Function spaces and isometrical extensions of bounded isometries of separable metric spaces

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1 Introduction

In this note, unless stated otherwise, we assume that all maps are continuous functions. Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ denote the set of integers, the set of natural numbers and the set of real numbers, respectively. Also, let $I, \Delta$ and $\mathbb{Q}$ be the unit interval $[0,1]$, a Cantor set and the Hilbert cube $I^\infty$, respectively. For any compact metric space $Z$, $C(Z)$ denotes the function space of all (continuous) maps from $Z$ to $\mathbb{R}$ with the supremum metric $\tilde{d}$, i.e.,

$$\tilde{d}(f,g) = \sup\{|f(z) - g(z)| \mid z \in Z\}$$

for $f, g \in C(Z)$.

A map $i : (X,d_X) \rightarrow (Y,d_Y)$ between separable metric spaces is an isometrical embedding from $(X,d_X)$ into $(Y,d_Y)$ if $i$ satisfies the condition $d_Y(i(x),i(x')) = d_X(x,x')$ for each $x, x' \in X$. A map $g : (X,d_X) \rightarrow (Y,d_Y)$ between separable metric spaces is an isometry if $g$ is surjective and $d_Y(g(x),g(x')) = d_X(x,x')$ for each $x, x' \in X$. For a separable metric space $(X,d)$, let $\text{Iso}(X)$ be the group of all isometries of $X$ equipped with the pointwise convergent topology, i.e.,

$$\text{Iso}(X) = \{g : X \rightarrow X \mid g \text{ is an isometry}\}.$$ 

A well-known theorem of Banach and Mazur is the result that $C(I)$ ($I = [0,1]$) is a universal space of separable metric spaces up to isometry (see [1,3,9]). Also, Urysohn [11] constructed a complete separable metric space $\mathbb{U}$ that is also universal up to isometry. In [12], Uspenskij proved that for any separable metric space $X$ there is a natural isometrical embedding $i : X \rightarrow \mathbb{U}$ such that $i$ induces a natural continuous monomorphism $i^* : \text{Iso}(X) \rightarrow \text{Iso}(\mathbb{U})$ satisfying that $i^*(g) \in \text{Iso}(\mathbb{U})$ is an extension of $g \in \text{Iso}(X)$ (see [2,3,5,7,12,13] for more detailed properties of $\mathbb{U}$).

In this note, we study the extension property of "bounded" isometries of separable metric spaces in function spaces $C(\mathbb{Q})$ and $C(\Delta)$. Also, we know that $C(I)$ does not have the extension property. Let $(X,d)$ be a separable metric space and $x_0 \in X$. A subgroup $G$ of $\text{Iso}(X)$ is bounded if diam $G(x_0) < \infty$, where $G(x_0) = \{g(x_0) \mid g \in G\}$ ($\subset X$). The definition of "bounded subgroup" of $\text{Iso}(X)$ does not depend on the choice of the point $x_0 \in X$. Also, each $g \in \text{Iso}(X)$ is bounded if diam$\{g^n(x_0)\}_{n \in \mathbb{Z}} \subset \mathbb{N} = \infty$. Note that if $(X,d)$ is bounded, i.e., diam$X < \infty$, then $\text{Iso}(X)$ itself is bounded. In particular, if $X$ is a compact metric space, then $\text{Iso}(X)$ is bounded. In [6], Mazur and Ulam proved that if $B$ and $B'$ are Banach spaces, then every isometry $T : B \rightarrow B'$ with $T(0) = 0$ is linealy...
isometric and moreover, Banach and Stone proved that if $X$ and $Y$ are compact Hausdorff spaces, then every isometry $T : C(X) \rightarrow C(Y)$ with $T(0) = 0$ is linear isometric and moreover, $T$ is induced by a homeomorphism $h : Y \rightarrow X$ (see [1,10]).

**Theorem 1.1.** (Banach [1] and Stone [10]) Let $X$ and $Y$ be compact Hausdorff spaces. Then the followings hold.

(1) $C(X)$ is isometric to $C(Y)$ if and only if $X$ is homeomorphic to $Y$.

(2) If $T : C(X) \rightarrow C(Y)$ is a linear isometry, then there is a homeomorphism $h : Y \rightarrow X$ and a (continuous) map $\alpha : Y \rightarrow \mathbb{R}$ with $|\alpha(y)| = 1$ for $y \in Y$ such that

$$(T(f))(y) = \alpha(y) \cdot (f \circ h)(y)$$

for $f \in C(X)$ and $y \in Y$. Moreover, if $Y$ is connected, $T(f) = f \circ h$ or $T(f) = -(f \circ h)$.

For any Banach space $B$, let

$$\text{LinIso}(B) = \{f \in \text{Iso}(B) | f \text{ is linear} \}.$$ 

Note that $\text{LinIso}(B)$ is bounded, because $\text{LinIso}(B)(0) = \{0\}$. 

## 2 Extensions of bounded isometries in function spaces

In this section, we assume that $(X, d)$ is a separable metric space and $x_0$ is a fixed point of $X$. In [9], Sierpiński considered the space

$$X' = \{f : X \rightarrow \mathbb{R} | f(x_0) = 0 \text{ and } |f(x) - f(y)| \leq d(x, y) \text{ for } x, y \in X\}$$

which is a topological space equipped with the pointwise convergent topology (see also [3]) and by use of the spaces $X'$, he proved that $C(I)$ is a universal space of separable metric spaces up to isometry. We modify the Sierpiński's method of [9]. In this paper, for any bounded subgroup $G$ of $\text{Iso}(X)$, we consider the following more general space

$$\tilde{X}(= \tilde{X}_G) = \{f : X \rightarrow \mathbb{R} | f(z) \in [-\text{diam}(G(x_0)), \text{diam}(G(x_0))] \text{ for } z \in G(x_0) \text{ and } |f(x) - f(y)| \leq d(x, y) \text{ for } x, y \in X\}$$

which is a topological space equipped with the pointwise convergent topology. We have the following lemmas.

**Lemma 2.1.** $\tilde{X}(= \tilde{X}_G)$ is a compact metric absolute retract (=$\text{AR}$). Moreover, if $g \in G$, then $\tilde{g} : \tilde{X} \rightarrow \tilde{X}$ is a homeomorphism, where $\tilde{g}$ is defined by $\tilde{g}(f) = f \circ g$ for $f \in \tilde{X}$.

**Lemma 2.2.** Suppose that $p_G : Z \rightarrow \tilde{X}(= \tilde{X}_G)$ is a map from a compact metric space $Z$ onto $\tilde{X}$ such that for each $g \in G$ there is a (lift) homeomorphism $L_g : Z \rightarrow Z$ satisfying the following commutative diagram.

$$
\begin{array}{ccc}
Z & \xrightarrow{L_g} & Z \\
p_G \downarrow & & \downarrow p_G \\
\tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X}
\end{array}
$$
Then there is an isometrical embedding $i_G : X \rightarrow C(Z)$ such that for each $g \in G$, the following commutative diagram holds:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
i_G \downarrow & & \downarrow i_G \\
C(Z) & \xrightarrow{\tilde{L}_g} & C(Z)
\end{array}
\]

where $\tilde{L}_g : C(Z) \rightarrow C(Z)$ is the isometry defined by $\tilde{L}_g(f) = f \circ L_g$ for $f \in C(Z)$. In particular, $\tilde{L}_g \in LinIso(C(Z))$ is an isometrical extension of $g \in G$.

Here we have the following theorem of $C(\mathbb{Q})$ which implies that $C(\mathbb{Q})$ is universal concerning isometrical extensions of bounded isometry groups of separable metric spaces.

**Theorem 2.3.** Let $(X, d)$ be a separable metric space and let $G$ be any bounded subgroup of $Iso(X)$. Then there is an isometrical embedding $i_G : X \rightarrow C(\mathbb{Q})$ such that $i_G$ induces a continuous monomorphism $i^*_G : G \rightarrow LinIso(C(\mathbb{Q}))$ such that $i^*_G(g) \in LinIso(C(\mathbb{Q}))$ is an extension of $g \in G$.

**Corollary 2.4.** Suppose that $(X, d)$ is a bounded separable metric space. Then there is an isometrical embedding $i : X \rightarrow C(\mathbb{Q})$ such that $i$ induces a continuous monomorphism $i^* : Iso(X) \rightarrow LinIso(C(\mathbb{Q}))$ such that $i^*(g) \in LinIso(C(\mathbb{Q}))$ is an extension of $g \in Iso(X)$.

Remark 1. Note that for any Banach space $B$, $LinIso(B)$ is a bounded group. Hence in this note, we can not omit the condition that $G$ is bounded.

If we observe the proof of Lemma 2.2, we see that some converse assertions of Lemma 2.2 are also true. In fact, we have the following.

**Proposition 2.5.** Suppose that $p_G : Z \rightarrow \bar{X}(= \bar{X}_G)$ is a map from a compact metric space $Z$ onto $\bar{X}$, $i_G : X \rightarrow C(Z)$ is the isometrical embedding as in the proof of Lemma 2.2 and $g \in G$. Let $L_g : Z \rightarrow Z$ be a homeomorphism. Then the followings hold.

1. The following diagram is commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{L_g} & Z \\
p_G \downarrow & & \downarrow p_G \\
\bar{X} & \xrightarrow{\bar{g}} & \bar{X}
\end{array}
\]

if and only if the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
i_G \downarrow & & \downarrow i_G \\
C(Z) & \xrightarrow{\tilde{L}_g} & C(Z)
\end{array}
\]

2. The following diagram is commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{L_g} & Z \\
p_G \downarrow & & \downarrow p_G \\
\bar{X} & \xrightarrow{\bar{g}} & \bar{X}
\end{array}
\]
if and only if the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow i_G & & \downarrow i_G \\
C(Z) & \xrightarrow{L_g} & C(Z)
\end{array}
\]

Example. Let \( X = \{x_i \mid i = 0, 1, 2\} \) be the set of three elements and let \( d \) be the metric on \( X \) defined by \( d(x_i, x_j) = r > 0 \) \( (i \neq j) \). Define the isometry \( g : X \to X \) by \( g(x_0) = x_0, g(x_1) = x_2 \) and \( g(x_2) = x_1 \). Let \( G = \{id_X, g\} \). Note that \( G(x_0) = \{x_0\} \). Then there is an isometrical embedding \( i_G : X \to C(\mathbb{Q}) \) such that there is no isometrical extension of \( g \) on \( C(\mathbb{Q}) \). In particular, \( C(\mathbb{Q}) \) is not equal to the Urysohn universal space \( U \), because that \( U \) has the following strong property: Any isometry between finite subsets of \( U \) can be extended to an isometry of \( U \).

Next we will consider the case of the function space \( C(\Delta) \). Let \( H(X) \) be the set of all homeomorphisms of a space \( X \).

**Proposition 2.6.** Let \( X \) be a compact metric space and let \( G \) be a countable subset of \( H(X) \). Then there is an onto map \( p_G : \Delta \to X \) such that for any \( g \in G \) there is a (lift) homeomorphism \( L_g : \Delta \to \Delta \) of \( \Delta \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
\Delta & \xrightarrow{L_g} & \Delta \\
p_G & & \downarrow p_G \\
X & \xrightarrow{g} & X
\end{array}
\]

Then we have the following theorem of \( C(\Delta) \).

**Theorem 2.7.** Let \((X, d)\) be any separable metric space and let \( G \) be a countable bounded subgroup of \( Iso(X) \). Then there is an isometrical embedding \( i_G : X \to C(\Delta) \) such that there exist a countable subgroup \( G^* \) of \( LinIso(C(\Delta)) \) and a continuous epimorphism \( r^* : G^* \to G \) such that each \( g^* \in G^* \) is an extension of \( r^*(g^*) \in G \). In particular, if \( g \in G \), then there is an extension \( g^* \in LinIso(C(\Delta)) \) of \( g \).

Remark 2. Note that the space \( H(\Delta) \) of all homeomorphisms of \( \Delta \) is homeomorphic to the space \( P \) of irrationals, and hence \( H(\Delta) \) is zero-dimensional. If \( G \) is any bounded subgroup of \( Iso(X) \) with \( \dim G \geq 1 \), there is no embedding from \( G \) to \( H(\Delta) \).

**Corollary 2.8.** Let \((X, d)\) be any separable metric space. If \( g \in Iso(X) \) is periodic i.e., \( g^n = id_X \) for some \( n \in \mathbb{N} \), then there is an isometrical embedding \( i_g : X \to C(\Delta) \) such that there is an extension \( g^* \in LinIso(C(\Delta)) \) of \( g \) with \( (g^*)^n = id_{C(\Delta)} \).

Finally, we consider the case of \( C(I) \). We have the following proposition of \( C(I) \).

**Proposition 2.9.** Let \((X, d)\) be any separable metric space and let \( g \in Iso(X) \) such that \( g \) has a periodic point \( x_0 \) with period \( n \in \mathbb{N} \). If \( n \geq 3 \), there is no isometrical embedding \( i \) from \( X \) to \( C(I) \) such that \( g \) has an extension in \( LinIso(C(I)) \).

Now, we have the following problem.

**Problem 2.10.** Let \((X, d)\) be any separable metric space. Is it true that there is an isometrical embedding \( i \) from \( X \) to \( C(\mathbb{Q}) \) such that each \( g \in Iso(X) \) has an extension which is an affine isometry of \( C(\mathbb{Q}) \) ?
References


