数理解析研究所講究録
第 1886 巻 2014 年 146-153

A portfolio model for the risk management in public pension*

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Abstract

The financial sustainability of public pension requires the reserve should be positive to pay the benefit in the demographic and economical environment change subject to the certain level of the income replacement ratio. Assuming the market asset and the income for pension follows Ito processes and we maximize the net present value of pension for the cohort, To guarantee the pension fund sustainability, we apply the martingale method of the optimal consumption and investment theory. We use the age-structured model to the pension population change.

Keywords: public pension, portfolio risk management, population cohort

1 Introduction

1.1 Risk management for a public pension

Welfare nations have been suffering the robust management of public pension as seen the reform in [4] and [5]. Japanese government has announced every five years the actuarial valuation of pension plan for 100 years, as seen [6]. It reviewed the long term financial viability under significant changes in the demographic and economical environment. The financial viability implies that the reserve of pension fund should be positive, under the condition that the replacement ratio is more the 50 %, which means that the average amount of pension is more than the average income of contributors.

The aging society with the low birth rate makes worse the balance of pension account in the near future with the prolonged deflationary economy in Japan.

Economic scenarios for the simulation are the rate of return of investment, the wage as the key factor of contribution and benefit, and inflation rate and interest rate.

The total pension fund was 144.4 trillion yen at 2009 and has decreased to 120 trillion at 2012. There exists the ambiguity in population projection by Government survey [6] as Table 1.1. Figure1.1 is the governmental projections of populations of pension.

The long term economic scenarios were assumed as Table1.1. In 2004 the government reform plan had set the maximum premium rate as 18.3% and benefit cutoff by the rate of decreasing contributors and aging rate. Their plan starts from 2012 and will end 2038.

1.2 A portfolio formulation for pension risk management

The objective satisfies the following considerations; 1) to minimize the government subsidy to pension fund, 2) to maximize the net present value of pension premium and benefit, 3) to

*This research was supported in part by the Grant-in-Aid for Scientific Research (No. 25350461) of the Japan Society for the Promotion of Science in 2013–2015.

Projection of 2055	2006 estimates	2012 estimate
Fertility	1.26	1.35
Male life expectancy	83.67	84.19
Female lefe expectancy	90.34	90.93

Table 1: Population projections

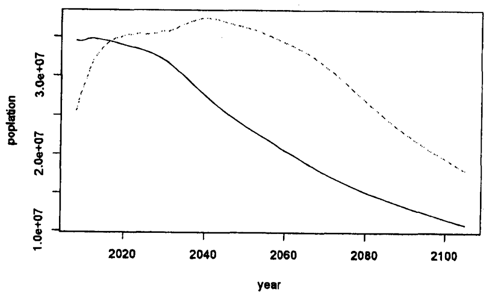

Figure 1: contributors(—) and beneficiaries (- - -)

minimize the difference among generations in the net values, 4) to perform income redistribution to the poor elderly.

The constraints are followings; 1) no default of pension system for 100 years, 2) The reserve fund at the end of plan should be enough to continue the system. 3) The income replacement ratio should be more than 50%.

2 The model

2.1 Policyholders

Let $p(t, y)$ denote the numbers of policyholders of the age y at t and ω_1 the starting age of paying premium, ω_2 the end age and starting of receiving benefit ω_3 the end age of beneficiary.

%	Inflation	wage	ROR	100years ratios
Scenario	π	x_l	r_l	-
High	1.0	1.9	3.2	61.2
Middle	1.0	1.5	3.1	55.6
Low	1.0	1.1	2.9	45.9
Abenomics	2.0	1.9	3.2	159.1

Table 2: Economic scenarios of 100 years

The total number of contributors satisfies:

$$\xi_t^1 = \int_{\omega_1}^{\omega_2} p(t, y) dy \tag{2.1}$$

The total number of beneficiaries:

$$\xi_t^2 = \int_{\omega_2}^{\omega_3} p(t, y) dy \tag{2.2}$$

The balance of total premium and benefit is assumed to be based on the average wage. Let H_t denote the average wage at t and a_t be the rate of premium. Let u_t be the total premium amount at t;

$$u_t := a_t H_t \xi_t^1$$

Let b_t be the benefit ratio to the average wage and s_t be the total benefit amount;

$$s_t := b_t H_t \xi_t^2.$$

We assume that a_t, b_t is predictable process and it satisfies self-finance strategies, which are in the condition of $0 < a_t, b_t < 1$. The balance of premium and benefit $q_t(a, b)H_t$ is defined as:

$$u_t - s_t = (a_t \xi_t^1 - b_t \xi_t^2) H_t =: q_t(a, b) H_t.$$

2.2 The portfolio

The pension portfolio consists of three assets; Market asset price satisfies the following Itô process:

$$dA_t / A_t = \mu_r(t) dt + \sigma_r(t) dW_t^r =: dr_t,$$

Human capital price(wage) process satisfies:

$$dH_t / H_t = \mu_x(t) dt + \sigma_x(t) dW_t^x =: dx_t,$$

The risk free rate r and the money market account e^{rt}.

Portfolio strategies of the pension fund are denoted as $\pi_t := (\phi_t, a_t, b_t, \beta_t)$; Let ϕ_t denote the investment amount of market asset, (a_t, b_t) denote the strategies for the human capital which means the policy of pension. Let $\beta_t > 0$ denote the government subsidy to pension fund at t and R_t denote the value of pension fund at t.

The portfolio value satisfies at t:

$$R_t = \phi_t A_t + q(a, b) H_t \tag{2.3}$$

The dynamics of fund is due to the predictability of strategies (ϕ_t, β_t):

$$dR_t = \phi_t dA_t + d(q(a, b) H_t) + \beta_t dt, \quad R_0 = \bar{R} \tag{2.4}$$

The population dynamics is assumed to be non stochastic but not satisfies self financing condition then the dynamics of the balance of premium and benefit of pension is as follows,

$$\begin{aligned} d(q(a, b) H_t) &= a_t (d\xi_t^1 H_t + \xi_t^1 dH_t) - b_t (d\xi_t^2 H_t + \xi_t^2 dH_t) \\ &= (a_t \xi_t^1 - b_t \xi_t^2) dH_t + (a_t d\xi_t^1 - b_t d\xi_t^2) H_t \end{aligned}$$

Let define the change of balance due to change of numbers of contributors and beneficiaries; $dq(a,b) := a_t d\xi_t^1 - b_t d\xi_t^2$, then

$$d(q(a,b)H_t) = q(a,b)dH_t + dq(a,b)H_t.$$

Subsitute it to (2.4) then

$$dR_t = \phi_t A_t dr_t + q(a,b)H_t dx_t + dq(a,b)H_t + \beta_t dt.$$

By using (2.3) we obtain:

$$dR_t = R_t dr_t + q(a,b)H_t(dx_t - dr_t) + dq(a,b)H_t + \beta_t dt.$$

From (2.1) and(2.2),

$$\begin{aligned} dq(a,b) &= a_t d\xi_t^1 - b_t d\xi_t^2 \\ &= \{a_t \int_{\omega_1}^{\omega_2^-} \frac{\partial p(t,y)}{\partial t} dy - b_t \int_{\omega_2}^{\omega_3} \frac{\partial p(t,y)}{\partial t} dy\} dt \end{aligned}$$

From the PDE of McKendrick-von Foerster

$$\frac{\partial p(t,y)}{\partial t} = -\frac{\partial p(t,y)}{\partial y} - \mu(t,y)p(t,y), \tag{2.5}$$

where $\mu(t,y)$ is decreasing speed of pension participant at t of the age y.

2.3 The cohort modeling

We use the method of characteristics in PDE which equals to use the cohort model of population.

Let $t = k + y$ and assume $\mu(y) = \mu(t,y)$ which means the decreasing number depends only the age. Let $v(k,y) := p(t,k)$ then From (2.5) we get:

$$dv(k,y) = -\mu(y)v(k,y)dy.$$

and the solution is

$$v(k,y) = v(k,0)\exp(\int_0^y \mu(s)ds)$$

In Figure 2.3 is seen the cohort model in a public pension plan for 100 years horizon and the ages of contribution (ω_1, ω_2) and benefit (ω_2, ω_3). The change of contributor is from (2.1):

$$d\xi_t^1 = -\int_{\omega_1}^{\omega_2^-} \mu(y)v(k,y)dy,$$

and the change of beneficiaries is similarily:

$$d\xi_t^2 = -\int_{\omega_2}^{\omega_3} \mu(y)v(k,y)dy.$$

Thus the change of pension balance satisfies ;

$$\begin{aligned} dq_t(a,b)H_t &= (a_t d\xi_t^1 - b_t d\xi_t^2)H_t \\ &= -\left(a_{k+y}\int_{\omega_1}^{\omega_2^-} \mu(y)v(k,y)dy - b_{k+y}\int_{\omega_2}^{\omega_3} \mu(y)v(k,y)\right)H_t, \end{aligned} \tag{2.6}$$

where $p(k+\omega_1, \omega_1)$ is the new entry numbers of contributors and $p(k+\omega_2, \omega_2)$ is the new entry umbers of beneficiaries.

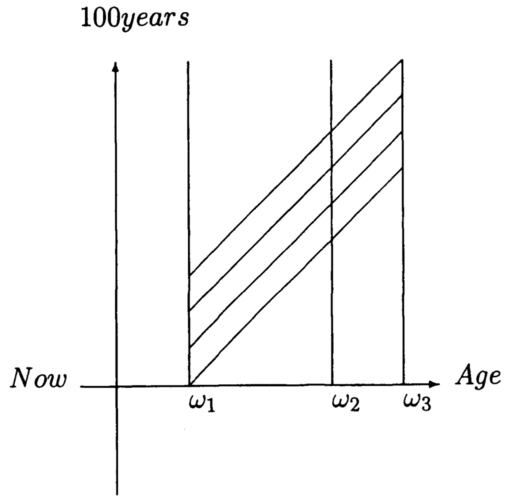

Figure 2: Cohorts and 100 year pension plan

3 The pension strategies

3.1 The present value of pension for a cohort

For the time horizon from 0 to T there are k cohort of $0 \leq k \leq T_l := T - (\omega_3 - \omega_1)$ who are their all contributions and benefits are within the planning period. These cohorts are new comers to the pension.

$$c(k) := -\int_{\omega_1}^{\omega_2^-} a_{y+k} H^*_{y+k} v(k,y) dy + \int_{\omega_2}^{\omega_3} b_{y+k} H^*_{y+k} v(k,y) dy \tag{3.1}$$

For existing pensionaries ($-\omega_3 < k < 0$), their premium and benefit was decided as a_c and b_c for the past :$y + k < 0$; They will follow the new premium and benefit from now $y + k \geq 0$. The net present value $c_e(k)$ should be positive;

$$\begin{aligned} c_e(k) = &-\int_{\omega_1}^{\omega_2^-} (a_c \mathbf{1}_{\{y+k<0\}} + a_{y+k} \mathbf{1}_{\{y+k\geq 0\}}) H^*_{y+k} v(k,y) dy \\ &+ \int_{\omega_2}^{\omega_3} (b_c \mathbf{1}_{\{y+k<0\}} + b_{y+k} \mathbf{1}_{\{y+k\geq 0\}}) H^*_{y+k} v(k,y) dy. \end{aligned} \tag{3.2}$$

For the future cohort ($T_l < k < T$) whose benefit will not finished before T, their net present value $c_p(k)$ should be positive;

$$\begin{aligned} c_p(k) = &-\int_{\omega_1}^{\omega_2^-} (a_{k+y} \mathbf{1}_{\{y+k<T\}} + \tilde{a}_{y+k} \mathbf{1}_{\{y+k\geq T\}}) H^*_{y+k} v(k,y) dy \\ &+ \int_{\omega_2}^{\omega_3} (b_{y+k} \mathbf{1}_{\{y+k<T\}} + \tilde{b}_{y+k} \mathbf{1}_{\{y+k\geq T\}}) H^*_{y+k} v(k,y) dy. \end{aligned} \tag{3.3}$$

The past balance of premium and benefit are accumulated as a part of the present reserve fund R_0;

$$\int_{-\omega_3}^{0} \{ \int_{\omega_1}^{\omega_2^-} a_c \mathbf{1}_{\{y+k<0\}} H^*_{y+k} v(k,y) dy - \int_{\omega_2}^{\omega_3} b_c \mathbf{1}_{\{y+k<0\}} H^*_{y+k} v(k,y) dy \} dk < R_0$$

3.2 The optimal problem of public pension

The objective function is to maximize the utility function of the new pension participant who are the cohort of $0 \leq k \leq T_l$, where $U_1(\cdot)$ is a utility function for present value of pension and $U_2(\cdot)$ is the utility of fund value at T:

$$\max_{\pi_t} E[\int_0^{T_l} U_1(c(k)dk + U_2(R_T))]$$

Beside constraints (3.2) and (3.3), we impose the following constraints seen in [5];

(1) No default of pension fund, which should satisfies the following;

$$R_t > 0 \quad \forall t \in [0, T],$$

(2) Government subsidy γ_t should be within the limitation;

$$E^Q[\int_t^T e^{-rs}\beta_s ds|\mathcal{F}_t] \leq \gamma_t.$$

At the planning time it is as follows;

$$E^Q[\int_0^T e^{-rs}\beta_s ds] \leq \gamma_0$$

(3) The economic rationality to hold the pension;

For the existing policyholders;

$$E^Q[c_e(k)] > 0, \quad k \in (-\omega_3, 0)$$

For the future policyholders;

$$E^Q[c_p(k)] > 0, \quad k > T - (\omega_3 - \omega_1)$$

4 Pension portfolio process under the risk neutral probability

4.1 Martingale method for the risk management

The risk management of pension should be no default which implies that $R_t > 0$ for all $t \in (0, T)$. It can be treated by the martingale method of optimal investment and consumption problem of Dana-Jeanblanc [2] ,pp137-144.

Let e^{rt} a numeraire, from (2.3) and (2.4)

$$dR_t e^{-rt} - R_t re^{-rt}dt = (\phi_t dA_t + q(a,b)dH_t + dq(a,b)H_t + \beta_t dt)e^{-rt} - (\phi_t A_t + q(a,b)H_t)re^{-rt} \quad (4.1)$$

Then

$$d(R_t/e^{rt}) = \phi_t d(A_t/e^{rt}) + q(a,b)d(H_t/e^{rt}) + dq(a,b)H_t e^{-rt} + \beta_t e^{-rt}dt$$

Let denote $R_t^* := R_t/e^{rt}, A_t^* = A_t/e^{rt}, H_t^* = H_t/e^{rt}$, then

$$dR_t^* = \phi_t dA_t^* + q(a,b)dH_t^* + dq(a,b)H_t^* + \beta_t e^{-rt}dt$$

The reserve fund at T becomes as follows:

$$R_T^* = R_0 + \int_0^T \phi_t dA_t^* + \int_0^T q(a,b)dH_t^* + \int_0^T H_t^* dq(a,b) + \int_0^T \beta_t e^{-rt}dt, \quad (4.2)$$

where H_t^* and A_t^* are martingales under the risk neutral measure Q.

4.2 The admissible strategies

The admissible strategies satisfying the constraint $R_t > 0$ as Dana et al [2]. We assume that the government subsidy to the pension fund should satisfies $E^{Q}[\int_t^T \beta_s ds + \int_t^T H_s dq(a,b)] \leq 0$. It is the assumption that the government subsidy is less than the deficit due to the population change of the aging with low fertility. Another assumption which is taken from Japanese pension report is that the final reserve fund covers the pension benefit;

$$R_T = H_T b_t \xi_t^2 \tag{4.3}$$

From (4.2)

$$R_t^* - \int_0^t H_s^* dq(a,b) - \int_0^t \beta_t e^{-rs} ds = R_0 + \int_0^t \phi_s dA_s^* + \int_0^t q(a,b) dH_s^* =: M_t \tag{4.4}$$

M_t is a positive Q-martingale which is under the risk neutral probability. Then

$$R_t^* = M_t + \int_0^t H_s^* dq(a,b) + \int_0^t \beta_t e^{-rs} ds \tag{4.5}$$

Using the conditional expectation of the martingale

$$\begin{aligned} R_t^* &= \underbrace{E^Q[R_T^* - \int_0^T H_s^* dq(a,b) - \int_0^T \beta_s e^{-rs} ds|\mathcal{F}_t]}_{M_t} + \int_0^t H_s^* dq(a,b) + \int_0^t \beta_s e^{-rs} ds \\ &= E^Q[R_T^* - \int_t^T H_t^* dq(a,b) - \int_t^T \beta_t e^{-rs} ds|\mathcal{F}_t] \end{aligned}$$

From the assumption $E^{Q}[\int_t^T \beta_s ds + \int_t^T H_s dq(a,b)] \leq 0$. H_t^* is Q-martingale then $E^Q[H_T^*] = H_0 > 0$. Therefore the the terminal reserve $E^Q[R_T^*] > 0$ by (4.3). From these two conditions

$$R_t > 0, \forall t \in (0,T)$$

and

$$E^Q[R_T^* - \int_0^T H_s^* dq(a,b)] \leq R_0 + \gamma_0$$

We restate the above as the following theorem;

Theorem 1 *if $E^{Q}[\int_t^T \beta_s ds + \int_t^T H_s dq(a,b)] \leq 0$ and if the final reserve satisfies (4.3), then*

$$R_t > 0, \forall t \in (0,T)$$

4.3 The optimal problem

From the consequence of (4.3), set the objective function as

$$\max_{\pi_t} E[\int_0^{T_l} U_1(c(k))dk].$$

The first constant is

$$E^Q[\int_t^T \beta_s ds + \int_t^T H_s dq(a,b)] \leq 0.$$

The second one is

$$E^Q[R_T^* - \int_0^T H_s^* dq(a,b)] \leq R_0 + \gamma_0.$$

Define the Radon-Nikodim derivative for the risk neutral measure: $L_t = dQ/dP|_{\mathcal{F}_t}$ then the constraint becomes under the original probability measure;

$$E[L_T R_T^* - \int_0^T L_t H_s^* dq(a,b)] \leq R_0 + \gamma_0$$

The third constraints are For the existing policyholders;

$$E^Q[c_e(k)] > 0, \quad k \in (-\omega_3, 0)$$

For the future policyholders;

$$E^Q[c_p(k)] > 0, \quad k > T - (\omega_3 - \omega_1)$$

This optimal problem will be solved in discretized version.

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