# Generic structures and model completeness

法政大学経営学部　池田　宏一郎*

Koichiro Ikeda

Faculty of Business Adiministration, Hosei University

The following question was raised at the seminar with Kikyo last year.

**Question 0.1** Is Hrushovski's strongly minimal structure ([5]) model complete?

I tried to solve this question using either one of the following two methods:

1. A characterization of model completeness for generic structures (Theorem 2.2);

2. Lindström's theorem (Fact 1.3).

Later, however, I found that in [4] Holland has already solved the question using Lindström's theorem. In this short note, we explain Theorem 2.2 and its application, and add a few questions.

# 1    Preliminaries

**Definition 1.1** A theory $T$ is said to be model complete, if whenever $M, N \models T$ and $M \subset N$, then $M \prec N$.

Let $T$ be a complete theory and $\mathcal{M}$ a big model. For $\bar{a} \in \mathcal{M}$, we denote $\mathrm{tp}_\exists(\bar{a}) = \{\psi(\bar{x}) : \mathcal{M} \models \psi(\bar{a}), \psi \text{ is an } \exists\text{-formula}\}$. The following is well-known.

**Note 1.2** $T$ is model complete $\Leftrightarrow$ for any $\bar{a} \in \mathcal{M}$, $\mathrm{tp}_\exists(\bar{a}) \vdash \mathrm{tp}(\bar{a})$.

The following theorem is known as a test for model completeness. .

**Fact 1.3 (Lindstörem)** Let $T$ be a $\forall\exists$-theory. If $T$ is $\lambda$-categorical for some $\lambda$, then it is model complete.

In what follows, we briefly explain the basics of generic structures. (For more details, see [3, 6]).

Let $L$ be a countable relational language, and let each $R \in L$ be reflexive and symmetric. Let $A, B, C, ...$ denote $L$-structures.

For each $R \in L$, let $\alpha_R \in (0, 1]$ be a real number. Then a predimension $\delta(A)$ of a finite $L$-structure $A$ is defined by

$$\delta(A) = |A| - \sum_{R \in L} \alpha_R |R^A|.$$

We denote $\delta(B/A) = \delta(B \cup A) - \delta(A)$.

For finite $A \subset B$, $A$ is said to be closed in $B$ (in symbol, $A \leq B$), if $\delta(X/A) \geq 0$ for any $X \subset B - A$. When $A, B$ are not necessarily finite, $A \leq B$ is defined by $A \cap X \leq X$ for any finite $X \subset B$.

For $A \subset B$, there is the smallest set $C \leq B$ containing $A$. We denote such a $C$ $\mathrm{cl}_B(A)$.

Let $\mathbf{K}^* = \{A \text{ finite} : \delta(A') \geq 0 \text{ for all } A' \subset A\}$.

**Definition 1.4** Let $\mathbf{K} \subset \mathbf{K}^*$. Then a countable $L$-structure $M$ is said to be a $(\mathbf{K}, \leq)$-generic structure, if it satisfies the following:

1. $A \in \mathbf{K}$ for any finite $A \subset M$;

2. If $A \leq B \in \mathbf{K}$ and $A \leq M$, then there is a $B'(\cong_A B)$ with $B' \leq M$;

3. $M$ has finite closure, i.e., $\mathrm{cl}_M(A)$ is finite for any finite $A \subset M$.

$(\mathbf{K}, \leq)$ is said to have the amalgamation property (AP), if whenever $A \leq B \in \mathbf{K}$ and $A \leq C \in \mathbf{K}$, then there are $B'(\cong_A B)$ and $C'(\cong_A C)$ with $B', C' \leq B' \cup C' \in \mathbf{K}$. If $(\mathbf{K}, \leq)$ is closed under substructures and has AP, then there exists the $(\mathbf{K}, \leq)$-generic structure. By back-and-forth method, we also have the following.

**Note 1.5** A generic structure $M$ is homogeneous over finite closed sets, i.e., for any finite $A, B$ with $A \cong_B$ and $A, B \leq M$, we have $\mathrm{tp}(A) = \mathrm{tp}(B)$.

# 2   A characterization of model completeness

**Definition 2.1** Let $A \leq B \in \mathbf{K}$ and $A \subset C \in \mathbf{K}$. Then $B$ is said to be amalgamatable to $C$ over $A$, if there is a $B'(\cong_A B)$ with $C \leq B' \cup C \in \mathbf{K}$.

**Theorem 2.2** Let $M$ be a saturated $(\mathbf{K}, \leq)$-generic structure. Then the following are equivalent.

1. $\text{Th}(M)$ is model complete;

2. If $A \subset C \in \mathbf{K}$ and $A \not\leq C$, then there is a $B \in \mathbf{K}$ with $A \leq B$ which is not amalgamatable to $C$ over $A$.

**Proof.**   $(2 \to 1)$ By Note 1.2, it is enough to show that, for any finite $F \subset M$,

$$\text{tp}_\exists(F) \vdash \text{tp}(F).$$

Take any finite $F \subset M$. Note that $A = \text{cl}(F)$ is finite. Let $(C_i)_{i \in \omega}$ be an enumeration of $\{C \in \mathbf{K} : A \subset C \text{ and } A \not\leq C\}$. By 2, for each $i \in \omega$, there is $B_i \in \mathbf{K}$ with $A \leq B_i$ which is not amalgamatable to $C_i$ over $A$. Then

$$\Sigma(Z) = \{\exists X \exists Y_0...\exists Y_n \bigwedge_{i \leq n}(XY_iZ \cong AB_iF) : n \in \omega\}.$$

is consistent. (Indeed, since $A = \text{cl}(F)$, each $B_i$ can be embedded into $M$ over $A$. So $F$ is a realization of $\Sigma$.) Since $\Sigma$ is a set of $\exists$-formulas, it is enough to show that

$$\Sigma \vdash \text{tp}(F).$$

Take any realization $F'$ of $\Sigma$ in $M$. Then

$$\Gamma(X) = \{\exists Y_i(XY_iF' \cong AB_iF) : i \in \omega\}$$

is consistent. Since $M$ is saturated, we can take a realization $A'(\subset M)$ of $\Gamma$. By Note 1.5, to show that $\text{tp}(F') = \text{tp}(F)$, it is enough to prove that

$$A' \leq M.$$

So suppose that $A' \not\leq M$. Let $C' = \text{cl}(A')$, and take $C$ with $CA \cong C'A'$. Clearly $A \not\leq C$, and so there is an $i \in \omega$ with $C = C_i$. Since $A'$ is a realization of $\Gamma$, there is a $B_i' \subset M$ with $A'B_i'F' \cong AB_iF$. Then $C' \leq B_i' \cup C' \in \mathbf{K}$. Hence $B_i'$ is amalgamatable to $C_i'$ over $A'$. A contradiction.

$(1 \to 2)$. Assume otherwise. Then there are $A, C \in \mathbf{K}$ with $A \subset C$ and $A \not\leq C$ such that any $B \in \mathbf{K}$ with $A \leq B$ is amalgamatable to $C$ over $A$. Since $M$ is generic, we can take an $A_0$ satisfying

$$A_0 \cong A \text{ and } A_0 \leq M.$$

Similarly we can take $A_1, C_1$ satisfying

$$A_1 C_1 \cong AC \text{ and } C_1 \leq M.$$

Then it suffices to show that

- $\text{tp}(A_0) \neq \text{tp}(A_1)$;

- $\text{tp}_\exists(A_0) \subset \text{tp}_\exists(A_1)$.

(Indeed, by model completeness, we have $\text{tp}_\exists(A_i) \vdash \text{tp}(A_i)$, and then we have a contradiction.) First, since $A_0 \leq M$ and $A_1 \not\leq M$, it is clear that $\text{tp}(A_0) \neq \text{tp}(A_1)$. Next we show that

$$\text{tp}_\exists(A_0) \subset \text{tp}_\exists(A_1).$$

Take any $\exists Y \phi(Y, X) \in \text{tp}_\exists(A_0)$, where $\phi$ is a quantifier-free formula. We can assume that $X \subset Y$. Take a realization $B_0$ of $\phi(Y, A_0)$ in $M$. Clearly $A_0 \leq B_0$. Take a $B_1 \in \mathbf{K}$ with $B_1 A_1 \cong B_0 A_0$. By our assumption, $B_1$ is amalgamatable to $C_1$ over $A_1$. Since $M$ is generic, we can take a $B_1' \subset M$ with $B_1' \cong_{C_1} B_1$. So we have $\models \phi(B_1', A_1)$. It follows that $\exists Y \phi(Y, X) \in \text{tp}_\exists(A_1)$.

$(\mathbf{K}, \leq)$ is said to be trivial, if $A \leq B$ for any $A, B$ with $A \subset B \in \mathbf{K}$.

We define that $(\mathbf{K}, \leq)$ has the strong amalgamation property (SAP), if whenever $A \leq B \in \mathbf{K}$ and $A \subset C \in \mathbf{K}$, then $B$ is amalgamatable to $C$ over $A$.

**Note 2.3** $(\mathbf{K}, \leq)$ is said to have the full amalgamation property (FAP), if whenever $A \leq B \in \mathbf{K}$ and $A \subset C \in \mathbf{K}$, then there is a $B'(\cong_A B)$ with $B' \oplus_A C \in \mathbf{K}$ ([3]). Clearly FAP implies SAP.

**Corollary 2.4** Let $M$ be a saturated $(\mathbf{K}, \leq)$-generic structure. If $(\mathbf{K}, \leq)$ is non-trivial and have SAP, then $\text{Th}(M)$ is not model complete.

**Proof.** Since $(\mathbf{K}, \leq)$ is non-trivial, there are $A, C \in \mathbf{K}$ with $A \subset C$ and $A \not\leq C$. Moreover, since $(\mathbf{K}, \leq)$ has SAP, any $B \in \mathbf{K}$ with $A \leq B$ is amalgamatable to $C$ over $A$. By Theorem 2.2, $\text{Th}(M)$ is not model complete.

**Example 2.5** Let $L$ consist of one binary relation $R$, and let $\alpha \in (0, 1]$ be rational. Let

$$\mathbf{K}^* = \{A \text{ finite} : \delta(A') \geq 0 \text{ for all } A' \subset A\}.$$

Since $(\mathbf{K}^*, \leq)$ has AP, there is the $(\mathbf{K}^*, \leq)$-generic structure. Moreover, since $\alpha$ is rational, $M$ is saturated. Also, it is seen that $(\mathbf{K}^*, \leq)$ is non-trivial and have SAP. By Corollary 2.4, $\text{Th}(M)$ is not model complete.

**Example 2.6 (Baldwin [1])** Let $L$ consist of one binary relation $R$, and let $\alpha = 1/2$. $\mathbf{K}^*$ is defined as in Example 2.5. For $A, B \in \mathbf{K}^*$ with $A \cap B = \emptyset$, $(B, A)$ is said to be a minimal pair, if

1. $\delta(B/A) = 0$;

2. $\delta(B'/A) > 0$ for any $B' \subset B$ with $B \neq B' \neq \emptyset$.

In addition, a minimal pair $(B, A)$ is said to be biminimal, if it satisfies the following.

3. For any $a \in A$ there is a $b \in B$ with $R(a, b)$.

For $a, b, c$ with $R(a, b) \wedge R(a, c)$, we call a pair $(a, bc)$ special. In particular, a special pair is biminimal.

Let $\mathcal{P}$ be a class of the biminimal pairs. Then let $\mu : \mathcal{P} \to \omega$ be a map satisfying the following:

- If $(Y, X) \in \mathcal{P}$ is special, then $\mu(Y, X) = 1$;

- Otherwise, $\mu(Y, X) > 2\delta(X)$ and $\mu(Y, X) > 2$.

Let

$$\mathbf{K} = \{A \in \mathbf{K}^* : \chi_A(Y/X) \leq \mu(Y, X) \text{ for any } (Y, X) \in \mathcal{P} \text{ with } X, Y \subset A\},$$

where $\chi_A(Y/X)$ denotes the maximal $n$ such that there exist pairwise disjoint $Y_1, ..., Y_n$ contained in $A$ with each $Y_i$ isomorphic to $Y$ over $X$.

It is checked that $(\mathbf{K}, \leq)$ has AP, and hence there is the (saturated) generic structure $M$. Moreover, it can be shown that $M$ is an $\aleph_1$-categorical non-Desarguesian projective plane.

Note that $(\mathbf{K}, \leq)$ is non-trivial, but it does not have SAP. For instance, let $A = \{a_1, a_2, a_3\}$ be a set with no relations, and let $b, c$ be elements such that $R(b, a_1) \wedge R(b, a_2) \wedge \neg R(b, a_3)$ and $R(c, a_i)$ for any $i = 1, 2, 3$. Let $B = A \cup \{b\}$ and $C = A \cup \{c\}$. Then we have $A \leq B \in \mathbf{K}$ and $A \subset C \in \mathbf{K}$, but $B$ is not amalgamatable to $C$ over $A$.

# 3 Questions

As it was previously mentioned, Holland has proved that Hrushovski's strongly minimal structure is model complete using Lindström's theorem. Then the first question is the following.

**Question 3.1** Can the model completeness of Hrushovski's strongly minimal structure be proved using Theorem 2.2?

On the other hand, Baldwin and Holland have obtained a similar result to that of Holland:

**Fact 3.2 (Baldwin-Holland [2])** Baldwin's projective plane is model complete in a language with additional constant symbols.

This result is proved using Lindström's theorem. However, whether his projective plane is model complete may be still open. For now the next question is the following.

**Question 3.3** Can the model completeness of Baldwin's projective plane be proved using Theorem 2.2?

# References

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