5.4.15]; in fact, they turn out to be the Bäcklund transforms of the surface, see [16, Thm. 7] and [21, Thm. 4.4]. A similar fact holds true for discrete constant mean curvature surfaces in the sense of [17].

In [12, Thm. 3.2], this condition for a Darboux transform of a constant mean curvature surface to have (the same) constant mean curvature again is shown to be a special case of a similar condition for (smooth) special isothermic surfaces of given type N. Here we shall analyze the situation for discrete special isothermic nets, giving rise to what we will call the "Bäcklund transformation" for special isothermic nets.

To this end we shall first recall the Darboux transformation for discrete isothermic nets, cf. [17] or [19, §§5.7.12 & 5.7.19]:

Definition 4.1. *Let $f : M \to S^3$ be a discrete isothermic net with its family of Calapso transformations T^λ. Then a* Darboux transform *of f is a discrete net*

$$\hat{f} : M \to S^3 \quad \text{so that} \quad \exists \mu \in \mathbb{R} : T^\mu \hat{f} \equiv const.$$

Equivalently, we can characterize a Darboux transform by the condition

$$\hat{f}_i = \Gamma^\mu_{ij} \hat{f}_j = \Gamma^{1-a_{ij}\mu}_{f_i,f_j} \hat{f}_j$$

for all edges (ij) of M: remember that the isothermic family of connections (2.15) descends to a family of connections on $M \times S^3$; thus a Darboux transform $\hat{f}$ can be thought of as a parallel section of a connection Γ^μ on $M \times S^3$ of the isothermic family of connections. Using a cross ratio identity (see [19, §4.9.11]), this condition can be reformulated as the cross ratio condition[22] (cf. (2.4))

$$[f_i; f_j; \hat{f}_j; \hat{f}_i] = 1 - [\hat{f}_j; f_i; \hat{f}_i; f_j] = a_{ij}\mu. \tag{4.1}$$

Clearly, a Darboux transform $\hat{f}$ of an isothermic net has a light cone lift $\hat{F} : M \to L^4$ so that

$$T^\mu \hat{F} \equiv const. \quad \Leftrightarrow \quad \hat{F}_i = \Gamma^\mu_{ij} \hat{F}_j \tag{4.2}$$

on all edges (ij) of M, that is, $\hat{F}$ is a Γ^μ-parallel lift of $\hat{f}$. Note that this is exactly the conserved quantity condition (3.1) for a light cone map $\hat{F}$ and a fixed value μ of the spectral parameter; we shall come back to this point later.

The crucial observation now is that the Darboux transformation produces discrete isothermic nets from isothermic nets, see [17] or [19, §5.7.12], where a proof relying on the hexahedron lemma [19, §4.9.13] was given; here we shall again give an alternative proof, relying on Lemma 2.5, that also provides us with a useful formula for the Calapso transformations of a Darboux transform, cf. [18] or [19, §5.7.35]:

Lemma 4.2. *A Darboux transform $\hat{f}$, $T^\mu \hat{f} \equiv const$, of a discrete isothermic net f with cross ratio factorizing function a is isothermic with the same cross ratio factorizing function $\hat{a} = a$ and with Calapso transformations*

$$\hat{T}^\lambda = T^\lambda \Gamma^{1-\lambda/\mu}_{f,\hat{f}}. \tag{4.3}$$

[22] Using quaternions or the Clifford algebra of $\mathbb{R}^3$ and thinking of $S^3 = \mathbb{R}^3 \cup \{\infty\}$, this is equivalent to the (discrete) Riccati type equation

$$d\hat{f}_{ij} = \mu\,(\hat{f} - f)_j\, df^*_{ij}\,(\hat{f} - f)_i,$$

where $df^*_{ij} = a_{ij}(df_{ij})^{-1}$ is the derivative of the Christoffel transform of f, cf. [19, §5.7.7]; the condition $\hat{f}_i = \Gamma^\mu_{ij}\hat{f}_j$ appears as the usual linearization from this Riccati equation: as "Darboux's linear system".

Proof. We shall show that the isothermic family of connections (2.15) of $\hat{f}$ is given by

$$\hat{\Gamma}^{\lambda}_{ij} := \Gamma^{1-\lambda a_{ij}}_{\hat{f}_i,\hat{f}_j} = (\Gamma^{1-\lambda/\mu}_{f,\hat{f}})^{-1}_i \Gamma^{\lambda}_{ij} (\Gamma^{1-\lambda/\mu}_{f,\hat{f}})_j,$$

which is then, with Γ^{λ}, clearly flat so that $\hat{f}$ is isothermic with cross ratio factorizing function $\hat{a} = a$ by Lemma 2.5; further, the formula for the Calapso transformations $\hat{T}^{\lambda}$ also follows directly.

Thus we have to show that

$$\Gamma^{1-\lambda/\mu}_{f_i,\hat{f}_i} \Gamma^{1-\lambda a_{ij}}_{\hat{f}_i,\hat{f}_j} = \Gamma^{1-\lambda a_{ij}}_{f_i,f_j} \Gamma^{1-\lambda/\mu}_{f_j,\hat{f}_j}.$$

But, with (4.1), this follows directly from Lemma 2.6. □

At this point we are now in perfect shape to discuss:

4.1. **Darboux transforms of special isothermic nets.** As outlined at the beginning of this section, we aim to obtain the Bäcklund transformations for special isothermic nets of type N as particular classes of Darboux transformations. The following theorem provides the essential criterion:

Theorem 4.3. *Let $\hat{f}$ be a Darboux transform, $T^{\mu}\hat{f} \equiv const$, of a special isothermic net f of type N with normalized polynomial conserved quantity $P(\lambda)$ of degree N. Then:*
(i) *$\hat{f}$ is special isothermic of type $\leq N+1$ and*
(ii) *$\hat{f}$ is special isothermic of type $\leq N$ as soon as $P(\mu) \in \hat{f}^{\perp}$.*

Note that, if $P_i(\mu) \in \hat{f}_i^{\perp}$ for some $i \in M$, then this holds true for all $i \in M$. Namely, if $\hat{F}$ is a light cone lift of $\hat{f}$ as in (4.2), then $P(\mu)$ and $\hat{F}$ are both parallel sections of the metric connection Γ^{μ}_{ij} on $M \times \mathbb{R}^{4,1}$ so that their scalar product $\langle P(\mu), \hat{F}\rangle \equiv const$.

Proof. We define

$$\hat{P}(\lambda) := (\lambda - \mu)\,\Gamma^{1-\lambda/\mu}_{\hat{f},f} P(\lambda). \tag{4.4}$$

Clearly $\hat{T}^{\lambda}\hat{P}(\lambda) \equiv const$ by (4.3) and, using (2.4), we obtain

$$\hat{P}(\lambda) = (\lambda-\mu)P(\lambda) - \frac{1}{\langle F,\hat{F}\rangle}\{\frac{\lambda(\lambda-\mu)}{\mu}\langle P(\lambda),F\rangle\hat{F} + \lambda\langle P(\lambda),\hat{F}\rangle F\},$$

which is polynomial of degree $\leq N+1$ as $\langle P(\lambda), F\rangle$ has degree $\leq N-1$ by (i) in Corollary 3.9. Finally, writing

$$P(\lambda) = \lambda^N Z + \cdots + Q \quad \text{and} \quad \hat{P}(\lambda) = \lambda^{N+1}\hat{Z} + \cdots + \hat{Q},$$

we find that

$$\lambda^{2N+2}|\hat{Z}|^2 + \cdots = |\hat{P}(\lambda)|^2 = (\lambda-\mu)^2\,|P(\lambda)|^2 = \lambda^{2N+2}|Z|^2 + \ldots$$

for all λ, so that $|\hat{Z}|^2 = |Z|^2$. Hence $\hat{P}(\lambda)$ is a normalized polynomial conserved quantity of degree $N+1$ and, therefore, $\hat{f}$ is special isothermic of type $\leq N+1$, proving (i).

To prove (ii) note that

$$\hat{P}(\mu) = -\mu\frac{\langle P(\mu),\hat{F}\rangle}{\langle F,\hat{F}\rangle}F = 0$$

when $P(\mu) \in \hat{f}^{\perp}$. Hence $\hat{f}$ has a polynomial conserved quantity of degree $\leq N$ by Lemma 3.3; in particular,

$$\hat{P}(\lambda) := \Gamma_{\hat{f},f}^{1-\lambda/\mu} P(\lambda) \tag{4.5}$$

provides a normalized polynomial conserved quantity of degree N for $\hat{f}$ showing that $\hat{f}$ is special isothermic of type $\leq N$. □

Definition 4.4. *A* Bäcklund transform $\hat{f}$ *of a special isothermic net* f *of type* N *with polynomial conserved quantity* $P(\lambda)$ *is a Darboux transform, that is,* $T^{\mu}\hat{f} \equiv const$*, so that* $P(\mu) \in \hat{f}^{\perp}$.

The Bäcklund transformation between special isothermic nets of the same type[23] is symmetric, that is, if $\hat{f}$ is a type N Bäcklund transform of a special isothermic net f of type N with polynomial conserved quantity $P(\lambda)$ then f is a Bäcklund transform of $\hat{f}$, where $\hat{P}(\lambda)$ is given by (4.5): firstly, f is a Darboux transform of $\hat{f}$ since the cross ratio condition (4.1) is symmetric in f and $\hat{f}$,

$$[\hat{f}_i; \hat{f}_j; f_j; f_i] = [f_i; f_j; \hat{f}_j; \hat{f}_i] = a_{ij}\mu,$$

by a cross ratio identity, see [19, §4.9.11]; and secondly, using (2.4), we find that f satisfies the Bäcklund condition[24] $\hat{P}(\mu) \in f^{\perp}$:

$$\hat{P}(\mu) = \lim_{\lambda\to\mu} \hat{P}(\lambda) = P(\mu) - \frac{\langle P(\mu), F\rangle}{\langle F, \hat{F}\rangle}\hat{F} \bmod f.$$

Recall that constructing a Darboux transform $\hat{f}$ of a given isothermic net amounts to determining a parallel isotropic section $\hat{F}$ of $M \times \mathbb{R}^{4,1}$ equipped with a connection Γ^{μ} in the isothermic family, see (4.2): after choosing an initial value there is a unique solution. As the condition on a Darboux transform of a special isothermic net to become a Bäcklund transform is preserved by the propagation it is sufficient to choose the initial value of the Darboux transform appropriately in order to obtain a Bäcklund transform.

Now note that (4.5) has exactly the same structure[25] as the conserved quantity condition (3.1), with $\frac{1}{\mu}$ taking the role of the a_{ij}. Thus, completely analogous to Lemma 3.7,

$$\hat{P}(\lambda) + \frac{\lambda}{\mu}\frac{\langle P(\lambda), F\rangle}{\langle F, \hat{F}\rangle}\hat{F} = P(\lambda) + \frac{\lambda}{\mu}\frac{\langle \hat{P}(\lambda), \hat{F}\rangle}{\langle F, \hat{F}\rangle}F, \tag{4.6}$$

where F and $\hat{F}$ are any[26] light cone lifts of f and $\hat{f}$, respectively. In particular, writing

$$P(\lambda) = \lambda^N Z + \lambda^{N-1} Y + \cdots + Q \quad \text{and} \quad \hat{P}(\lambda) = \lambda^N \hat{Z} + \lambda^{N-1}\hat{Y} + \cdots + \hat{Q},$$

[23] We know that the Bäcklund transform does not increase the type but, in special circumstances, if may decrease the type (by 1). We shall come back to this point later.

[24] The assumption that $\hat{f}$ is special isothermic of the same type N as f ensures that $\hat{P}(\lambda)$ has, with $P(\lambda)$, minimal degree and, in particular, that $\hat{P}(\mu) \neq 0$ so that the condition is meaningful.

[25] This is a common phenomenon in discrete differential geometry, often referred to as "multidimensional consistency", cf. [6] or [7]: the transformations of a discrete net are governed by exactly the same conditions as the net itself.

[26] A canonical choice would be to take F and $\hat{F}$ to satisfy (2.5) and $\langle F, \hat{F}\rangle \equiv \frac{1}{\mu}$: this is the "3D-consistency" of the condition to be discrete isothermic, see [7].

we find that, analogous to Corollary 3.8, the constant coefficients $\hat{Q} = Q$ and that, analogous to the geometric interpretation of Corollary 3.9,

$$S := Z + \frac{\langle \hat{Y}, \hat{F} \rangle}{\mu \langle F, \hat{F} \rangle} F = \hat{Z} + \frac{\langle Y, F \rangle}{\mu \langle F, \hat{F} \rangle} \hat{F} \tag{4.7}$$

yields a sphere congruence[27] that is enveloped by both f and $\hat{f}$ in the sense of Definition 3.10. Thus our Bäcklund transformation for special isothermic nets is a special type of Ribaucour transformation for discrete curvature line nets in Lie sphere geometry, see [6, Def. 22].

Theorem 4.5. *Let $\hat{f}$ be a Bäcklund transform of a special isothermic net f of type N, with respective polynomial conserved quantities*

$$P(\lambda) = \lambda^N Z + \lambda^{N-1} Y + \cdots + Q \quad \text{and} \quad \hat{P}(\lambda) = \Gamma_{\hat{f},f}^{1-\lambda/\mu} P(\lambda) = \lambda^N \hat{Z} + \lambda^{N-1} \hat{Y} + \cdots + \hat{Q}.$$

Then

- (i) $\hat{Q} = Q$, *and*
- (ii) $Z + \frac{\langle \hat{Y}, \hat{F} \rangle}{\mu \langle F, \hat{F} \rangle} F = \hat{Z} + \frac{\langle Y, F \rangle}{\mu \langle F, \hat{F} \rangle} \hat{F}$; *and,*
- (iii) *in particular, $Z + f$ and $\hat{Z} + \hat{f}$ give rise to a Ribaucour pair in the Lie geometric sense.*

4.2. **Bianchi permutability.** A key feature of Darboux-Bäcklund type transformations for smooth or discrete classes of surfaces is Bianchi permutability: given two transforms of a surface there is a fourth (often unique) surface, which is a simultaneous transform of the two initial transforms — thus providing the combinatorics of a quadrilateral for the transformation. We will refer to such quadrilaterals as *Bianchi quadrilaterals.*

In particular, such a theorem holds true for the Darboux transformation of (discrete) isothermic surfaces, see [17] or [19, §5.7.28], where the fourth surface is uniquely determined[28] by a cross ratio condition[29]:

Theorem 4.6. *If $\hat{f}_1$ and $\hat{f}_2$ are two Darboux transforms, $T^{\mu_i} \hat{f}_i \equiv const$, of a discrete isothermic net f, then*

$$f_{12} := \Gamma_{\hat{f}_1, \hat{f}_2}^{\mu_2/\mu_1} f$$

is a simultaneous Darboux transform of $\hat{f}_1$ and $\hat{f}_2$:

$$\hat{T}_1^{\mu_2} f_{12} \equiv const \quad \text{and} \quad \hat{T}_2^{\mu_1} f_{12} \equiv const.$$

---

[27]Indeed, a similar fact can be proved for Darboux transforms in general: given a sphere congruence Z enveloped by an isothermic net and a Darboux transform $\hat{f}$ of f, the sphere congruence

$$S := Z - \frac{\langle Z, \hat{F} \rangle}{\langle F, \hat{F} \rangle} F$$

will be enveloped by both f and $\hat{f}$. This is a fact about the existence of Ribaucour transforms for discrete curvature line nets in Lie geometry, cf. [6]. Note that this is a different "Darboux sphere congruence" than the one discussed in [17], which "lives" on the faces of the domain M.

[28]This is in contrast to the Ribaucour transformation of (discrete) principal nets, where a 1-parameter family of fourth surfaces exists, see [11] or [5].

[29]Note how repeated application of this theorem builds up a discrete isothermic net.

We wish to prove a similar theorem for the Bäcklund transformation of special isothermic nets of a given type N.

Thus let f be a special isothermic net with polynomial conserved quantity $P(\lambda)$ and let $\hat{f}_i$ be two Bäcklund transforms of f, i.e.,

$$T^{\mu_i}\hat{f}_i \equiv const \quad \text{and} \quad P(\mu_i) \in \hat{f}_i^{\perp}.$$

Clearly, by the permutability theorem, Theorem 4.6, for the Darboux transformation and (4.5), the simultaneous Bäcklund transform f_{12} of $\hat{f}_1$ and $\hat{f}_2$ is necessarily

$$f_{12} = \Gamma^{\mu_2/\mu_1}_{\hat{f}_1,\hat{f}_2} f \quad \text{with} \quad P_{12}(\lambda) = \Gamma^{1-\lambda/\mu_2}_{f_{12},\hat{f}_1}\hat{P}_1(\lambda) = \Gamma^{1-\lambda/\mu_1}_{f_{12},\hat{f}_2}\hat{P}_2(\lambda) \tag{4.8}$$

as a polynomial conserved quantity — just as in the proof of Lemma 4.2 it follows from Lemma 2.6 that

$$\Gamma^{1-\lambda/\mu_2}_{f_{12},\hat{f}_1}\Gamma^{1-\lambda/\mu_1}_{\hat{f}_1,f} = \Gamma^{(1-\lambda/\mu_1)/(1-\lambda/\mu_2)}_{\hat{f}_1,\hat{f}_2} = \Gamma^{1-\lambda/\mu_1}_{f_{12},\hat{f}_2}\Gamma^{1-\lambda/\mu_2}_{\hat{f}_2,f},$$

so that $P_{12}(\lambda)$ is well defined by (4.8). Now it is straightforward to see that

$$P_{12}(\mu_1) = \lim_{\lambda\to\mu_1} \Gamma^{(1-\lambda/\mu_1)/(1-\lambda/\mu_2)}_{\hat{f}_1,\hat{f}_2} P(\lambda) = P(\mu_1) - \frac{\langle P(\mu_1), \hat{F}_2\rangle}{\langle \hat{F}_1, \hat{F}_2\rangle}\hat{F}_1 + \ldots \hat{F}_2 \in \hat{f}_2^{\perp},$$

showing that $\hat{f}_2$ is a Bäcklund transform of f_{12}; similarly, $\hat{f}_1$ is also a Bäcklund transform of f_{12}. The symmetry of the Bäcklund transformation then completes the proof of the following

Theorem 4.7. *(Bianchi permutability) Given two Bäcklund transforms $\hat{f}_1$ and $\hat{f}_2$ with parameters μ_1 and μ_2, respectively, of a special isothermic net f of type N, there is a net f_{12} so that the four nets form a Bianchi quadrilateral: that is, f_{12} is a Bäcklund transform of $\hat{f}_1$ with parameter μ_2 and of $\hat{f}_2$ with parameter μ_1.*

Note that we have not used any new arguments to prove this theorem — indeed, using the similarity of the polynomial conserved quantity equations and the conditions governing the Bäcklund transformation, we could have formulated a proof based on the fact that "3D-consistency" of a "2D-system" implies higher dimensional consistency, see [7, Thm. 7]: in the case at hand we were interested in "4D-consistency". Thus, any higher dimensional permutability theorems can now be proved by purely combinatorial arguments[30]. For example, we can now argue that a "Bianchi cube" can be (uniquely) constructed from a special isothermic net and three Bäcklund transforms: the existence of the eighth Bäcklund transform is ensured by the very same fact that ensured the existence of a Darboux transform and the compatibility of the Bäcklund transformation with the construction, as discussed above.

4.3. **Complementary nets.** As we already noticed earlier, the equation (4.2) on a light cone map $\hat{F}$ to provide a Darboux transform of an isothermic net f is exactly the conserved quantity equation (3.1) for a fixed parameter μ. Consequently, any zero μ of $|P(\lambda)|^2$ provides a (light cone lift of a) Darboux transform $\hat{F} = P(\mu)$ of an isothermic net f with polynomial conserved quantity $P(\lambda)$. Moreover, since

[30]This is in contrast with the Ribaucour transformation, where 3-dimensional permutability, i.e., a "Bianchi cube" theorem, is the critical case as there is no uniqueness in the Bianchi quadrilateral.

$\langle \hat{F}, P(\mu)\rangle = |P(\mu)|^2 = 0$, this Darboux transform will, in fact, be a Bäcklund transform of f. In this section we shall discuss the role of these special Bäcklund transforms of a special isothermic net.

Definition 4.8. *Let f be a special isothermic net with polynomial conserved quantity $P(\lambda)$; those Bäcklund transforms $\hat{f}$ of f given by $\hat{F} = P(\mu)$, where $|P(\mu)|^2 = 0$, are the* complementary nets *of f.*

Clearly, a special isothermic net of type N has at most $2N$ complementary nets and, as $|P(\lambda)|^2$ is an even degree polynomial, there may be no (real) complementary nets: for example, let f be a type 1 special isothermic net with linear conserved quantity

$$P(\lambda) = \lambda Z + Q \quad \text{and} \quad H := -\langle Z, Q\rangle, \quad \kappa := -|Q|^2; \tag{4.9}$$

then the complementary nets of f are given by[31]

$$P(H \pm \sqrt{H^2+\kappa}) = (H \pm \sqrt{H^2+\kappa})\, Z + Q \tag{4.10}$$

so that the number of (real) complementary nets depends on the sign of $H^2 + \kappa$.

On the other hand, if enough complementary nets are known, then the corresponding polynomial conserved quantity can be reconstructed: first we observe that, given $N+1$ distinct parameter values $\lambda = \mu_0, \ldots, \mu_N$,

$$P(\lambda) = \sum_{n=0}^{N} P(\mu_n) \prod_{m\neq n} \frac{\lambda - \mu_m}{\mu_n - \mu_m}, \tag{4.11}$$

where $P(\mu_n)$ are Γ^{μ_n}-parallel and, with $\alpha_n := \prod_{m\neq n} \frac{1}{\mu_n - \mu_m}$, the leading coefficient

$$Z = \sum_{n=0}^{N} \alpha_n P(\mu_n) \in f^\perp$$

of $P(\lambda)$ is a (constant) linear combination of the $P(\mu_n)$. These are the assumptions that will allow us to reconstruct a polynomial conserved quantity from $N+1$ suitable Darboux transforms:

Lemma 4.9. *Let $\hat{F}^n$, $n = 0, \ldots, N$, be Γ^{μ_n}-parallel sections of $M \times \mathbb{R}^{4,1}$ for pairwise distinct μ_n and suppose that*

$$Z_i = \sum_{n=0}^{N} \alpha_n \hat{F}_i^n \in f_i^\perp$$

for some constants $\alpha_n \in \mathbb{R}$ and all $i \in M$. Then

$$P(\lambda) := \sum_{n=0}^{N} \alpha_n \hat{F}^n \prod_{m\neq n} (\lambda - \mu_m) \tag{4.12}$$

is a degree N polynomial conserved quantity for f with top degree coefficient Z; if $|Z_i|^2 = 1$ at some $i \in M$, then (4.12) *defines a normalized polynomial conserved quantity.*

[31]Note that $P(H \pm \sqrt{H^2+\kappa}) \neq 0$ since f is special isothermic of type 1 so that $P(\lambda)$ cannot have zeroes.

Note that, if $|Z|^2 > 0$, then f envelops the sphere congruence Z in the sense of Definition 3.10, as desired: incidence is given by assumption and touching is guaranteed because the sections $\hat{F}^n$ are Γ^{μ_n} parallel so that

$$dZ_{ij} = \sum_{n=0}^{N} \alpha_n \, d\hat{F}^n_{ij} = 0 \bmod f_i \oplus f_j.$$

This also implies directly that $|Z|^2 \equiv const$ on M.

Proof. It remains to show that $P(\lambda)$ is indeed a polynomial conserved quantity, that is, we wish to show that

$$\Gamma^\lambda_{ij} P_j(\lambda) - P_i(\lambda) = dP_{ij}(\lambda) + \frac{a_{ij}\lambda}{\langle F_i, F_j\rangle}\{\frac{1}{1-a_{ij}\lambda}\langle P_j(\lambda), F_i\rangle F_j - \langle P_j(\lambda), F_j\rangle F_i\} = 0$$

for all λ and each edge (ij) of M. Clearly, this equality holds true for $\lambda = \mu_n$, $n = 0, \dots, N$, since

$$\Gamma^{\mu_n}_{ij} P_j(\mu_n) = \alpha_n \Gamma^{\mu_n} \hat{F}^n_j \prod_{m \neq n} (\mu_n - \mu_m) = \alpha_n \hat{F}^n_i \prod_{m \neq n} (\mu_n - \mu_m) = P_i(\mu_n)$$

so that it holds for all λ as soon as we know that $\Gamma^\lambda_{ij} P_j(\lambda) - P_i(\lambda)$ is a degree N polynomial. By definition $P_i(\lambda)$ and $P_j(\lambda)$ are degree N polynomials, and $\langle P_j(\lambda), F_j\rangle$ is a degree $N-1$ polynomial by incidence, $Z_j \perp F_j$. Moreover,

$$\begin{aligned}
\langle P_j(\tfrac{1}{a_{ij}}), F_i\rangle &= \tfrac{1}{a_{ij}^N} \textstyle\prod_{m=0}^N (1 - a_{ij}\mu_m) \sum_{n=0}^N \frac{\alpha_n}{1-a_{ij}\mu_n} \langle \hat{F}^n_j, F_i\rangle \\
&= \tfrac{1}{a_{ij}^N} \textstyle\prod_{m=0}^N (1 - a_{ij}\mu_m) \sum_{n=0}^N \alpha_n \langle \hat{F}^n_i, F_i\rangle \\
&= \tfrac{1}{a_{ij}^N} \textstyle\prod_{m=0}^N (1 - a_{ij}\mu_m) \langle Z_i, F_i\rangle \\
&= 0
\end{aligned}$$

so that $\frac{1}{1-a_{ij}\lambda}\langle P_j(\lambda), F_i\rangle$ is also a degree $N-1$ polynomial. Hence the claim follows. □

Note that, in Lemma 4.9, we did not require the $\hat{F}^n$ to be isotropic sections of $M \times \mathbb{R}^{4,1}$, that is, we did not require them to be lifts of Darboux transforms of f.

Now suppose that μ is a simple zero of $|P(\lambda)|^2$ and let $\hat{F} = P(\mu)$ denote (a lift of) the corresponding complementary net of f. Then, from (4.5),

$$\hat{P}(\lambda) = P(\lambda) - \frac{\lambda}{\mu}\frac{\langle P(\lambda), F\rangle}{\langle P(\mu), F\rangle} P(\mu) - \frac{\lambda}{\lambda - \mu}\frac{\langle P(\lambda), P(\mu)\rangle}{\langle F, P(\mu)\rangle} F$$

such that

$$\hat{P}(\mu) = -\frac{\mu}{\langle P(\mu), F\rangle} \lim_{\lambda\to\mu} \frac{\langle P(\lambda), P(\mu)\rangle}{\lambda - \mu} F = -\frac{\mu}{2\langle P(\mu), F\rangle} \lim_{\lambda\to\mu} \frac{|P(\lambda)|^2}{\lambda - \mu} F$$

since

$$\frac{|P(\lambda)|^2}{\lambda - \mu} - 2\frac{\langle P(\lambda), P(\mu)\rangle}{\lambda - \mu} = (\lambda - \mu)\left|\frac{P(\lambda) - P(\mu)}{\lambda - \mu}\right|^2 \longrightarrow 0 \cdot |P'(\mu)|^2 = 0$$

as $\lambda \to \mu$. Consequently, if μ is a simple zero of $|P(\lambda)|^2$, then f is a complementary net of $\hat{f}$, that is, the notion of complementary nets is symmetric.

If, on the other hand, μ is a higher order zero of $|P(\lambda)|$, then $\hat{P}(\mu) = 0$ so that $\hat{f}$ is special isothermic of lower type than f by Lemma 3.3. Indeed, all type lowering

Bäcklund transformations arise in this way: suppose that $\hat{f}$ is a type $N-1$ Bäcklund transform of a special isothermic net f of type N — or, otherwise said, f is a Darboux transform of $\hat{f}$ which is not a Bäcklund transform. Then, their polynomial conserved quantities are related by (4.4):

$$P(\lambda) = (\lambda - \mu)\Gamma_{f,\hat{f}}^{1-\lambda/\mu}\hat{P}(\lambda) = (\lambda - \mu)(\hat{P}(\lambda) - \frac{\lambda}{\mu}\frac{\langle \hat{P}(\lambda), \hat{F}\rangle}{\langle F, \hat{F}\rangle} F) - \lambda\frac{\langle \hat{P}(\lambda), F\rangle}{\langle F, \hat{F}\rangle}\hat{F}$$

so that

$$P(\mu) = -\mu\frac{\langle \hat{P}(\mu), F\rangle}{\langle F, \hat{F}\rangle}\hat{F}$$

spans $\hat{f}$, which is therefore a complementary net of f.

Note that μ is a higher order zero of $|P(\lambda)|^2 = (\lambda - \mu)^2|\hat{P}(\lambda)|^2$.

We summarize these results:

Lemma 4.10. *Let f be special isothermic of type N with polynomial conserved quantity $P(\lambda)$.*

- (i) *If μ is a higher order zero of $|P(\lambda)|^2$, then $\hat{F} = P(\mu)$ defines a type $N-1$ Bäcklund transform of f.*
- (ii) *If $\hat{f}$ is a type $N-1$ Bäcklund transform of f, then $\hat{f}$ is a complementary net of f, $\hat{f} \ni P(\mu)$ for some μ, where μ is a higher order zero of $|P(\lambda)|^2$.*

As a consequence of this lemma, a special isothermic net of type N can have at most N Bäcklund transforms of type $N-1$ — generically, a Bäcklund transformation is between special isothermic nets of the same type, and is therefore symmetric as discussed above. For example, consider a special isothermic net f of type 1: by (4.10), f is a Darboux transform of a type 0 net, that is, of a spherical net (see Lemma 3.14), if and only if $H^2 + \kappa = 0$.

In the remainder of this section we shall discuss geometric properties of complementary nets of special isothermic nets of type 1 and 2.

First consider a type 1 special isothermic net with two complementary nets, i.e., $H^2 + \kappa > 0$ in (4.10); also, we assume $\kappa \neq 0$, excluding the degenerate case, where one of the complementary nets becomes constant. Then

$$\hat{F}^{\pm} := Z + \frac{1}{\mu_{\pm}}Q, \quad \text{where} \quad \mu_{\pm} := H \pm \sqrt{H^2 + \kappa}, \tag{4.13}$$

provides $\Gamma^{\mu_{\pm}}$-parallel light cone lifts of the two complementary nets so that

$$\hat{F}^{\pm} = \hat{F}^{\mp} \pm \frac{2\sqrt{H^2 + \kappa}}{\kappa}Q = \hat{F}^{\mp} - 2\frac{\langle \hat{F}^{\mp}, Q\rangle}{|Q|^2}Q.$$

Thus the two nets are Möbius equivalent and, more precisely, they are "antipodal" in the quadric of constant curvature given by Q, see [19, Section 1.4]. Moreover, the "orthogonal circles"

$$\hat{c}^{\pm} := \operatorname{span}\{F, Z, \hat{F}^{\pm}\}$$

of the Ribaucour pair[32] $(f, \hat{f}^{\pm})$, see Theorem 4.5, coincide in corresponding points:

$$\hat{c}_i^{\pm} = \text{span}\{F_i, Z_i, \hat{F}_i^{\pm}\} = \text{span}\{F_i, Z_i, Q\}.$$

For a type 2 special isothermic net f we obtain two similar properties: first let

$$\hat{F}^n = P(\mu_n) = \mu_n^2 Z + \mu_n Y + Q \quad (n = 1, 2)$$

denote (Γ^{μ_n}-parallel lifts of) two complementary nets of f; then the planes (in the quadric Q)

$$\hat{e}^n := \text{span}\{F, Z, \hat{F}^n, Q\}$$

of the orthogonal circles $\hat{c}^n$ coincide at corresponding points:

$$\hat{e}_i^n = \text{span}\{F_i, Z_i, \hat{F}_i^n, Q\} = \text{span}\{F_i, Z_i, Y_i, Q\}.$$

On the other hand, if

$$\hat{F}^n = P(\mu_n) = \mu_n^2 Z + \mu_n Y + Q \quad (n = 1, 2, 3)$$

provide three complementary nets of f, then

$$Q \in \hat{c}_i := \text{span}\{\hat{F}_i^n \,|\, n = 1, 2, 3\}$$

for all $i \in M$, that is, the circles through corresponding points of the three complementary nets are, in fact, straight lines in the quadric of constant curvature given by Q.

Reversing these observations yields four constructions for linear or quadratic conserved quantities from geometric configurations of Darboux transforms:

Theorem 4.11. *Let $\hat{f}^n$, $n = 1, 2$, be two Darboux transforms with different parameters μ_n of a discrete isothermic net f so that the circles*[33]

$$\hat{c}_{ij}^n := \text{span}\{F_i, F_j, \hat{F}_i^n\} = \text{span}\{F_i, F_j, \hat{F}_j^n\}$$

on the edges of M do not coincide for $n = 1, 2$. Suppose that the $\hat{f}^n$ are antipodal in a suitable non-Euclidean space form. Then f has a normalized linear conserved quantity.

Theorem 4.12. *Let $\hat{f}^n$, $n = 1, 2, 3$, be three Darboux transforms with different parameters μ_n of a discrete isothermic net f so that the circles*

$$\hat{c}_{ij}^n := \text{span}\{F_i, F_j, \hat{F}_i^n\} = \text{span}\{F_i, F_j, \hat{F}_j^n\}$$

on the edges of M are not cospherical for $n = 1, 2, 3$. Suppose that the circles

$$\hat{c}_i := \text{span}\{\hat{F}_i^n \,|\, n = 1, 2, 3\}$$

are straight lines in a suitable space form geometry. Then f has a normalized quadratic conserved quantity.

---

[32]In the smooth case, these orthogonal circles form a *cyclic system*, that is, they have a 1-parameter family of orthogonal surfaces so that any two orthogonal surfaces are Ribaucour transforms of each other.

[33]Note that $\text{span}\{F_i, F_j, \hat{F}_i^n\} = \text{span}\{F_i, F_j, \hat{F}_j^n\}$ since corresponding edges of a Darboux (or, more generally, Ribaucour pair) have concircular endpoints.

Proof. We prove both theorems. Thus let $N = 2$ or $N = 3$ and let $\hat{F}^n$, $n = 1, \dots, N$ denote Γ^{μ_n}-parallel light cone lifts of the Darboux transforms $\hat{f}^n$. Note that the non-degeneracy assumption for the edge circles c_{ij}^n ensures that the $\hat{F}_i^n$ or $\hat{F}_j^n$ are linearly independent $\operatorname{mod} f_i \oplus f_j$.

Now let $Q \in \mathbb{R}^{4,1} \setminus \{0\}$ denote the vector defining the space form. Then

$$Q = \sum_{n=1}^{N} \alpha_n \hat{F}^n$$

with suitable functions $\alpha_n : M \to \mathbb{R}$. First, we see that the α_n are constant:

$$0 = dQ_{ij} = \sum_{n=1}^{N} d(\alpha_n)_{ij} \hat{F}_{ij}^n + (\alpha_n)_{ij} d\hat{F}_{ij}^n = \sum_{n=1}^{N} d(\alpha_n)_{ij} \hat{F}_{ij}^n \bmod f_i \oplus f_j,$$

so that $d(\alpha_n)_{ij} = 0$ since $(\hat{F}^1, \dots, \hat{F}^N)$ are linearly independent $\operatorname{mod} f_i \oplus f_j$. Now,

$$0 = dQ_{ij} = \sum_{n=1}^{N} \alpha_n d\hat{F}_{ij}^n = \sum_{n=1}^{N} \alpha_n \mu_n \{\langle \hat{F}^n, F \rangle_j F_i - \langle \hat{F}^n, F \rangle_i F_j\},$$

showing that

$$Z := \sum_{n=1}^{N} \alpha_n \mu_n \hat{F}^n \perp f$$

defines a sphere congruence enveloped by f: note that $|Z|^2 > 0$ since the $\hat{F}^n$ and F are linearly independent. Hence f has a linear or quadratic conserved quantity if $N = 2$ or $N = 3$, respectively, by Lemma 4.9. □

Note that there are no corresponding theorems for the existence of higher degree polynomial conserved quantities for codimension 1 isothermic nets: the non-degeneracy assumption on the $\hat{F}_i^n$'s, F_i and F_j being linearly independent in $\mathbb{R}^{4,1}$ restricts N to numbers not greater than 3.

Theorem 4.13. *Let $\hat{f}^n$, $n = 1, 2$, be two Darboux transforms with different parameters μ_n of a discrete isothermic net f and let*

$$M \ni i \mapsto \hat{c}_i^n := \operatorname{span}\{F_i, Z_i, \hat{F}_i^n\}$$

denote the orthogonal circle congruences of the Ribaucour pairs $(f, \hat{f}^n)$, $n = 1, 2$, with respect to an enveloped sphere congruence[34] *Z and let*

$$(ij) \mapsto c_{ij} := \operatorname{span}\{F_i, Z_i, F_j\} = \operatorname{span}\{F_i, Z_j, F_j\}$$

denote the orthogonal edge circles[35]*. Further, let $Q \in \mathbb{R}^{4,1}$ define a quadric of constant curvature so that neither the circles c_{ij} nor the circles $\hat{c}_i$ are straight lines and let*

$$e_{ij} := \operatorname{span}\{F_i, Z_i, F_j, Q\} = \operatorname{span}\{F_i, Z_j, F_j, Q\} \quad \text{and} \quad \hat{e}_i^n := \operatorname{span}\{F_i, Z_i, \hat{F}_i, Q\}$$

denote the corresponding circle planes. Assume that $e_{ij} \neq \hat{e}_i^n$ for every edge and suppose that the circle planes $\hat{e}^1 = \hat{e}^2$. Then f has a quadratic conserved quantity.

[34]That is: Z defines the normal direction at each vertex of the net so that the notion of an "orthogonal circle" is well defined — thus, only the contact element $Z_i + \mathbb{R} F_i$ at each point i is needed.

[35]Note that $\operatorname{span}\{F_i, Z_i, F_j\} = \operatorname{span}\{F_i, Z_j, F_j\}$ by the enveloping condition Definition 3.10.

Theorem 4.14. *Let $\hat{f}^n$, $n = 1, 2$, be two Darboux transforms with different parameters μ_n of a discrete isothermic net f and let*

$$i \mapsto \hat{c}_i^n \quad \text{and} \quad (ij) \mapsto c_{ij}$$

denote the orthogonal circle congruences as in the previous theorem 4.13. Assume that $c_{ij} \neq \hat{c}_i^n$ for every edge and suppose that $\hat{c}^1 = \hat{c}^2$. Then f has a normalized linear conserved quantity.

The first of these two theorems is a discrete version of a famous classical theorem, cf. [14], [15, §84] and [12, Thms. 2.1 & 2.33].

Proof. Again, we prove both theorems as the proofs are very similar.

Let $\hat{F}^n$ denote Γ^{μ_n}-parallel lifts of the two Darboux transforms $\hat{f}^n$ and note that Q is Γ^0-parallel.

First consider the second theorem: as $\hat{c}^1 = \hat{c}^2$ we have

$$\hat{F}^2 = \alpha_1 \hat{F}^1 + \alpha\, F + \beta\, Z$$

with suitable functions α_1, α, β, where $\alpha_1 \neq 0$ since $\hat{f}^2 \neq f$. Hence

$$d(\alpha_1)_{ij} \hat{F}_i^1 = d\hat{F}_{ij}^2 - d(\alpha\, F + \beta\, Z)_{ij} - (\alpha_1)_j d\hat{F}_{ij}^1 = 0 \bmod c_{ij},$$

so that α_1 is constant since $\hat{c}_i^1 \neq c_{ij}$. Moreover, as $\hat{F}^1$, $\hat{F}^2$ and F are linearly independent,

$$\hat{F}^2 - \alpha_1 \hat{F}^1 = \alpha\, F + \beta\, Z$$

defines an enveloped sphere congruence and the claim follows from Lemma 4.9.

In the situation of the first theorem 4.13 we have

$$\hat{F}^2 = \alpha_0 Q + \alpha_1 \hat{F}^1 + \alpha\, F + \beta\, Z$$

and deduce

$$d(\alpha_1)_{ij} \hat{F}_i^1 = d\hat{F}_{ij}^2 - d(\alpha\, F + \beta\, Z)_{ij} - d(\alpha_0)_{ij} Q - (\alpha_1)_j d\hat{F}_{ij}^1 = 0 \bmod e_{ij},$$

so that, again, α_1 is constant since $\hat{e}_i^1 \neq e_{ij}$. Then

$$d(\alpha_0)_{ij} Q = d\hat{F}_{ij}^2 - d(\alpha\, F + \beta\, Z)_{ij} - (\alpha_1)_j d\hat{F}_{ij}^1 = 0 \bmod c_{ij},$$

showing that α_0 is constant as well, because c_{ij} was assumed to be a proper circle. Now

$$\hat{F}^2 - \alpha_1 \hat{F}^1 - \alpha_0 Q = \alpha\, F + \beta\, Z,$$

so that, again, the claim follows from Lemma 4.9. □

Note that, in the first theorem, we obtain a normalized quadratic conserved quantity as soon as we assume that the circles $\operatorname{span}\{\hat{F}^1, \hat{F}^2, F\}$ through corresponding points of f, $\hat{f}^1$ and $\hat{f}^2$ do not become straight lines in the quadric given by Q.

5. Discrete cmc nets in space forms

We are now prepared to define discrete cmc nets in space forms. A smooth isothermic surface f in the conformal 3-sphere has constant mean curvature H in a quadric of constant curvature $\kappa = -|Q|^2$, given by $Q \in \mathbb{R}^{4,1}$, if and only if it has a linear conserved quantity

$$P(\lambda) = \lambda Z + Q,$$

where Z is the mean curvature sphere congruence[36] of f, see [10] or [12, Thm. 2.27]: for smooth isothermic surfaces in the conformal 3-sphere (of codimension 1) it turns out that the top coefficient of a polynomial conserved quantity is necessarily its conformal Gauss map[37]. Note that, in the case of a spherical surface, the conformal Gauss map Z of f is constant and f has a constant conserved quantity $Q = Z$ (as in the discrete case: see Theorem 3.14), hence a linear conserved quantity, e.g.,

$$P(\lambda) = \lambda Z + Q = (\lambda + 1)\, Z.$$

In the discrete setting we use this characterization as a definition:

Definition 5.1. *A discrete isothermic net f will be called a* discrete cmc net *if it is special isothermic of type $N \leq 1$. In particular, if f is special isothermic of type* 1 *with normalized linear conserved quantity*

$$P(\lambda) = \lambda Z + Q,$$

we say that Z is the mean curvature sphere congruence *of f in the quadric*

$$\mathcal{Q} = \{Y \in L^4 \mid \langle Y, Q\rangle = -1\}$$

of constant curvature $\kappa = -|Q|^2$ and that f has (constant) mean curvature[38]

$$H := -\langle Z, Q\rangle.$$

Thus discrete cmc nets in space forms are special isothermic nets and a transformation theory is readily available to us, cf. [18]: the Bäcklund transformation, see Definition 4.4, yields a transformation for discrete cmc nets in a given space form preserving the mean curvature,

$$\tilde{Q} = Q \quad \text{and} \quad \tilde{H} = -\langle \tilde{Z}, \tilde{Q}\rangle = -\langle Z, Q\rangle = H$$

by Theorem 4.5, and satisfying Bianchi permutability by Theorem 4.7; the Calapso transformation provides a Lawson correspondence $f \mapsto f^\mu$ for discrete cmc nets, where both the mean and ambient curvature change,

$$\kappa^\mu = -|Q^\mu|^2 = \kappa + 2\mu H - \mu^2 \quad \text{and} \quad H^\mu = -\langle Z^\mu, Q^\mu\rangle = H - \mu$$

[36] The mean curvature sphere congruence of f, consisting of spheres touching f that have the same mean curvature as the surface at the touching points, can be defined using *any* ambient space form geometry: it can be characterized as the *conformal Gauss map* of f, i.e., the unique enveloped sphere congruence that induces the same conformal structure as f, or as the *central sphere congruence*, i.e., the congruence of spheres that exchange the curvature spheres (via inversion) or, equivalently, that have second order contact with the surface in orthogonal directions.

[37] Hence it follows directly that an isothermic surface with linear conserved quantity has constant mean curvature $H = -\langle Z, Q\rangle$ in the space form given by Q.

[38] A change $a \to \tilde{a} = \frac{a}{c}$ of the cross ratio factorizing function results in a change of the equation (3.1) for a linear conserved quantity, hence of the linear conserved quantity, see Lemma 3.6; in particular, $\tilde{Z} = Z$ and $\tilde{Q} = cQ$. Hence we obtain the effect of an ambient homothety: $\tilde{\kappa} = c^2\kappa$ and $\tilde{H} = cH$.

since $Z^\mu = T^\mu Z$ and $Q^\mu = T^\mu(\mu Z + Q)$ by Theorem 3.13, but

$$(H^\mu)^2 + \kappa^\mu = \kappa + H^2$$

remains invariant.

Definition 5.2. *The Calapso transformation for discrete cmc nets will also be called* Lawson correspondence.

The main goal of this section will be to show that our definition generalizes and truly extends previous definitions[39] from [3], [17], [4] and [18].

5.1. **Uniqueness and existence questions.** However, before addressing the relation of our definition with previous approaches we shall discuss the construction and uniqueness of linear conserved quantities for a given discrete isothermic net.

Clearly, given the value of a linear conserved quantity at one point of a discrete cmc net f with cross ratio factorizing function a, the linear conserved quantity is uniquely determined as it is a parallel section of the isothermic family of (flat) connections: given an initial value of a linear (or polynomial) conserved quantity $P(\lambda)$ we can use (3.1) to determine the values of $P(\lambda)$ at any point. Thus, starting with a linear quantity $\lambda Z + Q$ at some point of a discrete isothermic net, there is a unique parallel section of the isothermic family of connections[40]; but the parallel section may fail to be linear or even to be polynomial at other points of the isothermic net — hence we may not obtain a linear conserved quantity if we use the "wrong" initial value or if the isothermic net is not cmc.

Using instead the equivalent condition (3.2) from Lemma 3.7 (see also Corollary 3.8 and (i-ii) of Corollary 3.9), we learn that $P(\lambda) = \lambda Z + Q$ is a linear conserved quantity for an isothermic net f with cross ratio factorizing function a if and only if

$$Q \equiv const, \quad Z \perp F \quad \text{and} \quad Z_j = Z_i + \frac{a_{ij}}{\langle F_i, F_j\rangle}\{\langle Q, F_j\rangle F_i - \langle Q, F_i\rangle F_j\} \tag{5.1}$$

for any edge (ij) of the domain graph M. Again, we obtain a propagation formula that fixes the linear conserved quantity uniquely once an initial value of $P(\lambda)$ is given at some point of a discrete isothermic net; however, now the existence of Z as well as the incidence relation $Z \perp F$ are only satisfied if the isothermic net was, in fact, cmc and if the initial value of $P(\lambda)$ was chosen correctly.

First we address the integrability of the difference equation for the mean curvature sphere congruence Z,

$$Z_j = Z_i + \frac{a_{ij}}{\langle F_i, F_j\rangle}\{\langle Q, F_j\rangle F_i - \langle Q, F_i\rangle F_j\}. \tag{5.2}$$

This equation does clearly not depend on the choice of lift F of the isothermic net f; to simplify the computation we may choose a Moutard lift, satisfying (2.5), so that

$$\begin{aligned} Z_k &= Z_i + \langle Q, F_k - F_i\rangle F_j - \langle Q, F_j\rangle (F_k - F_i) \\ &= Z_i + \tfrac{a_{ij}-a_{il}}{\langle F_j, F_l\rangle}\{\langle Q, F_j\rangle F_l - \langle Q, F_l\rangle F_j\} \end{aligned} \tag{5.3}$$

[39] Note that the mean curvature in [22] is defined on the faces of a principal net and therefore different from the mean curvature defined here, living on the vertices.

[40] Here we use the flatness of the connections in the family.

on an elementary quadrilateral $(ijkl)$, where we used the Moutard equation (2.6). The last expression is symmetric in j and l; hence the propagation equation (5.2) is integrable.

Next we address the incidence relation $Z \perp F$. First consider an edge (ij): from (5.2)

$$\langle Z_j, F_j\rangle = \langle Z_i, F_j\rangle + a_{ij}\langle Q, F_j\rangle.$$

Consequently, incidence determines Z_i at the center $i = i_{(0,0)}$ of a non-spherical vertex star[41]: if we let $i_{(m,n)}$, $m, n \in \{-1, 0, 1\}$, denote the vertices of a 3×3-grid then the equations

$$(5.4) \qquad \langle Z_{i_{(0,0)}}, F_{i_{(0,0)}}\rangle = 0 \quad \text{and} \quad \langle Z_{i_{(0,0)}}, F_{i_{(m,n)}}\rangle = -a_{i_{(0,0)}i_{(m,n)}}\langle Q, F_{i_{(m,n)}}\rangle,$$

where $m^2 + n^2 = 1$, have a unique solution $Z_{i_{(0,0)}}$ since the vertices $F_{i_{(0,0)}}$ and $F_{i_{(m,n)}}$ of a non-spherical vertex star form a basis of $\mathbb{R}^{4,1}$. Thus an appropriate choice of the initial value $Z_{i_{(0,0)}}$ for the mean curvature sphere congruence at the center of a vertex star ensures that incidence is satisfied on all five vertices of the vertex star when using (5.2) to define Z on the corresponding 3×3-grid. At the diagonal vertices $F_{i_{(m,n)}}$, $m, n = \pm 1$, we obtain incidence without further conditions: let $(ijkl)$ denote an elementary quadrilateral; then, using (5.3), (2.6) and (2.5), we get

$$\begin{aligned}
\langle Z_k, F_k\rangle &= \langle Z_i + \tfrac{a_{ij}-a_{il}}{\langle F_j, F_l\rangle}\{\langle Q, F_j\rangle F_l - \langle Q, F_l\rangle F_j\}, F_i + \tfrac{a_{ij}-a_{il}}{\langle F_j, F_l\rangle}\{F_j - F_l\}\rangle \\
&= \langle Z_i, F_i\rangle + \tfrac{a_{ij}-a_{il}}{\langle F_j, F_l\rangle}\{\langle Z_i + a_{ij}Q, F_j\rangle - \langle Z_i + a_{il}Q, F_l\rangle\} \\
&= \langle Z_i, F_i\rangle + \tfrac{a_{ij}-a_{il}}{\langle F_j, F_l\rangle}\{\langle Z_j, F_j\rangle - \langle Z_l, F_l\rangle\} \\
&= 0,
\end{aligned}$$

that is, incidence of Z_k and f_k. Thus we have proved the following, cf. Corollary 3.4:

Lemma 5.3. *Let $f : \{(m,n) \,|\, m, n \in \{-1, 0, 1\}\} \to S^3$ be a non-spherical discrete isothermic 3×3-net and let $Q \in \mathbb{R}^{4,1} \setminus \{0\}$. Then f has a unique linear conserved quantity $P(\lambda) = \lambda Z + Q$.*

Note that, if the vertex star of $f_{(0,0)}$ is cospherical, then the corresponding 3×3-net is necessarily also cospherical since f is a discrete principal net, i.e., the vertices of its faces are concircular. Hence, assuming that f is non-spherical in Lemma 5.3, we have that the vertex star used to define Z at its center is non-spherical. Moreover, since any two adjacent vertex stars (i.e., vertex stars at the endpoints of an edge) of a discrete isothermic net have two face circles in common, they lie on the same sphere if they are cospherical — thus, an isothermic net is either spherical, hence type 0, or it has a non-spherical vertex star.

As a (degenerate) example consider the Moutard lift F of an isothermic 3×3-net, i.e., let F satisfy $\langle F_i, F_j\rangle = a_{ij}$ as in (2.5), and choose Q so that

$$(5.5) \qquad Q \perp F_{(1,0)} - F_{(-1,0)}, F_{(0,1)} - F_{(0,-1)}, F_{(0,1)} - F_{(1,0)} \quad \Leftrightarrow \quad \langle Q, F_{(m,n)}\rangle = c_0$$

for $m^2 + n^2 = 1$ and some $c_0 \in \mathbb{R}$; further let $c_1 := \langle Q, F_{(0,0)}\rangle$ and observe that, from (2.6),

$$\langle F_{(m,n)} - F_{(0,0)}, Q\rangle = \frac{a_{(0,0)(m,0)} - a_{(0,0)(0,n)}}{\langle F_{(m,0)}, F_{(0,n)}\rangle}\langle F_{(m,0)} - F_{(0,n)}, Q\rangle = 0$$

[41] Here we use the fact that any vertex has four neighbours; in a more general quad-graph this argument only works at vertices of degree 4, i.e., not at "umbilics" of a discrete net.

for $m, n = \pm 1$, so that

$$\langle F_{(m,n)}, Q\rangle = \begin{cases} c_0 & \text{if } m^2 + n^2 = 1, \\ c_1 & \text{if } m^2 - n^2 = 0. \end{cases}$$

Now, (5.4) yields $Z_{(0,0)} + c_0 F_{(0,0)} \perp F_{(0,0)}, F_{(m,n)}$ for $m^2+n^2 = 1$, that is, $Z_{(0,0)}+c_0F_{(0,0)}$ is orthogonal to a basis of $\mathbb{R}^{4,1}$, hence vanishes. Consequently, (5.2) yields

$$Z_{(m,n)} = \begin{cases} -c_0 F_{(m,n)} & \text{if } m^2 - n^2 = 0, \\ -c_1 F_{(m,n)} & \text{if } m^2 + n^2 = 1. \end{cases}$$

Thus $Z \parallel F$ becomes isotropic and yields another Moutard lift of f. Note that, if we choose Q so that $c_0 = 0$, i.e., so that Q represents the (unique) sphere containing the outer points of the vertex star, then Z vanishes for $m^2 - n^2 = 0$, that is, we are in the situation that we excluded from consideration in Lemma 3.11. This last situation is a worst-case scenario for a non-spherical isothermic net and that, by (5.2), Z does not vanish as soon as $\langle F, Q\rangle$ does not, that is, as soon as f does not hit the infinity boundary of the space form given by Q.

Now recall from Lemma 3.11 that, if Z is null but does not vanish, then it is necessarily a Moutard lift of f. In particular,

$$Z_{(m,n)} = \begin{cases} -c_0 F_{(m,n)} & \text{if } m^2 - n^2 = 0, \\ -c_1 F_{(m,n)} & \text{if } m^2 + n^2 = 1 \end{cases}$$

for some $c_0, c_1 \in \mathbb{R}$ and a Moutard lift F satisfying (2.5) as any two Moutard lifts are related in this way. Hence, for $m^2 + n^2 = 1$,

$$\langle F_{(m,n)}, Q\rangle = -\frac{1}{c_1}\langle Z_{(m,n)}, Q\rangle \equiv const,$$

so that we are back in the situation of (5.5). Thus avoiding (5.5) we can ensure that Z becomes spacelike and hence a suitable rescaling of the constructed linear conserved quantity will leave us with a *normalized* linear conserved quantity; hence, according to Definition 5.1:

Lemma 5.4. *Let $f : \{(m,n)\,|\,m,n \in \{-1,0,1\}\} \to S^3$ be a non-spherical discrete isothermic 3×3-net and choose $Q \in \mathbb{R}^{4,1}$ to satisfy, for $m,n \in \{-1,0,1\}$,*

$$(5.6) \qquad Q \not\perp F_{(m,n)} \quad \text{and} \quad Q \notin \{F_{(1,0)} - F_{(-1,0)}, F_{(0,1)} - F_{(0,-1)}, F_{(0,1)} - F_{(1,0)}\}^{\perp},$$

where F is a Moutard lift of f. Then f is a discrete cmc net in a space form defined by a suitable rescaling of Q.

Thus, as (5.6) imposes only open conditions on Q, any given isothermic 3×3-net is cmc in a 4-parameter family of possible space forms. Enlarging the net and using (5.2) to propagate Z will add more incidence conditions, hence conditions on Q; it is therefore natural to expect that a large enough generic isothermic net will have a unique linear conserved quantity and being cmc becomes a condition. In particular, the conditions (5.1) for a linear conserved quantity on an isothermic 5×5-net

$$f : \{(m,n)\,|\,m,n \in \{-2,-1,0,1,2\}\} \to S^3$$

can be reduced to the incidence conditions on an extended vertex star

$$\{f_{(0,0)}, f_{(\pm1,0)}, f_{(0,\pm1)}, f_{(\pm2,0)}, f_{(0,\pm2)}\}$$

by the arguments that proved Lemma 5.3: this yields nine linear equations for the linear conserved quantity at the center vertex $f_{(0,0)}$ of the net — hence we get existence of a linear conserved quantity on any isothermic 5×5-net, and we expect uniqueness up to scaling generically, i.e., using the scaling freedom to normalize the obtained linear conserved quantity we expect a generic isothermic 5×5-net to be a discrete cmc net in a unique way.

However, the following example shows that even "arbitrarily large" isothermic nets can be cmc in different space forms — and even with their respective mean curvature sphere congruences defining the same contact elements at each vertex.

For this purpose we reconsider our example (3.5). However, instead of thinking of f as part of a "zigzag-plane" as in (3.5), we now think of it as part of a discrete circular cylinder by letting

$$\alpha := \cos\varphi \quad \text{and} \quad \beta := \sin\varphi$$

for some $\varphi \in (0, \frac{\pi}{2})$, so that[42]

$$f_{(m,n)} = (\eta m, \frac{1+\cos\varphi}{2} + (-1)^n \frac{1-\cos\varphi}{2}, n\sin\varphi) = (\eta m, \cos n\varphi, \sin n\varphi). \tag{5.7}$$

Further we introduce[43]

$$f^*_{(m,n)} := (\eta m, -\cos n\varphi, -\sin n\varphi).$$

Now let

$$Z := \frac{1}{2}F^* - Q \quad \text{and} \quad Q := (1,0,0,0,-1), \tag{5.8}$$

where $F^* = (\frac{1+|f^*|^2}{2}, f^*, \frac{1-|f^*|^2}{2})$ denotes again the Euclidean lift of f^*, and note that

$$\langle F, F^* \rangle = -\frac{1}{2}|f - f^*|^2 \equiv -2,$$

so that $\langle \lambda Z + Q, F \rangle \equiv -1$. Now observe that $|f^*|^2 = |f|^2$ do not depend on n and hence

$$dF^*_{(m,n)(m+1,n)} - dF_{(m,n)(m+1,n)} = dF^*_{(m,n)(m,n+1)} + dF_{(m,n)(m,n+1)} = 0;$$

consequently, Z and Q define a linear conserved quantity by Lemma 3.7: for all edges (ij)

$$d(\lambda Z + Q)_{ij} - \frac{\lambda a_{ij}}{\langle F_i, F_j \rangle}\{\langle \lambda Z + Q, F \rangle_j F_i - \langle \lambda Z + Q, F \rangle_i F_j\} = 0,$$

where $a_{ij} = \pm\frac{1}{2}\langle F_i, F_j \rangle$ as in the example (3.5). Since $|Z|^2 \equiv 1$ the linear conserved quantity is normalized, characterizing f as a discrete net of constant mean curvature

$$H = -\langle Z, Q \rangle = \frac{1}{2}$$

in Euclidean space (as $|Q|^2 = 0$) according to Definition 5.1, as one would expect.

[42] Remember that $m, n \in \{-1, 0, 1\}$.

[43] This is the (parallel) Christoffel transform of f, which is the net of centers of the mean curvature sphere congruence in Euclidean space.

Superposition of the linear conserved quantities given by (3.6) and by (5.8) then yields a 1-parameter family of normalized linear conserved quantities for f, given by

$$Z_\alpha := \frac{1}{2}(F^* + \alpha(-1)^n F) - Q_0$$

and

$$Q_\alpha := Q_0 - \frac{2\alpha}{1-\cos\varphi}(\frac{1+\cos\varphi}{2}, 0, 1, 0, -\frac{1+\cos\varphi}{2}),$$

where $Q_0 = (1,0,0,0,-1)$ and $\alpha \in \mathbb{R}$. Thus the isothermic net has constant mean curvatures

$$H_\alpha = -\langle Z_\alpha, Q_\alpha\rangle = \frac{1}{2}(1+\alpha^2) - \frac{1+\cos\varphi}{1-\cos\varphi}\alpha$$

in the hyperbolic spaces

$$\mathcal{Q}_\alpha = \{Y \in L^4 \,|\, \langle Y, Q_\alpha\rangle = -1\}$$

of curvatures $\kappa_\alpha = -\frac{4\alpha^2}{(1-\cos\varphi)^2}$. Note that all Z_α define the same contact element at a vertex of f, that is, they all define the same "normal direction" at a vertex.

Finally observe that, as for the discrete net (3.5) with degenerate linear conserved quantity (3.6), the restriction of the domain to $\{-1,0,1\}^2$ is again not necessary: the circular cylinder in (5.7) can be defined on all of $\mathbb{Z}^2$ with linear conserved quantity given by (5.8). As the two nets, the "zigzag-plane" and the circular cylinder, coincide on $\mathbb{Z} \times \{-1,0,1\}$ we obtain an example of an isothermic $N \times 3$-net, $N \in \mathbb{N}$ arbitrary, that is cmc in different space forms.

5.2. **Relation with previous approaches.** At this point we are prepared to link the present approach to discrete cmc nets to previous approaches; in particular, we shall discuss how our definition relates to:

- the notion of discrete minimal surface in Euclidean space introduced in [3, Def. 7];
- the notion of discrete cmc net in Euclidean space introduced in [17, Sect. 5];
- the notion of discrete horospherical net (cmc 1 net) in hyperbolic space in [18, Def. 4.3];
- the definition of a net of constant mean curvature H in a space form of curvature κ, where $H^2 + \kappa \geq 0$, suggested in [18].

In this context we shall also discuss the relation of our notion of "mean curvature sphere" with that of [4, Sect. 4.5] and the notion of "central sphere congruence" in [7, Sect. 3].

First we wish to make contact with the definition of a discrete cmc net in Euclidean space given in [17]: recall that a discrete isothermic net $f : M^2 \to \mathbb{R}^3$ is called cmc in [17] if it has a parallel (isothermic) net, i.e., a simultaneous Christoffel and Darboux transform f^*. The (constant) distance of this parallel net yields (up to sign) the mean curvature of both nets,

$$H := \frac{1}{|f^* - f|}.$$

Note that any constant multiple of $\langle df, df^*\rangle$ is a cross ratio factorizing function[44], so that, without loss of generality,

$$a_{ij} = -\frac{H}{2}\,\langle df_{ij}, df^*_{ij}\rangle, \quad \text{that is,} \quad df^*_{ij} = -\frac{2a_{ij}}{H|df_{ij}|^2}\,df_{ij}.$$

Further, since $|f^* - f|^2 \equiv const$, (2.2) yields

$$(f^* - f)_{ij} \perp df_{ij}, df^*_{ij},$$

that is, $H\,(f^* - f)$ defines a (unit) normal field for f as well as for f^* in the sense of [22], cf. [17]. Consequently,

$$d(|f^*|^2)_{ij} = 2\langle f^*_{ij}, df^*_{ij}\rangle = -\frac{4a_{ij}}{H|df_{ij}|^2}\langle f_{ij}, df_{ij}\rangle = -\frac{2a_{ij}}{H|df_{ij}|^2}\,d(|f|^2)_{ij}.$$

Now suppose $f : M^2 \to \mathbb{R}^3$ is a discrete cmc net in this sense, with parallel cmc net f^* and (constant) mean curvature H. Let F and F^* denote the respective Euclidean lifts and let

$$Z := HF^* - \frac{1}{2H}Q \quad \text{and} \quad Q = (1, 0, 0, 0, -1)$$

as in the above example of a discrete circular cylinder, see (5.8). Then

$$\langle Z, F\rangle = -\frac{H}{2}|f^* - f|^2 + \frac{1}{2H} = 0$$

and

$$dZ_{ij} - \frac{a_{ij}}{\langle F_i, F_j\rangle}\,dF_{ij} = HdF^*_{ij} + \frac{2a_{ij}}{|df_{ij}|^2}\,dF_{ij} = 0,$$

so that $\lambda Z + Q$ defines a (normalized) linear conserved quantity of f by Lemma 3.7. Moreover,

$$\kappa = -|Q|^2 = 0 \quad \text{and} \quad -\langle Z, Q\rangle = H.$$

Conversely, suppose that f is a discrete isothermic net with cross ratio factorizing function a and normalized linear conserved quantity $\lambda Z + Q$ so that

$$\kappa = -|Q|^2 = 0 \quad \text{and} \quad H = -\langle Z, Q\rangle \neq 0.$$

Let

$$F^* := \frac{1}{H}(Z + \frac{1}{2H}Q) = \frac{1}{2H^2}(2H\,Z + Q)$$

denote the complementary net of f, see Definition 4.8. Clearly, F^* defines a Darboux transform of f,

$$T^{2H}F^* \equiv const,$$

and $\langle F^*, Q\rangle \equiv -1$. Choosing a Euclidean lift F of f, i.e., $\langle F, Q\rangle \equiv -1$, (3.2) yields

$$dF^*_{ij} = \frac{a_{ij}}{H\langle F_i, F_j\rangle}\,dF_{ij}. \tag{5.9}$$

[44]Given a cross ratio factorizing function a of an isothermic net one defines the Christoffel transform f^* by

$$df^*_{ij} = a_{ij}(df_{ij})^{-1} = -\tfrac{a_{ij}}{|df_{ij}|^2}\,df_{ij};$$

as a is only defined up to constant multiples, so is df^*. In the case of the parallel net of a discrete cmc net, however, there is a canonical scaling for f^*.

Now let, without loss of generality, $Q = (1,0,0,0,-1)$ so that

$$F^* = (\frac{1+|f^*|^2}{2}, f^*, \frac{1-|f^*|^2}{2}) \quad \text{and} \quad F = (\frac{1+|f|^2}{2}, f, \frac{1-|f|^2}{2}).$$

Then (5.9) gives

$$df^*_{ij} = -\frac{2a_{ij}}{H|df_{ij}|^2}\, df_{ij}.$$

Hence f^* is also the Christoffel transform of $f : M^2 \to \mathbb{R}^3$, so that f is a cmc net with parallel cmc net f^* in the sense of [17].

Thus we have proved the following

Theorem 5.5. *A discrete isothermic net in $\mathbb{R}^3$ is cmc in Euclidean space in the sense of* [17] *if and only if it is cmc in the sense of Definition 5.1.*

In [4, Sect. 4.5], a notion of mean curvature sphere for a discrete isothermic net in Euclidean space is introduced. We shall see that this mean curvature sphere is the same as our mean curvature sphere Z in the case of a discrete cmc net.

First recall [4, Def. 12]: given a vertex star $\{f_{i_{(m,n)}} \in \mathbb{R}^3 \,|\, m^2+n^2 \le 1\}$ of a discrete isothermic net with constant (negative) cross ratio function $q_{ijkl} = \frac{a_{ij}}{a_{il}}$ there is a unique point $c_{i_{(0,0)}}$ so that

$$(5.10) \qquad |f_{i_{(1,0)}} - c_{i_{(0,0)}}| = |f_{i_{(-1,0)}} - c_{i_{(0,0)}}| \quad \text{and} \quad |f_{i_{(0,1)}} - c_{i_{(0,0)}}| = |f_{i_{(0,-1)}} - c_{i_{(0,0)}}|$$

and

$$(5.11) \qquad \frac{|f_{i_{(1,0)}} - c_{i_{(0,0)}}|^2 - |f_{i_{(0,0)}} - c_{i_{(0,0)}}|^2}{a_{i_{(0,0)}i_{(1,0)}}} = \frac{|f_{i_{(0,1)}} - c_{i_{(0,0)}}|^2 - |f_{i_{(0,0)}} - c_{i_{(0,0)}}|^2}{a_{i_{(0,0)}i_{(0,1)}}}.$$

This point $c_{i_{(0,0)}}$ is the center of the mean curvature sphere at $f_{i_{(0,0)}}$ and its radius

$$(5.12) \qquad r_{i_{(0,0)}} := |f_{i_{(0,0)}} - c_{i_{(0,0)}}|.$$

Now suppose that f is a discrete cmc net in Euclidean space, with Q as above and mean curvature sphere congruence Z, and go back to (5.4): let F denote the Euclidean lift for f as before and write

$$Z = \frac{1}{r}(1+|c|^2-r^2, c, 1-|c|^2+r^2)$$

in terms of its center c and radius r; then

$$0 = \langle Z_{i_{(0,0)}}, F_{i_{(0,0)}} \rangle = -\frac{1}{2r_{i_{(0,0)}}}(|f_{i_{(0,0)}} - c_{i_{(0,0)}}|^2 - r^2_{i_{(0,0)}})$$

is equivalent to (5.12), while the remaining four equations of (5.4) read

$$1 = \frac{\langle Z_{i_{(0,0)}}, F_{i_{(m,n)}} \rangle}{a_{i_{(0,0)}i_{(m,n)}}} = -\frac{1}{2r_{i_{(0,0)}}} \frac{|f_{i_{(m,n)}} - c_{i_{(0,0)}}|^2 - r^2_{i_{(0,0)}}}{a_{i_{(0,0)}i_{(m,n)}}},$$

clearly implying (5.11); the equations (5.10) follow since the cross ratio function is constant, so that

$$a_{i_{(0,0)}i_{(1,0)}} = a_{i_{(0,0)}i_{(-1,0)}} \quad \text{and} \quad a_{i_{(0,0)}i_{(0,1)}} = a_{i_{(0,0)}i_{(0,-1)}}.$$

Hence we have proved:

Theorem 5.6. *The mean curvature sphere Z of a discrete cmc net $f : M \to \mathbb{R}^3$ in Euclidean space in the sense of Definition 5.1 is the mean curvature sphere of f in the sense of* [4, Sect. 4.5].

In fact, we have seen slightly more: the equations (5.4) define[45] the mean curvature sphere of [4] for any isothermic net in Euclidean space with constant cross ratio function[46].

This suggests to use (5.4) to define the *mean curvature sphere* of an isothermic net in any space form. Note that these mean curvature spheres of an isothermic net are, in contrast to the smooth case, *not* Möbius invariant: different choices of the ambient space form given by Q will, in general, lead to different mean curvature spheres Z. In particular, this also shows that the mean curvature sphere defined in this way is generally[47] different from the central sphere of [7], which is Möbius invariantly related to the isothermic net as it is characterized by incidence only.

Following ideas from [18], the Lawson correspondence of Definition 5.2 can be used to define discrete cmc nets in space forms with

$$H^2 + \kappa > 0$$

as Calapso transforms of cmc nets in Euclidean space. Using permutability theorems the above characterization of cmc nets in Euclidean space by the existence of a simultaneous Darboux and Christoffel transform, i.e., the existence of a parallel cmc net can then be carried over to other space forms to obtain an alternative characterization, similarly to the way in which horospherical nets in hyperbolic space can be characterized in two ways, see [18, Lemma 4.2] or [19, §5.7.37].

Namely, let f^* denote the parallel cmc net of a discrete cmc net f in Euclidean space and consider their Calapso transforms f^λ and $(f^*)^\lambda$ — which are only determined up to Möbius transformation. Then f^λ and $(f^*)^\lambda$ can be positioned in S^3 so that:

- (i) they form a Darboux pair with parameter $-\lambda$ since f and f^* form a Christoffel pair, see [18, Cor. 3.24] or [19, §5.7.34];
- (ii) they form a Darboux pair with parameter $\mu - \lambda$ since f and f^* form a Darboux pair with some parameter μ, see [18, Cor. 3.27] or [19, §5.7.35].

That is, there are two ways to position $(f^*)^\lambda$ in S^3 so that it is a Darboux transform of f^λ with different parameters or, otherwise said, f^λ has a pair of Möbius equivalent Darboux transforms. More precisely, given a Calapso transform of a discrete cmc net

[45] Besides the sphere, the equations (5.4) also determine the scaling of its representative in Minkowski space.

[46] Recently, a new approach has come into focus, where the "mean curvature" of a discrete principal net in $\mathbb{R}^3$ is defined, as a function on faces, via the area change of a face when varying through parallel nets, see [22] and [8]. This approach leads to the same class of discrete minimal or constant mean curvature nets as the one discussed here [9].

[47] Using a Moutard lift satisfying (2.5), as for (5.5), (2.6) and (5.4) yield

$$\langle Z_{i_{(0,0)}}, F_{i_{(m,n)}} \rangle = 0 \quad \Leftrightarrow \quad a_{i_{(0,0)} i_{(m,0)}} \langle Q, F_{i_{(m,0)}} \rangle = a_{i_{(0,0)} i_{(0,n)}} \langle Q, F_{i_{(0,n)}} \rangle$$

for $m, n = \pm 1$. Hence the "mean curvature sphere" defined by (5.4) is the "central sphere" of [7] (cf. Lemma 2.2) if and only if, for $m^2 + n^2 = 1$,

$$\langle Q, a_{i_{(0,0)} i_{(m,n)}} F_{i_{(m,n)}} \rangle = const.$$

in Euclidean space, (4.13) provides two antipodal Darboux transforms: the antipodal map identifies the ambient space form.

Theorem 4.11 provides the precise formulation of the converse: normalizing the linear conserved quantity from the proof,

$$(1+\lambda\mu_1)\alpha_1\hat{F}^1 + (1+\lambda\mu_2)\alpha_2\hat{F}^2,$$

we obtain $|P(\lambda)|^2 = \frac{(1+\lambda\mu_1)(1+\lambda\mu_2)}{\mu_1\mu_2}$ as its squared norm, hence

$$H^2+\kappa = (\frac{\mu_1+\mu_2}{2\mu_1\mu_2})^2 - \frac{1}{\mu_1\mu_2} = \frac{(\mu_1-\mu_2)^2}{4\mu_1^2\mu_2^2} > 0$$

since $\mu_1 \neq \mu_2$, so that the isothermic net f is indeed a Calapso transform of a cmc net in Euclidean space.

Finally we discuss minimal nets in Euclidean space and horospherical nets in hyperbolic space, that is, the discrete constant mean curvature nets with

$$H^2+\kappa = 0.$$

First recall that horospherical nets in hyperbolic space can be defined as Darboux transforms of their hyperbolic Gauss maps [18, Def. 4.3], that is, as Darboux transforms of a spherical net: hence Lemma 4.10 identifies them as the discrete cmc nets with

$$H^2+\kappa=0 \quad \text{and} \quad \kappa \neq 0$$

in the sense of Definition 5.1. On the other hand, horospherical nets can equivalently be characterized as Calapso transforms of a discrete minimal net in Euclidean space in the sense of [3] and, conversely, any discrete minimal net in Euclidean space gives rise to a Lawson family of horospherical nets, see [18, Lemma 4.2]. Hence we can reverse the argument to conclude that the discrete minimal nets in the sense of [3] are those discrete cmc nets with

$$H^2+\kappa=0 \quad \text{and} \quad \kappa = 0$$

in the sense of Definition 5.1. Note that [3, Thm. 8] provides two equivalent characterizations of discrete minimal nets in Euclidean space:

- (i) as Christoffel transforms of spherical isothermic nets (their "Gauss maps") and
- (ii) by the fact that their mean curvature sphere congruence consists of planes[48].

We summarize these discussions in the following

Theorem 5.7. *A discrete isothermic net is*

- (i) *minimal in* $\mathbb{R}^3$ *in the sense of* [3] *iff it is cmc with* $H=\kappa=0$ *in the sense of Definition 5.1;*
- (ii) *horospherical in the sense of* [18] *iff it is cmc with* $-H^2=\kappa<0$ *in the sense of Definition 5.1.*

---

[48]Assuming, as in [3], that $a_{ij} = \pm\delta$ a very similar computation as the one leading to Theorem 5.6 shows that our statement here is equivalent to [3, Def. 7]: with the Euclidean lift F and $Z = (d, n, -d)$, where n is the unit normal of the plane described by Z and $d = \langle n, f\rangle$ its distance from the origin, the equations (5.4) become

$$\langle n_{i_{(0,0)}}, f_{i_{(m,n)}} - f_{i_{(0,0)}}\rangle = \langle Z_{i_{(0,0)}}, F_{i_{(m,n)}}\rangle = a_{i_{(0,0)}i_{(m,n)}} = \pm\delta.$$

5.3. **Discrete cmc surfaces of revolution.** We conclude by discussing our example of a discrete surface of revolution in more detail: here we will be interested in the construction of discrete cmc surfaces of revolution with prescribed mean curvature in a given ambient space form.

Thus we consider a discrete surface of revolution

$$(m,n) \mapsto F_{(m,n)} = (-1)^m(M_m + \Phi_n C) \in \mathbb{R}^{2,1} \oplus \mathbb{R}^2, \tag{5.13}$$

see (2.8), with cross ratio factorizing function

$$a_{ij} = \alpha \langle F_i, F_j \rangle \tag{5.14}$$

since F is a Moutard lift[49]. By Corollary 3.16 a linear conserved quantity $P(\lambda) = \lambda Z + Q$ has the same rotational symmetry as the net if and only if the vector Q defining the ambient space form has this symmetry; in particular, the linear conserved quantity of an equivariant cmc net is rotationally symmetric with

$$P_{(m,n)}(\lambda) = P_m^{\perp}(\lambda) + (-1)^{m+1}\lambda p_m(\lambda)\Phi_n C = \lambda\{Z_m^{\perp} - \alpha\langle Q, M_m\rangle\, \Phi_n C\} + Q,$$

where $Z^{\perp}, Q \in \mathbb{R}^{2,1}$, see (3.8) and Lemma 3.6. Hence

$$\begin{array}{rclcl} Z_{(m,n)} & = & Z_m^{\perp} + \alpha\langle Q, M_m\rangle\, M_m & - & \alpha\langle Q, M_m\rangle\,(M_m + \Phi_n C) \\ & = & S_m & - & \alpha\langle Q, F_{(m,n+1)}\rangle\, F_{(m,n)}, \end{array} \tag{5.15}$$

where $S_m = S_{(m,n)(m,n+1)}$ is the curvature sphere (3.4) on an edge in the rotational direction: note that this family of curvature spheres does only depend on m, so that our discrete surface of revolution is the envelope of a 1-parameter family of spheres in the sense of Definition 3.10.

Our first aim is to formulate the condition (3.2) for a linear conserved quantity of the given form: clearly

$$0 = \langle Z, F\rangle = (-1)^m\langle S, M\rangle,$$

that is, the incidence relation is again expressed by orthogonality; further,

$$\begin{array}{rcl} 0 & = & dZ_{(m,n)(m+1,n)} + \alpha\{\langle Q, F_{(m,n)}\rangle\, F_{(m+1,n)} - \langle Q, F_{(m+1,n)}\rangle\, F_{(m,n)}\} \\ & = & dS_{m,m+1} - 2\alpha\langle Q, M_{m,m+1}\rangle\, dM_{m,m+1} \end{array}$$

and

$$0 = dZ_{(m,n)(m,n+1)} + \alpha\{\langle Q, F_{(m,n)}\rangle\, F_{(m,n+1)} - \langle Q, F_{(m,n+1)}\rangle\, F_{(m,n)}\}$$

are identically satisfied. Hence we obtain:

Lemma 5.8. *Let $Q \in \mathbb{R}^{2,1}$ and $Z_{(m,n)} := S_m - \alpha\langle Q, F_{(m,n)}\rangle F_{(m,n)}$, where F is a Moutard lift* (5.13) *of a discrete surface of revolution and $m \mapsto S_m \in \mathbb{R}^{2,1}$ is a discrete 1-parameter family of spheres. Then $P(\lambda) := \lambda Z + Q$ defines a linear conserved quantity for f with respect to* (5.14) *as a cross ratio factorizing function if and only if*

$$0 \equiv \langle S, M\rangle \quad \textit{and} \quad dS = 2\alpha\langle Q, M\rangle\, dM. \tag{5.16}$$

[49]We shall use the scaling freedom α in the cross ratio factorizing function later to normalize the linear conserved quantity that we will construct, see Lemma 3.6.

Note that the equations (5.16) determine S_m and S_{m+1} (and, hence, Z_m and Z_{m+1}) up to a common orthogonal offset from the plane[50] spanned by M_m and M_{m+1}. Also prescribing the mean curvature H,

$$\langle S, Q\rangle = \langle Z, Q\rangle + \alpha\langle M, Q\rangle^2 = -H + \alpha\langle M, Q\rangle^2, \tag{5.17}$$

we obtain six equations[51]

$$\begin{aligned}
\langle S_m, Q\rangle &= -H + \alpha\langle M_m, Q\rangle^2 \\
\langle S_{m+1}, Q\rangle &= -H + \alpha\langle M_{m+1}, Q\rangle^2 \\
\langle S_m, M_m\rangle &= 0 \\
\langle S_{m+1}, M_m\rangle &= -\alpha\langle Q, M_{m,m+1}\rangle\, |dM_{m,m+1}|^2 \\
\langle S_m, M_{m+1}\rangle &= -\alpha\langle Q, M_{m,m+1}\rangle\, |dM_{m,m+1}|^2 \\
\langle S_{m+1}, M_{m+1}\rangle &= 0
\end{aligned}$$

that determine S_m and S_{m+1} as soon as (Q, M_m, M_{m+1}) is a basis of $\mathbb{R}^{2,1}$.

In order to understand this restriction geometrically we analyze what happens when M_m, M_{m+1} and Q are linearly dependent, that is, when Q is in the $(1,1)$-plane spanned by M_m and M_{m+1}. First recall that the light cone in $\mathbb{R}^{2,1}$ is the axis of our discrete surface of revolution (2.8); similarly, for fixed n,

$$\mathrm{span}\{F_{(m,n)}, F_{(m+1,n)}, \Phi_n C\} = \mathrm{span}\{M_m, M_{m+1}, \Phi_n C\} =: c_{m,m+1}$$

defines the circle through $F_{(m,n)}$ and $F_{(m+1,n)}$ that intersects the axis orthogonally — which becomes a geodesic in the quadric $\mathcal{Q}$ of constant curvature (3.3) given by Q as soon as

$$Q \in c_{m,m+1}.$$

Thus prescribing an edge of a meridian curve, M_m and M_{m+1}, and a space form, Q, so that the orthogonal circle $c_{m,m+1}$ of the axis passing through $F_{(m,n)}$ and $F_{(m+1,n)}$ does *not* become a straight line, any choice of H will lead to a unique solution S_m and S_{m+1} of the equations (5.16) and (5.17), hence to a unique linear conserved quantity for the discrete net obtained by rotating the edge.

The top coefficient Z of a linear conserved quantity constructed from a solution S of (5.16) and (5.17) is, with S, spacelike since $S \perp M$ and has constant length by Lemma 3.5. The idea is then to use our scaling freedom α in the cross ratio factorizing function (5.14) to obtain a *normalized* linear conserved quantity, as sought in Definition 5.1: note that, in contrast to Lemma 3.6, where only the conserved quantity condition played a role, the prescribed mean curvature equation (5.17) causes a more complicated dependence of S on α. In particular[52],

$$|S|^2 = \frac{C^2}{A} H^2 - \frac{A}{\Delta}(\alpha + \frac{B}{A} H)^2,$$

[50]Recall that M in (2.8) takes values in one component of the hyperbolic quadric $|Y|^2 = -1$ in $\mathbb{R}^{2,1}$.

[51]Note that $d(\langle M, Q\rangle^2) = 2\langle M, Q\rangle\langle dM, Q\rangle$ by (2.2) so that (5.17) only adds one real equation to (5.16).

[52]Note that the symmetry of the formula confirms that $|S_m|^2 = |S_{m+1}|^2$.

where we let

$$\begin{array}{rcl} \Delta & := & |Q \wedge M_m \wedge M_{m+1}|^2, \\ C & := & |dM_{m,m+1}|^2 \langle M_{m,m+1}, Q\rangle, \\ B & := & -|dM_{m,m+1}|^2\{|M_{m,m+1}|^2\langle M_m, Q\rangle\langle M_{m+1}, Q\rangle + 2\langle M_{m,m+1}, Q\rangle^2\}, \\ A & := & -\frac{B^2-\Delta C^2}{|M_m\wedge M_{m+1}|^2}. \end{array}$$

Note that $|Q\wedge M_m\wedge M_{m+1}|^2, |M_m\wedge M_{m+1}|^2 < 0$. Also, the assumption $M_m, M_{m+1} \not\perp Q$ that the endpoints of our meridian edge do not lie in the infinity boundary of the space form defined by Q implies that B and C do not simultaneously vanish; as a consequence $A > 0$.

Thus the equation $|S|^2 = 1$ can be solved for α if and only if $C^2H^2 \leq A$.

Moreover, as α is a factor of our cross ratio factorizing function, we seek a non-zero solution of the equation: since $A = C^2H^2$ clearly implies $H \neq 0$ we need to exclude

$$A - C^2H^2 = B = 0 \quad \Rightarrow \quad H^2 = \frac{\Delta}{|M_m \wedge M_{m+1}|^2}.$$

Thus the equation $|S|^2 = 1$ has a non-zero solution α if and only if

$$C^2H^2 \leq A \quad \text{and} \quad C^2H^2 = A \Rightarrow H^2 \neq \frac{\Delta}{|M_m \wedge M_{m+1}|^2}. \tag{5.18}$$

Note that, in the case of strict inequality in (5.18), $C^2H^2 < A$, our construction will, in general, provide two different linear conserved quantities[53]. However, in the case $H = 0$ of a minimal surface, the equation $|S|^2 = 1$ has always exactly one non-zero solution α.

Lemma 5.9. *Prescribing an ambient space form and a mean curvature, the equivariant discrete surface of revolution obtained by rotating a single edge in the space form has a normalized linear conserved quantity as soon as:*
(i) *the orthogonal circle of the axis of revolution passing through the endpoints of the edge is not a straight line in the ambient constant curvature geometry, and*
(ii) *the mean curvature is not chosen too large; more precisely, the mean curvature H satisfies the constraint* (5.18).

Having equipped an initial edge of a meridian curve for a discrete equivariant cmc net with prescribed mean curvature H and ambient space form Q with a suitable mean curvature sphere at both endpoints, we shall now investigate how to propagate the meridian curve M and its enveloped 1-parameter family of spheres S to "build" a larger equivariant cmc net. That is, we aim to construct M_{m+1} and S_{m+1} from the data at the other endpoint of the edge, M_m, S_m, Q.

As a discrete analogue of a constant speed parametrization in the hyperbolic plane of the meridian curve[54] we prescribe a constant cross ratio factorizing function a along the meridian curve so that

$$\langle M_{m+1}, M_m\rangle = -(1 + \frac{a_{(m,n),(m+1,n)}}{\alpha}) \equiv -(1 + \frac{c}{\alpha})$$

[53]As a consequence, our discrete net will have a polynomial conserved quantity of degree 0 by Corollary 3.4 — which is not too surprising since the net is clearly spherical, cf. Theorem 3.14.

[54]Recall that a hyperbolic constant speed parametrization of the meridian curve of a (smooth) surface of revolution leads to a conformal curvature line parametrization of the surface up to constant rescaling of the parameters.

for all m and some $c \in \mathbb{R}$, see (5.14). Note that, since we seek M_m and M_{m+1} to take values in the same hyperbolic plane,

$$\frac{c}{\alpha} = \frac{1}{2}|dM_{m,m+1}|^2 > 0.$$

A second equation obtained from (5.16),

$$0 = \langle M_{m+1}, S_m \rangle + \alpha \langle Q, M_{m,m+1} \rangle |dM_{m,m+1}|^2 = \langle M_{m+1}, S_m + cQ \rangle + c\langle M_m, Q \rangle,$$

then confines M_{m+1} to a line in $\mathbb{R}^{2,1}$ as soon as M_m and $S_m + cQ$ are linearly independent or, equivalently, as soon as

$$X_m := S_m + c\,(Q + \langle Q, M_m \rangle\, M_m) \neq 0.$$

Observe that, requiring the meridian curve to not cross the axis of rotation into the other hyperbolic half plane nor to hit the infinity boundary $Q^\perp$ of the ambient space form[55],

$$0 \neq c\,\langle Q, M_m \rangle \{\langle M_{m+1}, M_m \rangle - 1\} = \langle M_{m+1}, X_m \rangle,$$

so that $X_m \neq 0$ describes a sphere intersecting the axis of rotation orthogonally and containing the point of the meridian curve given by M_m since $X_m \perp M_m$. In particular, $X \neq 0$ at both endpoints of a "proper" meridian curve edge.

Thus

$$M_{m+1} = \{\frac{\alpha + c}{\alpha} M_m - \frac{((\alpha + c)^2 - \alpha^2)\langle Q, M_m \rangle}{\alpha} \frac{X_m}{|X_m|^2}\} + t\,\frac{Y_m}{|Y_m|^2}$$

for a suitable $t \in \mathbb{R}$, where Y_m is orthogonal to X_m in the (Euclidean) plane $M_m^\perp$ in $\mathbb{R}^{2,1}$ and has the same length,

$$Y_m \perp M_m, X_m \quad \text{and} \quad |Y_m|^2 = |X_m|^2.$$

As we wish M to take values in the hyperbolic plane

$$0 \stackrel{!}{=} 1 + |M_{m+1}|^2 = \tfrac{1}{|X_m|^2}\{t^2 - \tfrac{((\alpha+c)^2-\alpha^2)(1-2cH-c^2\kappa)}{\alpha^2}\},$$

where we have used (5.17), so that we obtain two candidates for M_{m+1} as soon as

$$1 - 2cH - c^2\kappa > 0. \tag{5.19}$$

When propagating the meridian curve with constant cross ratio factorizing function, then one of the two solutions must give the predecessor M_{m-1} of M_m; hence the propagation of a meridian curve is unique if it is possible. Observe that the condition (5.19) does not depend on m; hence it is automatically satisfied as soon as a "seed" meridian curve contains more than one edge[56]

Once M_{m+1} is constructed (5.16) yields

$$S_{m+1} := S_m + 2\alpha \langle Q, M_{m,m+1} \rangle\, dM_{m,m+1};$$

---

[55]We do, however, not exclude the possibility of the meridian curve crossing the infinity boundary.

[56]In case we have *one* edge of a seed meridian curve we infer that the quadratic equation has at least one solution, i.e.,

$$1 - 2cH - c^2\kappa \geq 0.$$

In the case $1 - 2cH - c^2\kappa = 0$ of one solution, the described propagation procedure will then alternate the two endpoints of one edge.

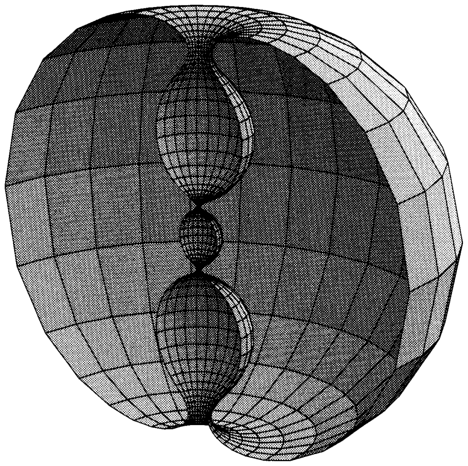

FIGURE 5.1. Discrete cmc torus in S^3

it is then straightforward to verify incidence[57]

$$\langle S_{m+1}, M_{m+1}\rangle = \langle M_{m+1}, S_m + cQ\rangle + c\langle Q, M_m\rangle = 0$$

and the mean curvature (5.17) being constant,

$$\langle Q, S_{m+1}\rangle - \alpha\langle Q, M_{m+1}\rangle = \langle Q, S_m\rangle - \alpha\langle Q, M_m\rangle.$$

Hence, by Lemma 5.8, we have succeeded in propagating the meridian curve of a discrete cmc net of revolution:

Lemma 5.10. *Let a point and unit normal of the meridian curve of a discrete cmc net of revolution in a space form be given. Prescribing a (constant) cross ratio factorizing function c and a Moutard factor α, the meridian curve can be propagated uniquely in either of two directions to obtain a discrete cmc surface of revolution as long as:*
(i) $\frac{c}{\alpha} > 0$ *and the point of the meridian curve does not lie in the infinity boundary of the space form, and*
(ii) $1 - 2cH - c^2\kappa > 0$, *where the mean curvature H is given by* (5.17) *and κ is the ambient curvature.*

These two lemmas now provide a method of construction for discrete cmc nets of revolution in a prescribed ambient space form and with prescribed mean curvature:

Construction: Choose an ambient space form $Q \in \mathbb{R}^{2,1}$ and a mean curvature $H \in \mathbb{R}$. Choose an initial edge $M_0, M_1 \in H^2 \subset \mathbb{R}^{2,1}$ so that:

- (i) $\langle Q, M_0\rangle, \langle Q, M_1\rangle \neq 0$, that is, M_0 and M_1 do not lie in the infinity boundary of the space form[58] Q;
- (ii) (Q, M_0, M_1) is a basis of $\mathbb{R}^{2,1}$, that is, the straight line in H^2 through M_0 and M_1 is not straight in the chosen space form Q; and

[57]Recall that the equations (5.16) also ensure that $|S|^2$ is constant.

[58]Note that this condition is sufficient but not necessary, as Figure 1.1 suggests.

- (iii) the constraint (5.18) is satisfied with the chosen mean and ambient curvatures.

Next, construct the spheres $S_0, S_1 \in S^2 \subset \mathbb{R}^{2,1}$ to satisfy the conserved quantity and mean curvature conditions (5.16) and (5.17), see Lemma 5.9; generically, there is a choice of two such sets of spheres.

Then, propagate this "seed" meridian curve using a constant cross ratio factorizing function, see Lemma 5.10; this yields a unique and proper propagation of the meridian curve if (5.19) is satisfied.

A variant of this construction was used to obtain the discrete cmc torus in S^3 shown in Figure 5.1 as well as to construct the discrete minimal net in two copies of H^3 shown in Figure 1.1: the plane shown in the figure indicates the common infinity boundary of the two copies of the ambient H^3.

References

[1] M. Babich, A. Bobenko: *Willmore tori with umbilic lines and minimal surfaces in hyperbolic space*; Duke Math. J. 72, 151–185 (1993)

[2] L. Bianchi: *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche*; Ann. Mat. 11, 93–157 (1904)

[3] A. Bobenko, U. Pinkall: *Discrete isothermic surfaces*; J. Reine Angew. Math. 475, 187–208 (1996)

[4] A. Bobenko, U. Pinkall: *Discretization of surfaces and integrable systems*; Oxf. Lect. Ser. Math. Appl. 16, 3–58 (1999)

[5] A. Bobenko, U. Hertrich-Jeromin: *Orthogonal nets and Clifford algebras*; Tôhoku Math. Publ. 20, 7–22 (2001)

[6] A. Bobenko, Y. Suris: *On Organizing Principles of Discrete Differential Geometry. Geometry of spheres*; Russian Math. Surveys 62, 1–43 (2007)

[7] A. Bobenko, Y. Suris: *Isothermic surfaces in sphere geometries as Moutard nets*; Proc. Royal Soc. A 463, 3171–3193 (2007)

[8] A. Bobenko, Y. Liu, H. Pottmann, J. Wallner, W. Wang: *Geometry of multi-layer structures for architecture*; ACM Trans. Graphics 26 (2007), SIGGRAPH 2007

[9] A. Bobenko: *Discrete minimal surfaces from quadrilaterals*; Talk at the International Congress on Minimal and Constant Mean Curvature Surfaces, Buzios, August 19–24, 2007 (2007)

[10] F. Burstall, D. Calderbank: *Conformal submanifold geometry*; Manuscript 2007

[11] F. Burstall, U. Hertrich-Jeromin: *The Ribaucour transform in Lie sphere geometry*; Differ. Geom. Appl. 24, 503–520 (2006)

[12] F. Burstall, S. Santos: *Special isothermic surfaces of type d*; J. London Math. Soc. 85, 571–591 (2012)

[13] E. Christoffel: *Ueber einige allgemeine Eigenschaften der Minimumsflächen*; Crelle's J. 67, 218–228 (1867)

[14] G. Darboux: *Sur les surfaces isothermiques*; Ann. Sci. Éc. Norm. Sup. 16, 491–508 (1899)

[15] L.P. Eisenhart: *Transformations of surfaces*; Princeton University Press, Princeton (1923)

[16] U. Hertrich-Jeromin, F. Pedit: *Remarks on the Darboux transform of isothermic surfaces*; Doc. Math. J. DMV 2, 313–333 (1997)

[17] U. Hertrich-Jeromin, T. Hoffmann, U. Pinkall: *A discrete version of the Darboux transform for isothermic surfaces*; Oxf. Lect. Ser. Math. Appl. 16, 59–81 (1999)

[18] U. Hertrich-Jeromin: *Transformations of discrete isothermic nets and discrete cmc-1 surfaces in hyperbolic space*; Manuscr. Math. 102, 465–486 (2000)

[19] U. Hertrich-Jeromin: *Introduction to Möbius differential geometry*; London Math. Soc. Lect. Note Ser. 300, Cambridge University Press (2003)

[20] T. Hoffmann: *Discrete curves and surfaces*; PhD thesis, Technische Universität Berlin (2000)

[21] J. Inoguchi, S. Kobayashi: *Characterizations of Bianchi-Bäcklund transformations of constant mean curvature surfaces*; Int. J. Math. 16, 101–110 (2005)

[22] W. Schief: *On a maximum principle for minimal surfaces and their integrable discrete counterparts*; J. Geom. Phys. 56, 1484–1495 (2006)

Francis E. Burstall, Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY (United Kingdom), Email: f.e.burstall@bath.ac.uk

Udo Hertrich-Jeromin, Institut fur Diskrete Mathematik und Geometrie, TU-Vienna, A-1040 Vienna (Austria), Email: uhj@geometrie.tuwien.ac.at

Wayne Rossman, Department of Mathematics, Kobe University, Rokko, Kobe 657-8501 (Japan), Email: wayne@math.kobe-u.ac.jp

Susana D. Santos, Departamento de Matematica, Faculdade de Ciencias, CMAF, Universidade de Lisboa, 1749-016 Lisboa (Portugal), Email: susantos@ptmat.fc.ul.pt