A characterization of \( n \)-dependent theories

Kota Takeuchi
Graduate school of Pure and Applied Sciences, University of Tsukuba

1 Introduction

The notion of \( n \)-dependent property, a generalization of dependent property (\( NIP \)), were introduced by Shelah in [1] (2009). Only few basic properties of the \( n \)-dependent property are known, although Shelah showed an interesting result on definable groups for 2-dependent theories [2]. In this article, we show a characterization of \( n \)-dependent theories by using counting types over finite sets:

**Theorem 1.** Let \( \varphi(x, y_1, \ldots, y_n) \) be an \( L \)-formula. \( \varphi \) is \( n \)-dependent if and only if there is a constant \( \epsilon > 0 \) such that \( |S_\varphi(\Pi_i A_i)| < 2^{k^{n-\epsilon}} \) for sufficiently large \( k \in \omega \) and for all \( |A_i| = k \).

Then we see boolean combinations of \( n \)-dependent theories are again \( n \)-dependent as a corollary of the characterization. This characterization gives a partial answer for a conjecture on the number of types in \( n \)-dependent theories by Shelah in [1]:

**Conjecture 2** (S. Shelah). Let \( \varphi(x, y_1, \ldots, y_n) \) be an \( L \)-formula and let \( m = \text{len}(x) \). \( \varphi \) is \( n \)-dependent if and only if \( |S_\varphi(\Pi_i A_i)| < 2^{mk^{n-1}} \) for all \( |A_i| = k \).

(Note that Shelah’s conjecture is immediately false where \( n = 1 \), and you can check that it is also false where \( n \neq 1 \) with a little discussion. So I think we should replace \( "mk^{n-1}" \) by something like \( "\beta(\log k)k^{n-1}" \) (\( \beta \) depends on \( n, \varphi \)), to make a sense.)

One of the most important results in this article is a generalization of Sauer-Shelah lemma, a famous combinatorial lemma, discussed in section 3. One will notice that the characterization and (generalized) Sauer-Shelah lemma are two sides of the same coin. This report is a partial result of a study with A. Chernikov and D. Palacín on \( n \)-dependent theories.
2 Preliminaries

When we discuss on model theoretic topic, we will use ordinal notation in model theory: $\varphi(x), \psi(y), \ldots, M, N, \ldots, A, B, \ldots$ are used for formulas, models and subsets of models, except $x, y, \ldots$ and $a, b, \ldots$ are used for tuples of variables and elements, respectively. We work under the big model of a complete $L$-theory $T$, so every model and set of elements are contained in it.

When we discuss on combinatorial situation, we will use $X, Y, \ldots$ for (universal) sets, $V, W, \ldots$ for subsets of $X, Y, \ldots$ and $v, w, \ldots$ for elements in $V, W, \ldots$.

First of all, we give a definition of $n$-dependence.

Definition 3. 1. Let $\varphi(x, y_{1}, \ldots, y_{n})$ be an $L$-formula. The formula $\varphi$ is said to be $n$-independent if there are sets $A_{i} (1 \leq i \leq n)$ such that for every disjoint subsets $X$ and $Y \subset \Pi_{i} A_{i}$ there is a tuple $b$ satisfies $\models \bigwedge_{(a_{1}, \ldots, a_{n}) \in X} \varphi(b, a_{1}, \ldots, a_{n}) \wedge \bigwedge_{(a_{1}, \ldots, a_{n}) \in Y} \neg \varphi(b, a_{1}, \ldots, a_{n})$. $n$-dependence is defined by the negation of $n$-independence.

2. Let $T$ be an $L$-theory. $T$ is said to be $n$-dependent (or, have $n$-dependent property) if every formula $\varphi(x, y_{1}, \ldots, y_{n})$ is $n$-dependent.

Note that $\varphi(x, y)$ is 1-dependent if and only if $\varphi(x, a)$ is independent for some $a$. It is immediate that $n$-dependence implies $(n + 1)$-dependence, so $n$-dependent property is a generalization of NIP.

Definition 4. Let $\varphi(x, y_{1}, \ldots, y_{n})$ be an $L$-formula and $A_{i} (1 \leq i \leq n)$ a set of parameters. Let $B \subset \Pi_{i} A_{i}$.

1. A $\varphi$-types over $B$ is a maximal consistent set of formulas $\varphi(x, a_{1}, \ldots, a_{n})$ and $\neg \varphi(x, a_{1}, \ldots, a_{n})$ with $(a_{1}, \ldots, a_{n}) \in B$.

2. $S_{\varphi}(B)$ is the set of all $\varphi$-types over $B$.

For the proof of the main result, we'll use a graph theoretic fact, as bellow.

Definition 5. Let $n \geq 1$ be a natural number. An $n$-partite $n$-hypergraph $(V, E)$ is an $n$-uniform hypergraph satisfying the following:

- $V$ is a disjoint union of sets $V_{i} (1 \leq i \leq n)$.

- If $E(v_{1}, \ldots, v_{n})$ holds then $v_{i} \in V_{i}$.
We say \((V, E)\) has size \(k\) if \(|V_i| = k\) for all \(i\). An \(n\)-partite \(n\)-hypergraph \((V, E)\) is said to be complete if there is no \(n\)-partite \(n\)-hypergraph \((V, E')\) with \(E' \supsetneq E\). If \(n = 1\), the \(n\)-hypergraph \((V, E)\) is just a set \(V\) and a subset \(E \subseteq V\), and it is complete if \(E = V\).

Let \(G\) be an \(n\)-partite \(n\)-hypergraph of size \(k\). If \(G\) is complete, then it has \(k^n\) edges, and immediately contains copies of complete \(n\)-partite \(n\)-hypergraphs of size \(< k\). The following fact shows that there is \(\varepsilon\) (not depending on the choice of \(G\)) such that if \(G\) has \(k^{n-\varepsilon}\) edges then it contains a copy of complete \(n\)-partite \(n\)-hypergraph of size \(d\).

**Fact 6** (Erdős[3]). Let \(d, n > 1\) be natural numbers. Then for sufficiently large \(k > n_0\), the following condition holds: Let \((V, E)\) be an \(n\)-partite \(n\)-hypergraph of size \(k\). If \(|E| \geq k^{n-\varepsilon}\) with \(\varepsilon = d^{1-n}\) then \((V, E)\) contains a copy of a complete \(n\)-partite \(n\)-hypergraph of size \(d\).

**Remark 7.** Fact 6 given in [3] doesn’t hold where \(n = 1\), because \(k^{n-\varepsilon} = k^0 = 1\). So we replace the lower bound \(k^{n-\varepsilon}\) by \(dk^{n-\varepsilon}\), then the fact holds for all \(n \geq 1\). This replacement is necessary for our main lemma to include Sauer-Shelah lemma. But it make the inequation in the main lemma more complex.

Our characterization of \(n\)-dependent property is related to a combinatorial proposition, called Sauer-Shelah lemma. To explain this lemma, we need to introduce some notions in combinatorics. Most of the following is proved in Hang Q. Ngo’s online note [4] and [5].

**Definition 8.** Let \(X\) be a set.

1. A set system \(\mathcal{H}\) on \(X\) is a subset of the power set \(\mathcal{P}(X)\) of \(X\).
2. \(\mathcal{H} \cap V := \{W \cap V : W \in \mathcal{H}\}\) for \(V \subseteq X\).
3. We say \(V \subseteq X\) is shuttered by \(\mathcal{H}\) if \(\mathcal{H} \cap V = \mathcal{P}(V)\).

**Definition 9.** Let \(X\) be an infinite set and \(\mathcal{H}\) a set system on \(X\).

1. \(\pi_{\mathcal{H}}(k) := \max\{|\mathcal{H} \cap V| : |V| = k\}\). The function \(\pi_{\mathcal{H}} : \mathbb{N} \rightarrow \mathbb{N}\) is called a shutter function.
2. VC-dimension (Vapnik-Chervonenkis dimension): \(VC(\mathcal{H}) = \max\{k : \pi_{\mathcal{H}}(k) = 2^k\}\).
Fact 10 (Sauer-Shelah lemma). Let $\mathcal{H}$ be a set system on an infinite set $X$. Suppose that $VC(\mathcal{H}) = d < \infty$. Then for $n > d$,

$$\pi_{\mathcal{H}}(k) \leq \sum_{i=1}^{d} \binom{k}{i} \leq \left(\frac{ek}{d}\right)^{d} = O(2^{d\log_{2}(k)}).$$

By using Sauer-Shelah lemma, we have the following:

Fact 11. Let $\varphi(x, y)$ be an $L$-formula. $\varphi(x, y)$ is dependent if and only if there is $d$ such that for all $k > d$, $|S_{\varphi}(A)| \leq \left(\frac{ek}{d}\right)^{d} = O(2^{d\log_{2}(k)})$, where $|A| = k$.

One of elegant proofs of Sauer-Shelah lemma is given by Shifting technique, as below.

Fact 12. Let $X$ be a finite set and $\mathcal{H}$ a set system on $X$. Then we can find a set system $\mathcal{G}$ on $X$ such that

- $|\mathcal{H}| = |\mathcal{G}|$,
- if $V \subset X$ is shuttered by $\mathcal{G}$ then $V$ is shuttered by $\mathcal{H}$,
- $\mathcal{G}$ is closed under taking subset.

3 A generalization of Sauer-Shelah lemma

In this section, we prove an inequation like Sauer-Shelah lemma. There may be better bound for our inequation, but still it is useful enough to apply to $n$-dependent theories.

We'll generalize the notions in the previous section to higher dimension. Suppose $n \geq 1$. Let $X_i$ $(1 \leq i \leq n)$ be sets of size $m \in \omega \cup \{\omega\}$ and let $X = \Pi_i X_i$. Let $\mathcal{H}$ be a set system on $X$. (Note that $|X| = m^n$, and if $X$ is shuttered by $\mathcal{H}$ then $|\mathcal{H}| = 2^{m^n}$.)

Definition 13. 1. $\pi_{\mathcal{H},n}(k) := \max\{|\mathcal{H} \cap V| : V = \Pi_i V_i, V_i \subset X_i, |V_i| = k\}$.

2. $VC_n$-dimension: $VC_n(\mathcal{H}) = \max\{k : \pi_{\mathcal{H},n}(k) = 2^{kn}\}$. 
Lemma 14 (Main lemma). 1. (precise form) Let $n \geq 1$ and let $VC_n(\mathcal{H}) = d < \infty$. For sufficiently large $k$, we have

$$\pi_{\mathcal{H},n}(k) \leq \sum_{i=0}^{D(k)} \binom{k^n}{i} \leq \left( \frac{ek^n}{D(k)} \right)^{D(k)} = O(2^{D(k)(\epsilon \log_2 k + \log_2 (e/(d+1)))}),$$

where $D(k) = (d+1)k^{n-\epsilon} - 1$ and $\epsilon = (d+1)^{1-n}$. Especially, if $n = 1$ then $\epsilon = 1$ and $D(k) = d$, so we have Sauer-Shelah lemma.

2. (simpler form 1) Let $VC_n(\mathcal{H}) = d < \infty$ and let $\epsilon = (d+1)^{1-n}$. There is $\beta$ (depends only on $d$ and $n$) such that $\pi_{\mathcal{H},n}(k) \leq 2^{\beta n^k \log k}$ for sufficiently large $k$.

3. (simpler form 2) Let $VC_n(\mathcal{H}) = d < \infty$. There is $\epsilon'$ (depends only on $d$ and $n$) such that $\pi_{\mathcal{H},n}(k) \leq 2^{k-\epsilon'}$ for sufficiently large $k$.

Proof. The simpler form is immediately shown from the precise form by taking $\beta > (d+1)\epsilon$ and $\epsilon' < \epsilon$. We'll show the first item. Let $X = \Pi_x X_i$ and $\mathcal{H}$ a set system on $X$. Let $V_i \subset X_i$ be a set of size $k$ and let $\mathcal{H}_0 = \mathcal{H} \cap V$ with $V = \Pi_i V_i$. We'll check $|\mathcal{H}_0| \leq \sum_{i=0}^{(d+1)k^{n-\epsilon} - 1} \binom{k^n}{i}$. By the shifting technique in Fact 12, we can find $\mathcal{G}$ satisfying

- $|\mathcal{H}_0| = |\mathcal{G}|$,
- if $W \subset V$ is shuttered by $\mathcal{G}$ then $V$ is shuttered by $\mathcal{H}_0$,
- $\mathcal{G}$ is closed under taking subset. (Hence if $W \in \mathcal{G}$ then $W$ is shuttered by $\mathcal{H}_0$.)

Consider any subset $W \subset V$ with $W = \Pi_i W_i$ and $|W_i| = d + 1$. Since $VC_n(\mathcal{H}) = d < \infty$, $\mathcal{G}$ cannot contain $W$, otherwise $W$ is also shuttered by $\mathcal{H}_0$, hence by $\mathcal{H}$, contradicting to the assumption $VC_n(\mathcal{H}) = d$. Take an element $W' \in \mathcal{G}$. Then we have $W \not\subset W'$ because $\mathcal{G}$ is closed under taking subset. We may regards $W'$ as an $n$-partite $n$-hypergraph of size $k$ with vertices $V_1 \cup \ldots \cup V_n$ and edges $W'$. Then $W'$ has no complete $n$-partite $n$-hyper subgraph of size $d + 1$. So, by Fact 6 and Remark 7, the number $|W'|$ of edges must be bounded by $(d+1)k^{n-\epsilon}$ where $\epsilon = (d+1)^{1-n}$. Then we have

$$\mathcal{G} \subset \{W' \subset V : |W'| \leq (d+1)k^{n-\epsilon} - 1\}.$$
and
\[ |G| \leq |\{W' \subset V : |W'| \leq (d+1)k^{n-\epsilon} - 1\}| \leq \sum_{i=0}^{(d+1)k^{n-\epsilon}-1} \binom{k^n}{i}. \]

The rest of the inequation is shown by a general inequation \( \sum_{i=0}^{s} \binom{t}{i} \leq (et/s)^{s} \) for \( t > s \in \mathbb{N} \).

\[ \square \]

Note that if \( VC_n(\mathcal{H}) = \infty \) then \( \pi_{\mathcal{H},n}(k) = 2^{k^n} \) for all \( k \). So we have the following dichotomy:

**Corollary 15.** Let \( \mathcal{H} \) be a set system on \( X = \Pi_{i=1}^{n}X_i \) with \( |X_i| = \omega \). One of the following holds.

1. \( \pi_{\mathcal{H},n}(k) = 2^{k^n} \) for all \( k \).

2. There is \( \epsilon' > 0 \) such that for sufficiently large \( k \), \( \pi_{\mathcal{H},n}(k) < 2^{k^{n-\epsilon'}} \).

## 4 Characterizing \( n \)-dependent property

First we recall the definition of \( n \)-dependent property.

**Definition 16.** Let \( \varphi(x, y_1, \ldots, y_n) \) be an \( L \)-formula. The formula \( \varphi \) is said to be \( n \)-independent if there are sets \( A_i \) (\( 1 \leq i \leq n \)) such that for every disjoint subsets \( X \) and \( Y \subset \Pi_i A_i \), there is a tuple \( b \) satisfies \( \models \bigwedge_{(a_1, \ldots, a_n) \in X} \varphi(b, a_1, \ldots, a_n) \wedge \bigwedge_{(a_1, \ldots, a_n) \in Y} \neg \varphi(b, a_1, \ldots, a_n) \). \( n \)-dependence is defined by the negation of \( n \)-independence.

Let \( A = \Pi_i A_i \) be a set of parameters with \( A_i \) of size \( k \) and let \( \varphi(x, y_1, \ldots, y_n) \) be an \( L \)-formula. We want to measure the size of the set \( S_{\varphi}(A) \) of \( \varphi \)-types over \( A \).

**Definition 17.** Let \( M \) be an \( \omega \)-saturated model of \( T \) and let \( \varphi(x, y_1, \ldots, y_n) \) be an \( L \)-formula. We define

1. For \( p \in S_{\varphi}(\Pi_i M^{|y_i|}) \), we define \( (\Pi M)_p \subset \Pi_i M^{|y_i|} \) by \( \{(a_1, \ldots, a_n) \in \Pi_i M^{|y_i|} : \varphi(x, a_1, \ldots, a_n) \in p\} \).

2. A set system \( \mathcal{H}_{\varphi} \) on \( \Pi_i M^{|y_i|} \) is the set \( \{(\Pi M)_p : p \in S_{\varphi}(\Pi_i M^{|y_i|})\} \).
Remark 18. Let $A \subset \Pi_i M^{\left|y_i\right|}$. The following are immediate from the definitions.

1. $M_p \cap A = M_q \cap A$ if and only if $p|A = q|A$.

2. $\left|\mathcal{H}_\varphi \cap A\right| = |S_\varphi(A)|$.

3. $\varphi$ is $n$-dependent if and only if $VC_n(\mathcal{H}) = d < \infty$.

With the above remark, we can calculate the number of types by counting $\mathcal{H}_\varphi \cap A$.

Theorem 19. Let $\varphi(x, y_1, \ldots, y_n)$ be an $L$-formula. The following are equivalent.

1. $\varphi$ is $n$-dependent.

2. For sufficiently large $k$, if $A = \Pi_i A_i$ with $|A_i| = k$, then $|S_\varphi(A)| \leq \sum_{i=0}^{D(k)} \binom{k^n}{i} \leq \left(\frac{ek^n}{D(k)}\right)^{D(k)} = O\left(2^{D(k)(\epsilon \log_2 k + \log_2 (e/(d+1)))}\right)$, where $D(k) = (d+1)^{1-n}$ and $\epsilon = (d+1)^{1-n}$. Especially, the case $n = 1$ implies the well known characterization of dependent property.

3. Let $\epsilon = (d+1)^{1-n}$. There is $\beta$ such that for sufficiently large $k$, $|S_\varphi(A)| \leq 2^{\beta k^{n-\epsilon}}$ for all $A = \Pi_i A_i$ with $|A_i| = k$.

4. There is $\epsilon'$ such that for sufficiently large $k$, $|S_\varphi(A)| \leq 2^{\epsilon' k^{n-\epsilon}}$ for all all $A = \Pi_i A_i$ with $|A_i| = k$.

Proof. Immediately shown from Lemma 14 and Remark 18.

Corollary 20. $n$-dependent formulas are closed under taking boolean combinations.

Proof. Let $\varphi(x, y_1, \ldots, y_n)$ and $\psi(x, y_1, \ldots, y_n)$ be $n$-dependent formulas. By the definition, the negation of $n$-dependent formula is $n$-dependent. On the other hand, $|S_{\varphi \land \psi}(A)| \leq |S_\varphi(A)| \times |S_\psi(A)| \leq 2^{k^n - \epsilon'} \times 2^{k^n - \epsilon''} \leq 2^{k^n - \epsilon'''}$ for some $\epsilon', \epsilon''$ and $\epsilon'''$. So $\varphi \land \psi$ is $n$-dependent.
References


