数理解析研究所講究録
第 1881 巻 2014 年 115-120

Stationary measures of the KPZ equation

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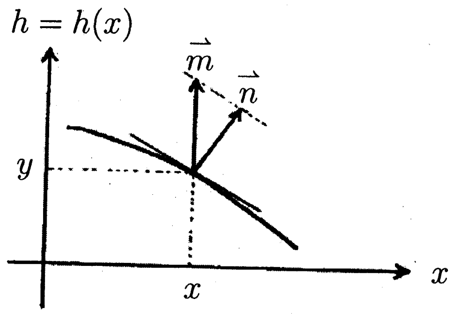
1 The KPZ equation

Kardar, Parisi and Zhang [4] introduced a nonlinear PDE with an additive stochastic term, which is called the KPZ equation, as a model for growing interfaces with random fluctuations. Here, we briefly review its derivation in a one-dimensional setting. We first recall the work of Professor Hiroshi Matano. He discussed in [5] with Nakamura and Lou a motion of interfaces (or curves) located in a two-dimensional cylinder, which grows upward with normal velocity:

(1.1) $$V = \kappa + A,$$

where κ is the curvature and $A > 0$ is a constant. Two edges of the curve perpendicularly contact to oscillatory boundaries of the cylinder. His main interest was the homogenization problem at the boundary.

The interfacial dynamics can be described as an equation for its height function $h(t,x)$ assuming that the interface in $\mathbb{R}^2$ is represented as a graph $\{(x,y) \in \mathbb{R}^2; y = h(t,x), x \in \mathbb{R}\}$.



The normal vector $\vec{n}$ to the curve $\mathcal{C}_h = \{y = h(x)\}$ at the point (x,y) is given by

$$\vec{n} = \frac{1}{\left(1 + (\partial_x h(x))^2\right)^{1/2}} \begin{pmatrix} -\partial_x h(x) \\ 1 \end{pmatrix}.$$

This is easily seen from $\vec{n} \perp \vec{t}$ and $|\vec{n}| = 1$, where $\vec{t}$ is the tangent vector to $\mathcal{C}_h$ given by

$$\vec{t} = \begin{pmatrix} 1 \\ \partial_x h(x) \end{pmatrix}.$$

The interfacial growth to the direction $\vec{n}$ is equivalent to the growth of the height function h to the vertical direction $\vec{m}$, where

$$\vec{m} = \begin{pmatrix} 0 \\ \left(1 + (\partial_x h(x))^2\right)^{1/2} \end{pmatrix}, \tag{1.2}$$

which is obtained noting that $(\vec{m} - \vec{n}) \perp \vec{n}$.

It is well-known that the curvature of the curve $\{y = h(x)\}$ at (x, y) is given by

$$\kappa = \frac{\partial_x^2 h(x)}{\left(1 + (\partial_x h(x))^2\right)^{3/2}}. \tag{1.3}$$

Therefore, from (1.2) and (1.3), the interface growing equation with normal velocity $V = \kappa + A$ can be written as

$$\partial_t h = \left\{ \frac{\partial_x^2 h}{(1 + (\partial_x h)^2)^{3/2}} + A \right\} (1 + (\partial_x h)^2)^{1/2},$$

that is,

$$\partial_t h = \frac{\partial_x^2 h}{1 + (\partial_x h)^2} + A(1 + (\partial_x h)^2)^{1/2},$$

for the height function $h = h(t, x)$, cf. [5].

If we consider $\tilde{h} := h - At$ instead of h by subtracting the constant growth factor At and write $\tilde{h}$ as h again, we obtain that

$$\begin{aligned} \partial_t h &= \frac{\partial_x^2 h}{1 + (\partial_x h)^2} + A\left\{(1 + (\partial_x h)^2)^{1/2} - 1\right\} \\ &\simeq \partial_x^2 h + \frac{A}{2}(\partial_x h)^2, \end{aligned}$$

that is,

$$\partial_t h = \partial_x^2 h + \frac{A}{2}(\partial_x h)^2, \tag{1.4}$$

at least if the slope $|\partial_x h|$ is small. Note that $u := \partial_x h$ is a solution of (viscous) Burgers equation.

The KPZ equation is then obtained from (1.4) by taking the fluctuation effects due to noises into account:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{R}, \tag{1.5}$$

where $h = h(t,x,\omega)$ and $\dot{W}(t,x) = \dot{W}(t,x,\omega)$ is the space-time Gaussian white noise defined on a certain probability space $(\Omega, \mathcal{F}, P)$ with mean 0 and correlation function:

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(x-y)\delta(t-s). \tag{1.6}$$

We take $A = 1$ and put $\frac{1}{2}$ in front of $\partial_x^2 h$. The correlation structure (1.6) heuristically means that $\dot{W}(t,x)$ are independent if (t,x) are different. This is natural from physical viewpoint.

2 Solvability of the KPZ equation (1.5)

Let us consider linear stochastic partial differential equations (SPDEs in short) on $\mathbb{R}^d$ replacing $\frac{1}{2}\partial_x^2$ by higher order differential operators $\mathcal{A}$ and dropping nonlinear term:

$$\partial_t h = \mathcal{A}h + \dot{W}(t,x), \quad x \in \mathbb{R}^d, \tag{2.1}$$

where $\dot{W}(t,x)$ is the space-time Gaussian white noise defined on $\mathbb{R}^d$ similarly as above and $\mathcal{A} = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C_b^\infty(\mathbb{R}^d)$, $m \in \mathbb{N}$, $D^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^d}\right)^{\alpha_d}$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$. The coefficients satisfy the uniform ellipticity condition:

$$\inf_{x,\sigma\in\mathbb{R}^d, |\sigma|=1} (-1)^{m+1} \sum_{|\alpha|=2m} a_\alpha(x)\sigma^\alpha > 0,$$

where $\sigma^\alpha = \sigma_1^{\alpha_1} \cdots \sigma_d^{\alpha_d}$ for $\sigma = (\sigma_1, \dots, \sigma_d) \in \mathbb{R}^d$. The solutions $h(t,x)$ are sometimes called Ornstein-Uhlenbeck processes. The solution of (2.1) is defined in a generalized functions' sense (by multiplying test functions $\varphi \in C_0^\infty(\mathbb{R})$) or in a mild form (via Duhamel's principle):

$$h(t) = e^{t\mathcal{A}}h(0) + \int_0^t e^{(t-s)\mathcal{A}} dW(s).$$

The last term is defined as a stochastic integral.

It is known that, if $2m > d$,

$$h(t,x) \in \cap_{\delta>0} C^{\alpha-\delta,\beta-\delta}((0,\infty)\times\mathbb{R}^d), \quad \text{a.s.},$$

where $\alpha = \frac{2m-d}{4m}$ and $\beta = \frac{2m-d}{2}$; see [1]. The necessity of the condition "$2m > d$" can be seen also from

$$E\left[\left\{\int_0^t e^{(t-s)\mathcal{A}}dW(s)\right\}^2\right] = \int_0^t ds \int_{\mathbb{R}^d} p^2(t-s,x,y)dy$$
$$= \int_0^t p(2s,x,x)ds \asymp \int_0^t s^{-\frac{d}{2m}}ds < \infty \quad \text{if and only if } d < 2m,$$

where $p(t,x,y)$ is the fundamental solution of $\partial_t - \mathcal{A}$. For the first line, we applied Itô isometry for the stochastic integrals:

$$E\left[\left\{\int_0^t \int_{\mathbb{R}^d} \varphi(s,y,\omega)dW(s,y)\right\}^2\right] = E\left[\int_0^t ds \int_{\mathbb{R}^d} \varphi^2(s,y,\omega)dy\right].$$

Coming back to the KPZ equation, the linear SPDE:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \dot{W}(t,x), \quad x \in \mathbb{R},$$

obtained by dropping the nonlinear term has a solution $h \in \bigcap_{\delta>0} C^{\frac{1}{4}-\delta,\frac{1}{2}-\delta}([0,\infty)\times\mathbb{R})$ a.s. (by taking $m = d = 1$). Therefore, there is no way to define the term $(\partial_x h)^2$ in (1.5) in a usual sense. In fact, it requires a renormalization. See (3.4) below. Hairer [3] recently gave a meaning to the KPZ equation (1.5) with $(\partial_x h)^2$ replaced by $(\partial_x h)^2 - \infty$ based on the rough path theory.

3 Cole-Hopf solution and linear stochastic heat equation

Consider the linear stochastic heat equation for $Z = Z(t,x,\omega)$:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x), \quad x \in \mathbb{R}, \tag{3.1}$$

with a multiplicative noise defined in Itô's sense. The solution $Z(t)$ of (3.1) can be defined in a generalized functions' sense or in a mild form:

$$Z(t,x) = \int_{\mathbb{R}} p(t,x,y)Z(0,y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x,y)Z(s,y)dW(s,y),$$

where $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}e^{-(y-x)^2/(2t)}$ is the heat kernel. It is known that these two notions are equivalent, and there exists a unique solution $Z(t)$ such that $Z(t) \in C([0,\infty),\mathcal{C}_{\mathrm{tem}})$ a.s., where

$$\mathcal{C}_{\mathrm{tem}} = \left\{Z \in C(\mathbb{R},\mathbb{R}); \|Z\|_r = \sup_{x\in\mathbb{R}} e^{-r|x|}|Z(x)| < \infty \text{ for every } r > 0\right\},$$

Moreover, a strong comparison theorem is known for (3.1): If $Z(0,x) \geq 0$ for every $x \in \mathbb{R}$ and $Z(0,x) > 0$ for some $x \in \mathbb{R}$, then $Z(t) \in C((0,\infty), \mathcal{C}_+)$ a.s., where $\mathcal{C}_+ = C\big(\mathbb{R}, (0,\infty)\big)$. Therefore, we can define the Cole-Hopf transformation for $Z(t,x)$:

$$h(t,x) := \log Z(t,x). \tag{3.2}$$

Heuristic derivation of the KPZ equation (with renormalization factor $\delta_x(x)$) from the stochastic heat equation (3.1) under the Cole-Hopf transformation (3.2) goes as follows. First we recall Itô's formula for $h = f(Z)$:

$$dh = f'(Z)dZ + \frac{1}{2}f''(Z)(dZ)^2, \tag{3.3}$$

and, from (3.1), since $dW(t,x)dW(t,y) = \delta(x-y)dt$, we can compute as

$$\begin{aligned}(dZ(t,x))^2 &= (ZdW(t,x))^2 \\ &= Z^2\delta_x(x)dt.\end{aligned}$$

Under the Cole-Hopf transformation (3.2), we take $f(z) = \log z$, and noting that $(\log z)' = z^{-1}$ and $(\log z)'' = -z^{-2}$, Itô's formula (3.3) proves that

$$\begin{aligned}\partial_t h &= Z^{-1}\partial_t Z - \tfrac{1}{2}Z^{-2}(\partial_t Z)^2 \\ &= Z^{-1}\left(\tfrac{1}{2}\partial_x^2 Z + Z\dot{W}\right) - \tfrac{1}{2}\delta_x(x) \\ &= \tfrac{1}{2}Z^{-1}\partial_x^2 Z + \dot{W} - \tfrac{1}{2}\delta_x(x).\end{aligned}$$

The second equality follows from (3.1). However, since $h = \log Z$, a simple computation shows that

$$Z^{-1}\partial_x^2 Z = \partial_x^2 h + (\partial_x h)^2.$$

This leads to the KPZ equation with renormalization factor:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x), \quad x \in \mathbb{R}. \tag{3.4}$$

The function $h(t,x)$ defined by (3.2) is meaningful and called the Cole-Hopf solution to the KPZ equation, although the equation (1.5) does not make sense.

4 Main results

It is important to know the asymptotic behavior of the solutions of the KPZ equation as $t \to \infty$. The goal is to give a class of stationary (= invariant) measures.

Let $\mu^c, c \in \mathbb{R}$ be the distribution of $e^{B(x)+cx}, x \in \mathbb{R}$ on $\mathcal{C}_+$, where $B(x)$ is the two-sided Brownian motion such that $\mu^c(B(0) \in dx) = dx$. Let ν^c be the distribution of $B(x) + cx$ on $\mathcal{C}$. Note that these are not probability measures but infinite measures.

Theorem 4.1. *$\{\mu^c\}_{c\in\mathbb{R}}$ are stationary under the stochastic heat equation* (3.1), *i.e., if $Z(0) \overset{law}{=} \mu^c$, then $Z(t) \overset{law}{=} \mu^c$ for all $t \geq 0$ and $c \in \mathbb{R}$.*

Corollary 4.2. *$\{\nu^c\}_{c\in\mathbb{R}}$ are stationary under the Cole-Hopf solution to the KPZ equation.*

Corollary 4.2 is immediate from Theorem 4.1. Note that c means the average tilt of the interfaces, and we have different stationary measures for different average tilts. The proofs are given in [2] based on a method of stochastic analysis.

Remark 4.1. *Since only leading terms are taken in the equation,* (1.5) *has a scale invariance at least at a heuristic level. Recently, Sasamoto and Spohn [6] succeeded to prove the $\frac{1}{3}$-law (instead of the $\frac{1}{2}$-law in usual central limit theorem) for the Cole-Hopf solution of the KPZ equation, which was conjectured by [4], and derived the so-called Tracy-Widom distributions (instead of Gaussian distributions in CLT) in the limit.*

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