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On the Set of Ideal Points in Computable Metric Spaces

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1 Introduction

When extending computability from the discrete to the continuum, it is natural that topology plays the fundamental role. For example, every computable function must be continuous. The theory of computability on the continuum is called computable analysis[5]. As the basic framework of this theory, the concept of computable metric space is important. The use of computer simulations works well for capturing the behavior of many natural phenomena. For the improvement of computer simulation, it is necessary to study the theoretical limits of simulation. Concerning recent survey of this field, we can refer to [2]. We give consideration to the structure of ideal points in computable metric spaces.

The classical computability is defined for partial functions $f :\subset N \to N$ and for subsets of N via Turing machines. A set $E \subset N$ is said to be recursively enumerative (r.e.) if there is an algorithm $\varphi : N \to N$ enumerating E, or equivalently if there exists an algorithm for determining an element n is a member of E. We can translate computability from natural numbers to rational numbers by using an effective numbering $Q = \{q_0, q_1, \ldots, q_n, \ldots\}$.

2 Computable metric space

The computability of real numbers is defined by Turing in 1936. $x \in \mathbf{R}$ is computable if the set $\{i \in N : q_i < x\}$ and $\{i \in N : q_i > x\}$ are r.e. This is equivalent to the following:

Definition 1. A real number $x \in R$ is computable if there is an algorithm $\mathcal{A} : N \to N$ such that $|q_{\mathcal{A}(n)} - x| < 2^{-n}$ for all $n \in N$.

π and e are computable. There are only countably many computable real numbers, since the set of algorithms is a countable set.

Definition 2. A sequence of real numbers $\{x_i\}_{i=0}^{\infty}$ is said to be uniformly computable if there exists an algorithm $\mathcal{A} : N \times N \to N$ such that for any input (n, m), $\mathcal{A}(n, m)$ satisfies $|x_n - q_{\mathcal{A}(n,m)}| < 2^{-m}$.

These concepts are generalized to computable metric spaces. Recently, algorithmic randomness is discussed on computable metric spaces[4].

Definition 3. A computable metric space is a triple (X, d, S), where

- (X, d) is a separable complete metric space.
- $S = \{s_i | i \in N\}$ is a numbered dense subset of X (S is called the set of ideal points).
- The real numbers $(d(s_i, s_j))_{i,j}$ are all computable, uniformly in $i, j \in N$, i.e. there is an algorithm $\mathcal{A} : N \times N \times N \to N$ such that for any $i, j \in N$, $|d(s_i, s_j) - q_{\mathcal{A}(i,j,n)}| < 2^{-n}$ $(n = 0, 1, 2, \dots)$.

Example. As fundamental examples of computable metric space, we can give the following.
(1) (R^n, d_{R^n}, Q^n), where (R^n, d_{R^n}) is the n-dimensional Euclidean space.
(2) For a computable metric space (X, d, S), let $H(X)$ be the set of all nonempty compact subset of X, $F(S)$ be the set of all nonempty finite subset of S. Then $(H(X), h, F(S))$ is a computable metric space, where h is the Hausdorff metric on $H(X)$.
(3) For a computable metric space (X, d, S), let $\mathcal{M}(X)$ be the metric space of all probability Borel measures over X with the Prokhorov metric

$$\pi(\mu, \nu) = \inf\{\epsilon > 0 : \mu(A) \le \nu(A^{\epsilon}) + \epsilon \quad \text{for every Borel set } A\},$$

where $A^{\epsilon} = \{x \in X : d(x, A) < \epsilon\}$. Let $\mathcal{D}$ be the set of those probability measures that are concentrated in finitely many points of S and assign rational values to them. Then $(\mathcal{M}(X), \pi, \mathcal{D})$ is a computable metric space.

Definition 4. Let (X, d, S) be a computable metric space. A point $x \in X$ is computable if there is an algorithm $\mathcal{A} : N \to N$ such that $d(x, s_{\mathcal{A}(n)}) < 2^{-n}$ for all $n \in N$.

Example. It is obvious that a real number $x \in R$ is computable if and only if x is computable in (R, d_R, Q).

Definition 5. A sequence $\{x_i\}$ said to be uniformly computable in (X, d, S) if there exists an algorithm $\mathcal{A} : N \times N \to N$ such that for any input (k, n), $\mathcal{A}(k, n)$ satisfies $d(x_k, s_{\mathcal{A}(k,n)}) < 2^{-n}$.

Let $(X, d_X, S_X), (Y, d_Y, S_Y)$ be computable metric spaces. For a mapping $f : X \to Y$, the computability of f is also defined as the one above for a real function .

Definition 6. A function $\phi : N \to N$ is called an oracle representing $x \in X$ if the condition $d(s_{\phi(n)}, x) < 2^{-n}$ is satisfied for each $n = 0, 1, 2, \ldots$.

Definition 7. Let $f : X \to Y$ be a mapping. If there exists an oracle algorithm $\mathcal{A}^{(\cdot)} : N \to N$ such that for any $x \in X$ and any oracle ϕ representing x, the condition $d(f(x), s_{\mathcal{A}^{\phi}(n)}) < 2^{-n}$ is satisfied for each input $n = 0, 1, 2, \ldots$.

An oracle algorithm is an algorithm which is allowed to refer to elements $\phi(k)$ of the oracle sequence on the way of computation. In other word, a mapping $f : X \to Y$ is computable if and only if the following diagram commutes

$$\begin{array}{ccc} \phi & \longmapsto & \mathcal{A}^{\phi} \\ N^N & \longrightarrow & N^N \\ \downarrow & & \downarrow \ \text{representing} \\ X & \longrightarrow & Y \\ x & \longmapsto & f(x) \end{array}$$

It is well known that computable mappings are continuous.

3 Computable compact set

Concerning computability on compact set, there are several definitions. We adopt here the following.

Definition 8. A compact set $C \subset X$ is said to be computable if C is computable as a point in $(H(X), h, F(S))$.

Penrose's problem: Is the Mandelbrot set computable ?

We can consult [3] about this problem.

As a method of generating fractal figures, the method of iterated function systems is well known. We can show that every figure generated by an IFS is computable if all contraction mappings in this system are computable.

Definition 9. An iterated function system (IFS) $\{X; w_n, n = 1, 2, \ldots, N\}$ is a complete metric space (X, d) together with a finite set of contraction mappings $w_n : X \to X$ [1].

For an IFS $\{X; w_n, n = 1, 2, \ldots, N\}$, the transformation $W : H(X) \to H(X)$ is defined by $W(B) = \cup_{n=1}^{N} w_n(B)$ for all $B \in H(X)$. Since W becomes a contraction mapping, there exists the fixed point A called the attractor which satisfies that $A = \lim_{n\to\infty} W^n(B)$ for any $B \in H(X)$.

Theorem 1. *Let (X, d, S) be a computable metric space. If a computable mapping $f : X \to X$ is contractive, then the fixed point b_f of f is computable.*

Corollary 1. *Let $\{X; w_n, n = 1, 2, \ldots, N\}$ be an IFS on a computable metric space (X, d, S). If w_n $(n = 1, 2, \ldots, N)$ are all computable, then the attractor A of this IFS is computable.*

Remark: Under a slightly different definition of computable compact set, this corollary has been proved for IFS on R^n by H. Kamo and K. Kawamura [4].

4 Addition and subtraction of ideal points

In this section, we study the computable metric spaces obtained by adding or subtracting ideal points.

Theorem 2. *For a uniformly computable sequence $\{x_i\}$ in a computable metric space (X, d, S), let $S' = S \cup \{x_i\}$. Then (X, d, S') is a computable metric space such that $x \in X$ is computable in (X, d, S) if and only if x is computable in (X, d, S').*

In fact, let $S' = \{s'_i : i \in N\}$ where $s'_{2n} = s_n, s'_{2n+1} = x_n$ for $n = 0, 1, \ldots$. It is proved that $(d(s'_i, s'_j))_{i,j}$ are computable uniformly in i, j.

Theorem 3. *Let $\{(X, d, S), f\}$ be a dynamical system consisting of a computable metric space (X, d, S) and a computable mapping $f : X \to X$. Let $S' = \cup_{n=0}^{\infty} f^n(S) = \{f^j(s_i) : i, j \in N\}$. Then S' is uniformly computable. Hence $\{(X, d, S'), f\}$ is a dynamical system on the computable metric space (X, d, S') satisfying $f(S') = S'$.*

It suffices to show that $d(f^{j_1}(s_{i_1}), f^{j_2}(s_{i_2}))_{i_1,j_1,i_2,j_2}$ are all computable, uniformly in i_1, j_1, i_2, j_2.

Let A be a closed subset of a computable metric space(X, d, S). In general, $(A, d|_{A\times A}, S \cap A)$ is not a computable metric space.

Definition 10. Let A be a closed subset of a computable metric space(X, d, S). If there exists a uniformly computable sequence $\{x_n\}$ such that $\{x_n\}$ is dense in A, then A is called a generalized computable closed subset and $(A, d|_{A\times A}, \{x_n\})$ is called a generalized computable closed subspace of (X, d, S).

Theorem 4. *Let C be a computable compact subset of a computable metric space (X, d, S). Then C is a generalized computable closed subset.*

Since $\{B(s_i, q_j) : s_i \in S, q_j \in Q_+\} \sim N \times N$, there exists an effective numbering $\{B_n\} = \{B(s_i, q_j) : s_i \in S, q_j \in Q_+\}$.

Theorem 5. *Let A be a closed subset of (X, d, S). A is a generalized computable closed subset of (X, d, S) if and only if $\{n : B_n \cap A \neq \phi\}$ is r.e.*

Let (X, d, S) be a computable metric space. If $s_i \in S$ is not isolated, then $(X, d, S - \{s_i\})$ is a computable metric space.

Definition 11. Let (X, d, S) be a computable metric space. If any $s_i \in S$ is isolated in (X, d) , then (X, d, S) is called an i-minimum computable metric space.

Theorem 6. *Let (X, d, S) be a computable metric space. Then there exists an i-minimum computable metric space (X', d', S') which satisfies*
(1) (X, d, S) is a generalized computable closed subspace of (X', d', S').
(2) For $x \in X$, x is computable in (X, d, S) if and only if x is computable in (X', d', S').

In fact, X' can be constructed as a subspace of $X \times [0, 1]$. Let
$X' = (X \times \{0\}) \cup \bigcup_{n=1}^{\infty} \{(s_i, 2^{-n}) : i = 0, 1, \ldots, n-1\}$,
$S' = \bigcup_{n=1}^{\infty} \{(s_i, 2^{-n}) : i = 0, 1, \ldots, n-1\}$,
$d'((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), |y_1 - y_2|\}$.
Then (X', d', S') satisfies conditions of theorem 6.

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