On a function space with the hypograph topology

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1 Introduction

The study of topologies on function spaces plays a significant role in geometric functional analysis. Since function spaces are frequently infinite-dimensional, the theory of infinite-dimensional topology has made meaningful contributions to it. Indeed, several function spaces have been shown to be homeomorphic to typical infinite-dimensional spaces. From the end of 1980s to the beginning of 1990s, many researchers investigated topological types of function spaces of real-valued continuous functions on countable spaces endowed with the pointwise convergence topology, see [8]. In this article, we define a hypograph of a map from a compact metrizable space to a dendrite and discuss the topology of the hypograph space. We can consider that hypograph spaces give certain geometric aspect to function spaces with the pointwise convergence topology. This article is a résumé of the joint work with K. Sakai and H. Yang [6].

Throughout the article, all maps are continuous, but functions are not necessarily continuous. Let $X$ be a compact metrizable space and $Y$ be a dendrite with an end point $0$. Recall that a dendrite is a Peano continuum, namely a connected, locally connected, compact metrizable space, containing no simple closed curves. An end point of a space has an arbitrarily small open neighborhood whose boundary is a singleton. It is well-known that each pair of distinct points of a dendrite is connected by the unique arc [12, Chapter V, (1.2)]. We denote the unique arc of two points $x, y$ in the dendrite $Y$ by $[x, y]$, where it is the constant path if $x = y$.

For each function $f: X \to Y$, we define the hypograph $\downarrow f$ of $f$ as follows:

$$\downarrow f = \bigcup_{x \in X} \{x\} \times [0, f(x)] \subset X \times Y.$$  

When $f$ is continuous, the hypograph $\downarrow f$ is closed in $X \times Y$. We denote the set of maps from $X$ to $Y$ by $C(X, Y)$ and the hyperspace of non-empty closed sets in $X \times Y$ endowed with the Vietoris topology by $\text{Cld}(X \times Y)$. Then we have

$$\downarrow C(X, Y) = \{\downarrow f \mid f \in C(X, Y)\} \subset \text{Cld}(X \times Y).$$

Let $\overline{\downarrow C(X, Y)}$ be the closure of $\downarrow C(X, Y)$ in $\text{Cld}(X \times Y)$. In the case that $Y$ is the closed unit interval $I = [0, 1]$ and $0 = 0$, we can regard

$$\downarrow \text{USC}(X, I) = \{\downarrow f \mid f: X \to I \text{ is upper semi-continuous}\}$$

as the subspace in $\text{Cld}(X \times I)$. Let $Q = I^\mathbb{N}$ be the Hilbert cube and $c_0 = \{(x_i)_{i \in \mathbb{N}} \in Q \mid \lim_{i \to \infty} x_i = 0\}$. Z. Yang and X. Zhou [10, 11] showed the following theorem:
Theorem 1.1. Suppose that the set of isolated points of X is not dense. Then \( \downarrow \text{USC}(X, I) = \downarrow C(X, I) \) and the pair \((\downarrow \text{USC}(X, I), \downarrow C(X, I))\) is homeomorphic to \((Q, c_0)\).

For spaces \( W_1 \) and \( W_2 \), the symbol \((W_1, W_2)\) means that \( W_2 \subset W_1 \). A pair \((W_1, W_2)\) of spaces is homeomorphic to \((Z_1, Z_2)\) if there exists a homeomorphism \( f : W_1 \to Z_1 \) such that \( f(W_2) = Z_2 \).

We generalize their result as follows:

Main Theorem. If X is infinite and locally connected, then the pair \((\downarrow C(X, Y), \downarrow C(X, Y))\) is homeomorphic to \((Q, c_0)\).

2 Preliminaries

The topological characterizations for pairs of infinite-dimensional spaces goes back to the uniqueness of cap sets and f-d cap sets due to R.D. Anderson [1], and now, has reached the one of absorbing pairs for each Borel class, refer to [2, 3]. In this section, we shall introduce the notion of strong universality and absorbing pair for the proof of the main theorem. For each open cover \( \mathcal{U} \) of a space \( Z \), a map \( f : W \to Z \) is \( \mathcal{U} \)-close to \( g : W \to Z \) provided that for any \( w \in W \), both of \( f(w) \) and \( g(w) \) are contained in some \( U \in \mathcal{U} \). When \( Z = (Z, d) \) is a metric space, for each \( \epsilon > 0 \), a map \( f \) is \( \mathcal{U} \)-close to \( g \) if \( d(f(w), g(w)) < \epsilon \) for all \( w \in W \). Let \((W_1, W_2)\) be a pair of spaces, and \( C_1 \) and \( C_2 \) be classes of spaces. We say that \((W_1, W_2)\) is strongly \((C_1, C_2)\)-universal if the following condition holds:

(su) Let \( Z_1 \in C_1 \), \( Z_2 \in C_2 \), \( K \) a closed subset of \( Z_1 \), and \( f : Z_1 \to W_1 \) a map such that the restriction \( f|_K \) of \( K \) is a \( Z \)-embedding. Then for every open cover \( \mathcal{U} \) of \( W_1 \), there exists a \( Z \)-embedding \( g : Z_1 \to W_1 \) such that \( g \) is \( \mathcal{U} \)-close to \( f \), \( g|_K = f|_K \) and \( g^{-1}(W_2) \backslash K = Z_2 \backslash K \).

It is said that a closed subset \( A \) of \( W \) is a \( Z \)-set in \( W \) if for each open cover \( \mathcal{U} \) of \( W \), there exists a map \( f : W \to W \) such that \( f \) is \( \mathcal{U} \)-close to the identity map \( \text{id}_W \) and \( f(W) \cap A = \emptyset \). A countable union of \( Z \)-sets is called a \( Z_\sigma \)-set. In addition, a \( Z \)-embedding is an embedding whose image is a \( Z \)-set. A pair \((W_1, W_2)\) is \((C_1, C_2)\)-absorbing provided that the following conditions are satisfied:

1. \( W_1 \in C_1 \) and \( W_2 \in C_2 \);
2. \( W_2 \) is contained in a \( Z_\sigma \)-set in \( W_1 \);
3. \((W_1, W_2)\) is strongly \((C_1, C_2)\)-universal.

Denote the class of compact metrizable spaces by \( M_0 \), and the one of separable metrizable absolute \( F_{\sigma \delta} \)-spaces by \( F_{\sigma \delta} \). According to Theorem 1.7.6 of [3], the following can be established.

Theorem 2.1. Let \( W_1 \) and \( Z_1 \) be topological copies of the Hilbert cube \( Q \). If pairs \((W_1, W_2)\) and \((Z_1, Z_2)\) are \((M_0, F_{\sigma \delta})\)-absorbing, then they are homeomorphic.

The following fact is well known.

Fact 1. The pair \((Q, c_0)\) is \((M_0, F_{\sigma \delta})\)-absorbing.

Combining Theorem 2.1 with Fact 1, we need to show the following conditions:

1. \( \overline{\downarrow C(X, Y)} \) is homeomorphic to \( Q \) and \( \downarrow C(X, Y) \) is an \( F_{\sigma \delta} \)-set in \( \overline{\downarrow C(X, Y)} \);
2. \( \downarrow C(X, Y) \) is contained in a \( Z_\sigma \)-set in \( \overline{\downarrow C(X, Y)} \);
3. \( (\overline{\downarrow C(X, Y)}, \downarrow C(X, Y)) \) is strongly \((M_0, F_{\sigma \delta})\)-universal.
3 The space $\downarrow C(X, Y)$ is homeomorphic to the Hilbert cube

This section is devoted to proving the following theorem:

**Theorem 3.1.** If $X$ has no isolated points, then $\downarrow C(X, Y)$ is homeomorphic to $Q$.

First, we have the following proposition:

**Proposition 3.2.** If $X$ has no isolated points, then $\downarrow C(X, Y)$ is an AR.

*Sketch of proof.* Observe that $\downarrow C(X, Y)$ is a Peano continuum. According to the Wojdyła\-lewski\-skie\-wski theorem [13], see Theorem 5.3.14 of [7], the hyperspace $Cld(\downarrow C(X, Y))$ is an AR. Then we have the retraction

$$\bigcup : Cl\overline{d}(Cl\overline{d}(X \times Y)) \ni A \mapsto \bigcup A \in Cl\overline{d}(X \times Y)$$

and $\bigcup(Cl\overline{d}(\downarrow C(X, Y))) = \downarrow C(X, Y)$. It follows that $\downarrow C(X, Y)$ is a retract of $Cl\overline{d}(\downarrow C(X, Y))$, which implies that $\downarrow C(X, Y)$ is an AR. □

We say that a subset $Z$ is homotopy dense in a space $W$ if there exists a homotopy $h : W \times I \to W$ such that $h(w, 0) = w$ and $h(w, t) \in Z$ for every $w \in W$ and $t > 0$. Using the same technique as [5, Theorem 4.1], we have the following:

**Proposition 3.3.** If $X$ has no isolated points, then $\downarrow C(X, Y)$ is homotopy dense in $\downarrow C(X, Y)$.

Let $d_X$ and $d_Y$ be admissible metrics on $X$ and $Y$, respectively. We use an admissible metric $\rho$ on $X \times Y$ as follows:

$$\rho((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}$$

for each $x, x' \in X$ and $y, y' \in Y$.

Since $X$ and $Y$ are compact, the hyperspace $Cl\overline{d}(X \times Y)$ admits the Hausdorff metric $\rho_H$ induced by $\rho$. For each $A \in Cl\overline{d}(X \times Y)$, we define a set-valued function $A : X \to Cl\overline{d}(Y) \cup \{\emptyset\}$ as follows:

$$A(x) = \{y \in Y \mid (x, y) \in A\} \in Cl\overline{d}(Y) \cup \{\emptyset\}.$$

The following is the key lemma of this article.

**Lemma 3.4 (The Digging Lemma).** Let $\phi : Z \to Cl\overline{d}(X, Y)$ be a map of a paracompact Hausdorff space $Z$. If $X$ has a non-isolated point $x_\infty$, then for each map $\epsilon : Z \to (0, 1)$, there exist maps $\psi : Z \to Cl\overline{d}(X, Y)$ and $\delta : Z \to (0, 1)$ such that for each $z \in Z$,

(a) $\rho_H(\phi(z), \psi(z)) < \epsilon(z)$,

(b) $\psi(z)(x) = \{\emptyset\}$ for all $x \in X$ with $d_X(x, x_\infty) < \delta(z)$.

A space $Z$ has the disjoint cells property provided that for any maps $f, g : Q \to Z$ of the Hilbert cube and any open cover $U$ of $Z$, there exist maps $f', g' : Q \to Z$ such that $f'$ and $g'$ are $U$-close to $f$ and $g$, respectively, and $f'(Q) \cap g'(Q) = \emptyset$.

**Proposition 3.5.** If $X$ has no isolated points, then $\downarrow C(X, Y)$ has the disjoint cells property.
Sketch of proof. Let \( f, g : Q \to \overline{\downarrow C(X, Y)} \) be maps and \( \epsilon > 0 \). Since \( \downarrow C(X, Y) \) is homotopy dense in \( \overline{\downarrow C(X, Y)} \) by Proposition 3.3, we can obtain maps \( f' : Q \to \downarrow C(X, Y) \) that is \( \epsilon \)-close to \( f \), and \( g' : Q \to \downarrow C(X, Y) \) that is \( \epsilon/3 \)-close to \( g \). Taking a non-isolated point \( x_\infty \in X \) and applying the Digging Lemma 3.4, we can find a map \( g'' : Q \to \downarrow C(X, Y) \) such that \( g'' \) is \( \epsilon/3 \)-close to \( g' \) and \( g''(z)(x_\infty) = \{0\} \) for all \( z \in Q \). Define a map \( g''' : Q \to \overline{\downarrow C(X, Y)} \setminus \downarrow C(X, Y) \) as follows:

\[
g'''(z) = g''(z) \cup \{x_0\} \times \{y \in Y \mid d_Y(y, 0) \leq \epsilon/3\}.
\]

Then \( f' \) and \( g''' \) are \( \epsilon \)-close to \( f \) and \( g \), respectively, and \( f'(Q) \cap g'''(Q) = \emptyset \). \( \Box \)

Combining Propositions 3.2 and 3.5 with Toruńczyk's characterization of the Hilbert cube [9], we can obtain Theorem 3.1.

4 The space \( \downarrow C(X, Y) \) is an \( F_{\sigma\delta} \)-set in \( \overline{\downarrow C(X, Y)} \)

In this section, we show the following proposition:

**Proposition 4.1.** The space \( \downarrow C(X, Y) \) is an \( F_{\sigma\delta} \)-set in \( \overline{\downarrow C(X, Y)} \).

**Sketch of proof.** For each \( \delta, \epsilon > 0 \), define \( A(\delta, \epsilon) \subset \overline{\downarrow C(X, Y)} \) as follows:

- \( A \in A(\delta, \epsilon) \) provided that for each \( x_1, x_2 \in X \) with \( d_X(x_1, x_2) < \delta \), if \( y_i \in A(x_i) \) and \( y_i \notin [0, z_i] \) for any \( z_i \in A(x_i) \setminus \{y_i\}, i = 1, 2 \), then \( d_Y(y_1, y_2) \leq \epsilon \).

Then it is closed in \( \overline{\downarrow C(X, Y)} \) and we have

\[
\downarrow C(X, Y) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/m, 1/n).
\]

Hence \( \downarrow C(X, Y) \) is an \( F_{\sigma\delta} \)-set in \( \overline{\downarrow C(X, Y)} \). \( \Box \)

5 The space \( \downarrow C(X, Y) \) is contained in a \( Z_{\sigma} \)-set in \( \overline{\downarrow C(X, Y)} \)

We use the following lemma for detecting \( Z \)-sets in \( \overline{\downarrow C(X, Y)} \).

**Lemma 5.1.** Suppose that \( F = E \cup Z \) is a closed set in \( \overline{\downarrow C(X, Y)} \) such that \( Z \) is a \( Z \)-set in \( \overline{\downarrow C(X, Y)} \), and for each \( A \in E \), there exists a point \( a \in X \) with \( A(a) = \{0\} \). Then \( F \) is a \( Z \)-set in \( \overline{\downarrow C(X, Y)} \).

**Proposition 5.2.** If \( X \) has no isolated points, then \( \downarrow C(X, Y) \) is contained in some \( Z_{\sigma} \)-set in \( \overline{\downarrow C(X, Y)} \).

**Sketch of proof.** Take a countable dense set \( D = \{d_n \mid n \in \mathbb{N}\} \) in \( X \). For each \( n, m \in \mathbb{N} \),

\[
F_{n,m} = \{f \in \downarrow C(X, Y) \mid d_Y(f(d_n), 0) \geq 1/m\}
\]

is a \( Z \)-set in \( \downarrow C(X, Y) \) due to the Digging Lemma 3.4. Then the closure \( F_{n,m} \) is a \( Z \)-set in \( \overline{\downarrow C(X, Y)} \) because \( \downarrow C(X, Y) \) is homotopy dense in \( \overline{\downarrow C(X, Y)} \) by Proposition 3.3. Moreover, we have

\[
F = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} (\downarrow C(X, Y) \setminus F_{n,m}) = \{X \times \{0\}\}.
\]

It follows from Lemma 5.1 that the closure \( F \) is a \( Z \)-set in \( \overline{\downarrow C(X, Y)} \). \( \Box \)
6 The pair $(\downarrow C(X, Y), \downarrow C(X, Y))$ is strongly $(M_0, F_{\sigma\delta})$-universal

We need the following lemma to verify the strong $(M_0, F_{\sigma\delta})$-universality of $(\downarrow C(X, Y), \downarrow C(X, Y))$.

**Lemma 6.1.** Let $x_m, x_\infty \in X$, $m \in \mathbb{N}$, such that $\{r_m = d_X(x_m, x_\infty)\}_{m \in \mathbb{N}}$ is a strictly decreasing sequence converging to 0, and let $y_0 \in Y \setminus \{0\}$ such that $d_Y(0, y_0) \leq 1$. Suppose that $g : Z \to Q$ is an injection from a space $Z$ into the Hilbert cube $Q$ and $\delta : Z \to (0, 1)$ is a map. Then there exists a map $\Phi : Z \to \overline{\downarrow C(X, [0, y_0])}$ satisfying the following conditions:

1. $\Phi$ is injective;
2. $\rho_H(\Phi(z), X \times \{0\}) \leq \delta(z)$ for all $z \in Z$;
3. $\Phi(z)(x) = \{0\}$ for all $x \in X$ with $d_X(x, x_\infty) \geq r_{2k}$ and $z \in Z$ with $2^{-k} \leq \delta(z) \leq 2^{-k+1}$, $k \in \mathbb{N}$;
4. $z \in g^{-1}(c_0)$ if and only if $\Phi(z) \in \downarrow C(X, [0, y_0])$;
5. $\Phi(z)(x_\infty) = \{y \in [0, y_0] \mid d_Y(y, 0) \leq \delta(z)\}$ for all $z \in Z$.

**Proposition 6.2.** If $X$ has no isolated points, then $(\downarrow C(X, Y), \downarrow C(X, Y))$ is strongly $(M_0, F_{\sigma\delta})$-universal.

**Sketch of proof.** Let $Z \in M_0, C \in F_{\sigma\delta}$, $K$ a closed subset of $Z$, $0 < \epsilon$ and $\Phi : Z \to \overline{\downarrow C(X, Y)}$ a map such that $\Phi|_K$ is a $Z$-embedding. We shall construct a $Z$-embedding $\Psi : Z \to \overline{\downarrow C(X, Y)}$ so that $\Psi$ is $\epsilon$-close to $\Phi$, $\Psi|_K = \Phi|_K$ and $\Psi^{-1}(\overline{\downarrow C(X, Y)}) \setminus K = C \setminus K$.

Since $\Phi(K)$ is a $Z$-set in $\overline{\downarrow C(X, Y)}$, we may assume that $\Phi(K) \cap \Phi(Z \setminus K) = \emptyset$. Define $\delta(z) = \min\{\epsilon, \rho_H(\Phi(z), \Phi(K))\}/4$. Since $\downarrow C(X, Y)$ is homotopy dense in $\overline{\downarrow C(X, Y)}$ by Proposition 3.3, there exists $h : Z \to \overline{\downarrow C(X, Y)}$ such that $\rho_H(h(z), \Phi(z)) \leq \delta(z)$ and $h(Z \setminus K) \subset \downarrow C(X, Y)$.

Take a non-isolated point $x_\infty \in X$. By the Digging Lemma 3.4, we can obtain $\psi : Z \setminus K \to \downarrow C(X, Y)$ and $r : Z \setminus K \to (0, 1)$ so that

(a) $\rho_H(h(z), \psi(z)) \leq \delta(z)$,
(b) $\psi(z)(x) = \{0\}$ for all $x \in X$ with $d_X(x, x_\infty) < r(z)$.

Let $Z_k = \{z \in Z \mid 2^{-k} \leq \delta(z) \leq 2^{-k+1}\} \subset Z \setminus K$. Since $x_\infty$ is a non-isolated point, we can choose $x_m \in X \setminus \{x_\infty\}$ so that $r_m = d_X(x_m, x_\infty) < \min\{1/m, d_X(x_{m-1}, x_\infty), r(z) \mid z \in Z_m\}$. Since $(Q, c_0)$ is strongly $(M_0, F_{\sigma\delta})$-universal by Fact 1, we can take an embedding $g : Z \to Q$ so that $g^{-1}(c_0) = C$. Choose $y_0 \in Y \setminus \{0\}$ with $d_Y(0, y_0) \leq 1$. Using Lemma 6.1, we can obtain $\psi' : Z \setminus K \to \overline{\downarrow C(X, [0, y_0])}$ satisfying the following conditions:

1. $\psi'$ is injective;
2. $\rho_H(\psi'(z), X \times \{0\}) \leq \delta(z)$ for all $z \in Z \setminus K$;
3. $\psi'(z)(x) = \{0\}$ for all $x \in X$ with $d_X(x, x_\infty) \geq r_{2k}$ and $z \in Z_k, k \in \mathbb{N}$;
4. $z \in C \setminus K$ if and only if $\psi'(z) \in \downarrow C(X, [0, y_0])$;
5. $\psi'(z)(x_\infty) = \{y \in [0, y_0] \mid d_Y(y, 0) \leq \delta(z)\}$ for all $z \in Z \setminus K$.

Define $\Psi|_{Z \setminus K}$ by $\Psi(z) = \psi(z) \cup \psi'(z)$. □
7 Remarks

In this section, we will give some remarks on the main theorem. For more details, refer to [4]. Z. Yang and X. Zhou [11] proved the stronger result as follows:

**Theorem 7.1.** The pair $(\downarrow \text{USC}(X, I), \downarrow \text{C}(X, I))$ is homeomorphic to $(\mathbb{Q}, c_0)$ if and only if the set of isolated points of $X$ is not dense.

It is unknown whether the same result holds or not in the general case. However, the author [4] shows the following theorem (Z. Yang [10] proved the case that $Y = I$).

**Theorem 7.2.** The space $\downarrow \text{C}(X, Y)$ is a Bare space if and only if the set of isolated points of $X$ is dense.

**Sketch of proof.** The “only if” part follows from the same argument as Section 5. In fact, if the set of isolated points of $X$ is not dense, then $\downarrow \text{C}(X, Y)$ is a $Z_\alpha$-set in itself, and hence it is not a Bare space.

Next, we show the “if” part. Let $X_0$ be the set of isolated points in $X$ and $\mathcal{F}$ be the finite subsets of $X_0$. For each $F \in \mathcal{F}$ and $n \in \mathbb{N}$, we define

$$U_{F,n} = \{A \in \overline{\downarrow \text{C}(X,Y)} | d_Y(y, 0) < 1/n \text{ for all } x \in X \setminus F \text{ and } y \in A(x)\}.$$

Then $U_{F,n}$ is open in $\overline{\downarrow \text{C}(X,Y)}$ and $U_n = \bigcup_{F \in \mathcal{F}} U_{F,n}$ is dense in $\overline{\downarrow \text{C}(X,Y)}$. Observe that the $G_\delta$-set $G = \bigcap_{n \in \mathbb{N}} U_n \subset \downarrow \text{C}(X,Y)$ is a Baire space and dense in $\downarrow \text{C}(X,Y)$. Consequently, $\downarrow \text{C}(X,Y)$ is a Baire space. \( \square \)

**References**


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