REGARDING 2-CHAiNS WITH 1-SHELL BOUNDARIES IN ROSY THEORIES

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1. INTRODUCTION

In [6], Hrushovski developed connections between amalgamation properties and definable groupoids for a stable theory: if a stable theory $T$ fails the 3-uniqueness property, then there exists a definable groupoid. J. Goodrick and A. Kolesnikov constructed such groupoid in [5]. Furthermore J. Goodrick, B. Kim, and A. Kolesnikov developed homology groups $H_n$ associated to a family of amalgamation functors and computed the group $H_2$ for strong types in stable theories. In particular, they showed that if $T$ has $n$-CA based on $A = acl(A)$ for $n \geq 3$, then $H_{n-2} = 0$ for $p \in S(A)$, thus $H_1(p) = 0$ holds for any simple $T$.

In this article, we work with amenable families of functors and corresponding homology groups from [3],[4] to show $H_1(p) = 0$ holds for a rosy $T$, where $p$ is a Lascar type and classify all the possible 2-chains with a 1-shell boundary in a nontrivial amenable collection of functors.

This article is only intended to present a summary of the results from [7],[8] and we do not include all the details of the proofs.

BASIC DEFINITIONS

In this section, we recall the basic definitions and facts which are established in [3],[4]. Throughout, $s$ denotes some finite set of natural numbers. A subset $X \subseteq P(s)$ is called downward closed if whenever $u \subseteq v \in X$, then $u \in X$. Then as an ordered (by inclusion) set, $X$ is a category. Before defining an amenable family of functors, we introduce some notations. We fix a category $\mathcal{C}$. Given a functor $f : X \rightarrow \mathcal{C}$ and $u \subseteq v \in X$, $f^u_v := f(t_{u,v}) \in Mor_{\mathcal{C}}(f(u), f(v))$ where $t_{u,v}$ is the single inclusion map in $Mor(u,v)$.

Definition 1.1. (1) Let $X$ be a downward closed subset of $P(s)$ and let $t \in X$. The symbol $X|_t$ denotes the set

\[ \{u \in P(s|t) \mid t \cup u \in X\} \subseteq X. \]

(2) For $s, t$, and $X$ as above, let $f : X \rightarrow \mathcal{C}$ be a functor. Then the localization of $f$ at $t$ is the functor $f|_t : X|_t \rightarrow \mathcal{C}$ such that

\[ f|_t(v) = f(t \cup v), \]

and $(f|_t)^u_v = f^u_v|_t$, for any $u \subseteq v \in X|_t$.

(3) Let $X \subseteq P(s)$ and $Y \subseteq P(t)$ be downward closed subsets, where $s$ and $t$ are finite sets of natural numbers. Let $f : X \rightarrow \mathcal{C}$ and $g : Y \rightarrow \mathcal{C}$ be functors. We say $g$ is a permutation of $f$ if there is a bijection (not necessarily order-preserving) $\sigma : s \rightarrow t$ such that $Y = \{ \sigma(u) : u \in X\}$ and for $v \subseteq w \in Y$, $g(w) = f(\sigma^{-1}(w))$ and $(g)^w_v = f^\sigma_{\sigma^{-1}(w)}$. In this case we write $g = f \circ \sigma^{-1}$.

We say that $f$ and $g$ are isomorphic if there are an order-preserving bijection $\tau : s \rightarrow t$ such that $Y = \{ \tau(u) : u \in X\}$ and a family of morphisms $\{ h_u : f(u) \rightarrow g(\tau(u)) \mid u \in X \}$ from $Mor(\mathcal{C})$ such that for any $u \subseteq v \in X$,

\[ h_v \circ f^u_v = g^\tau_{\tau(v)} \circ h_u. \]

For example $f$ and $f \circ \sigma^{-1}$ are isomorphic when $\sigma$ is order-preserving.
Remark 1.2. It easily follows that for a downward closed $X \subseteq \mathcal{P}(s)$ and $t \in X$, we have

$$X|_t = X \cap \mathcal{P}(s \setminus t) \iff X = \bigcup \{\mathcal{P}(u) \mid t \subseteq u \in X\};$$

and in that case $X|_t = \bigcup \{\mathcal{P}(u \setminus t) \mid t \subseteq u \in X\}$.

Definition 1.3. Let $A$ be a non-empty collection of functors $f : X \to C$ for various non-empty downward-closed subsets $X \subseteq \mathcal{P}(s)$ for all finite sets $s$ of natural numbers. We say that $A$ is amenable if it satisfies all of the following properties:

1. (Closed under isomorphisms and permutations) If $f : X \to C$ is in $A$, then every functor $g : Y \to C$ which is either a permutation of $f$ or is isomorphic to $f$ is also in $A$.

2. (Closed under restrictions and unions) Given a functor $f : X \to C$, $f \in A$ if and only if for every $u \in X$, we have that $f \mid \mathcal{P}(u) \in A$.

3. (Closed under localizations) Suppose that $f : X \to C$ is in $A$. Then for any $t \in X$, $f|_t : X|_t \to C$ is also in $A$.

4. (Extensions of localizations are localizations of extensions.) Let $f : X \to C$ be in $A$, and let $t \in X \subseteq \mathcal{P}(s)$ be such that $X|_t = X \cap \mathcal{P}(s \setminus t)$ (see Remark 1.2). Suppose that the localization $f|_t : X|_t \to C$ has an extension $g : Z \to C$ in $A$ for some $(X|_t \subseteq Z \subseteq \mathcal{P}(s \setminus t))$. Then there is a functor $g_0 : Z_0 \to C$ in $A$ such that $Z_0 = \{u \cup v : u \in Z, v \subseteq t\}, f \subseteq g_0$, and $g_0|_t = g$.

Definition 1.4. Let $B \in \text{Ob}(C)$ and suppose $f(\emptyset) = B$. We say that $f$ is over $B$ and we let $A_B$ denote the set of all functors $f \in A$ that are over $B$.

Let $A$ be a non-empty amenable collection of functors mapping into the category $C$.

Definition 1.5. Let $n \geq 0$ be a natural number. A (regular) $n$-simplex in $C$ is a functor $f : \mathcal{P}(s) \to C$ for some set $s \subseteq \omega$ with $|s| = n + 1$. The set $s$ is called the support of $f$, or supp($f$).

Let $S_n(A; B)$ denote the collection of all regular $n$-simplices in $A_B$. Then put $S(A; B) := \bigcup S_n(A; B)$ and $S(A) := \bigcup_{B \in \text{Ob}(C)} S(A; B)$.

Let $C_n(A; B)$ denote the free abelian group generated by $S_n(A; B)$; its elements are called $n$-chains in $A_B$, or $n$-chains over $B$. Similarly, we define $C(A; B) := \bigcup C_n(A; B)$ and $C(A) := \bigcup_{B \in \text{Ob}(C)} C(A; B)$.

If $c$ is an $n$-chain in the form $\sum_{1 \leq i \leq k} n_i f_i$, where the $f_i$'s are distinct $n$-simplices and the $n_i$'s are nonzero integers, then we define the length of $c$ as $|c| = |n_1| + \cdots + |n_k|$ and the support of $c$ as the union of the supports of $f_i$'s.

Of course $c$ can be sometimes written as $(c + g) - g$, but $|c|$ and the support of $c$ are always uniquely computed in its standard form.

We use $a, b, c, \ldots, f, g, h, \ldots, \alpha, \beta, \ldots$ to denote simplices and chains. Now we will define the boundary operators and the homology groups.

Definition 1.6. If $n \geq 1$ and $0 \leq i \leq n$, then the $i$-th boundary operator $\partial^i_n : C_n(A; B) \to C_{n-1}(A; B)$ is defined so that if $f$ is a regular $n$-simplex with domain $\mathcal{P}(s)$ with $s = \{s_0 < \cdots < s_n\}$, then

$$\partial^i_n(f) = f \mid \mathcal{P}(s \setminus \{s_i\})$$

and extended linearly to a group map on all of $C_n(A; B)$.

If $n \geq 1$ and $0 \leq i \leq n$, then the boundary map $\partial_n : C_n(A; B) \to C_{n-1}(A; B)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial^i_n(c).$$

We write $\partial^i$ and $\partial$ for $\partial^i_n$ and $\partial_n$, respectively, if $n$ is clear from context.

Definition 1.7. The kernel of $\partial_n$ is denoted $Z_n(A; B)$, and its elements are called $(n-)$cycles. The image of $\partial_{n+1}$ in $C_n(A; B)$ is denoted $B_n(A; B)$. The elements of $B_n(A; B)$ are called $(n-)$boundaries.
Since \( \partial_n \circ \partial_{n+1} = 0 \), \( B_n(A;B) \subseteq Z_n(A;B) \) and we can define simplicial homology groups relative to \( A_B \).

**Definition 1.8.** The \( n \)th (simplicial) homology group of \( A \) over \( B \) is

\[
H_n(A;B) := Z_n(A;B)/B_n(A;B).
\]

**Remark/Definition 1.9.** Let \( \sigma : \mathbb{N} \to \mathbb{N} \) be a bijection. Then \( \sigma \) induces an automorphism \( \sigma^* : C_n(A,B) \to C_n(A,B) \) as follows: Let \( c = \sum_i n_i f_i \in C_n(A,B) \), where each \( n \)-simplex \( f_i \) with \( s_i := \text{supp}(f_i) = \{s_{i,0} < \cdots < s_{i,n}\} \). Let \( \sigma_i := \sigma | s_i \) and let \( t_i := \sigma_i(s_i) = \{t_0 < \cdots < t_n\} \). We define

\[
\sigma^*(c) := \sum_i n_i (-1)^{|\sigma_i|} f_i \circ \sigma_i^{-1}
\]

(see Definition 1.1(3)) with \( |\sigma_i| = |\sigma'| \) (so \( = 0 \) or \( 1 \)), where \( \sigma' \in \text{Sym}(n+1) \) such that for \( j \leq n \), \( \sigma_i(s_{i,j}) = t_{\sigma'(j)} \).

Now \( \sigma^* \) commutes with \( \partial \), i.e.,

\[
\partial(\sigma^*(c)) = \sigma^*(\partial(c)).
\]

This can be inductively shown after verifying when \( \sigma \) is a transposition.

Next we define the amalgamation properties. Notice that for \( n = \{0, \ldots, n-1\} \), we use \( \mathcal{P}^{-}(n) \) is \( \mathcal{P}(n) \setminus \{n\} \).

**Definition 1.10.** Let \( A \) be a non-empty amenable family of functors into a category \( C \) and let \( n \geq 1 \).

1. \( A \) has \( n \)-amalgamation if for any functor \( f : \mathcal{P}^{-}(n) \to C, f \in A \), there is an \( n \)-simplex \( g \supseteq f \) such that \( g \in A \).
2. \( A \) has \( n \)-complete amalgamation or \( n \)-CA if \( A \) has \( k \)-amalgamation for every \( k \) with \( 1 \leq k \leq n \).
3. \( A \) has strong \( 2 \)-amalgamation if whenever \( f : \mathcal{P}(s) \to C, g : \mathcal{P}(t) \to C \) are simplices in \( A \) and \( f \upharpoonright \mathcal{P}(s \cap t) = g \upharpoonright \mathcal{P}(s \cap t) \), then \( f \cup g \) can be extended to a simplex \( h : \mathcal{P}(s \cup t) \to C \) in \( A \).

**Definition 1.11.** An amenable family of functors \( A \) is called non-trivial if it is non-empty and satisfies the strong 2-amalgamation property.

It easily follows that any non-trivial amenable family of functors contains an \( n \)-simplex for each \( n \geq 1 \). In the rest of the paper, we shall only work with a non-trivial amenable family \( A \) of functors into \( C \).

**Definition 1.12.** If \( n \geq 1 \), an \( n \)-shell is an \( n \)-chain \( c \) of the form

\[
\pm \sum_{0 \leq i < n+1} (-1)^i f_i,
\]

where \( f_0, \ldots, f_{n+1} \) are \( n \)-simplices such that whenever \( 0 \leq i < j \leq n+1 \), we have \( \partial^i f_j = \partial^{i-1} f_i \).

**Remark/Definition 1.13.** The boundary of an \( (n+1) \)-simplex is an \( n \)-shell, and the boundary of any \( n \)-shell is 0. Note that \( A \) has \( (n+2) \)-amalgamation if any \( n \)-shell is a boundary of an \( (n+1) \)-simplex. For an \( (n+1) \)-chain \( c \) having an \( n \)-shell boundary, \( |c| \) is always an odd integer.

Now we introduce a weaker notion than 3-amalgamation: \( A \) has weak 3-amalgamation over \( B \) if any 1-shell over \( B \) is the boundary of a 2-chain over \( B \) of length \( \leq 3 \).

The details of the following fact and corollaries can be found in [3],[4].

**Fact 1.14.** If \( A \) has \( (n+1) \)-CA for some \( n \geq 1 \), then

\[
H_n(A;B) = \{ [c] : c \text{ is an } n \text{-shell over } B \text{ with support } n+2 \}.
\]

Since \( A \) already has 2-amalgamation, we have that \( H_1(A;B) \) is trivial if any 1-shell over \( B \) is the boundary of some 2-chain over \( B \).
Corollary 1.15. Assume that $T$ has $n$-CA over $A = \text{acl}(A)$ for some $n \geq 3$. Then $H_{n-2}(p) = 0$ for $p \in S(A)$.

Corollary 1.16. If $T$ is simple, then $H_1(p) = 0$ for any complete type $p$ in $T$.

From now on, we work with a large saturated model $M = M^{eq}$ whose theory $T$ is rosy. Recall that $T$ is rosy if there is a ternary independence relation $\perp$ on the small sets of $M$ satisfying the basic independence properties [1],[2]. We take $\perp$ here as thorn-independence.

Now fix an algebraically closed set $B = \text{acl}(B)$, and let $C_B$ denote the category of all small subsets of $M$ containing $B$ and morphisms are elementary maps over $B$ (i.e., fixing $B$ pointwise). For a functor $f : X \to C_B$ and $u \subseteq v \in X$, we write $f^u_v(u) := f^u_v(f(u)) \subseteq f(v)$. We now fix $p(x) \in S(B)$ where the tuple $x$ may possibly have an infinite arity.

Definition 1.17. A closed independent functor in $p$ is a functor $f : X \to C_B$ such that:
1. $X$ is a downward-closed subset of $\mathcal{P}(s)$ for some finite $s \subseteq \omega$; $f(\emptyset) \supseteq B$; and for $i \in s$, $f([i])$ is of the form $\text{acl}(Cb)$, where $b(\models p)$ is independent with $C = f^\emptyset([i])$ over $B$.
2. For all non-empty $u \in X$, we have

\[ f(u) = \text{acl}(B \cup \bigcup_{i \in u} f^i_u([i])); \]

and \( f^i_u([i]) \mid i \in u \) is independent over $f^\emptyset([\emptyset])$.

Let $\mathcal{A}(p)$ denote all closed independent functors in $p$.

Fact 1.18. $\mathcal{A}(p)$ is a non-trivial amenable family of functors.

2. MAIN RESULT : $H_1(p) = 0$ IN ROSY THEORIES

We have $H_1(p) = 0$ for any Lascar strong type which follows from the fact that Lascar distances are finite in rosy theories. Meanwhile the same holds for a simple $T$ due to 3-amalgamation and Fact 1.14. For given $f : X \to C_B$ in $\mathcal{A}(p)_B$ (so $f(\emptyset) = B$), and $u = \{i_0 < \cdots < i_k\} \in X$, we write $f(u) = [a_0, \ldots, a_k]$, where $a_j \models p$, $f(u) = \text{acl}(B, a_0 \ldots a_k)$, and $\text{acl}(a_jB) = f^i_{u}([i])$. Thus \( \{a_0, \ldots, a_k\} \) is independent over $B$.

Theorem 2.1. If $B = M$ is a model, then $\mathcal{A}(p)$ has weak 3-amalgamation over $M$. Therefore $H_1(p) = 0$.

Definition 2.2. Let a set $B$ and tuples $a, b$ be such that $a =_B b$. By the Lascar distance over $B$ of $a$ and $b$, denoted by $d_B(a, b)$, we mean the smallest natural number $n$ such that there are tuples $a = a_0, \ldots, a_n = b$, where for each $a_ia_{i+1}(i < n)$ begins some $B$-indiscernible sequence.

Theorem 2.3. Suppose that the strong type $p$ is the Lascar (strong) type. Then $H_1(p) = 0$.

Proof. For notational simplicity we may assume $B$ to be $\emptyset$. Given a 1-shell $f = a_{12} - a_{02} + a_{01}$ where each $a_{ij} : \mathcal{P}([i, j]) \to C_B$ is a 1-simplex in $S_1(\mathcal{A}(p))$, we want to find a 2-chain $g$ such that $\partial g = f$.

Again there is no harm to assume that $a_{01}([1]) = [b] = a_{12}([1])$ and $a_{02}([2]) = [c] = a_{12}([2])$, and $a_{01}([0]) := [d], a_{02}([0]) := [d']$. By the extension axiom, we can assume that $\{b, c, d, d'\}$ is independent. Let $d, d' \models p$ such that $d(d, d') = n$. So we have $d = d_0, \ldots, d_n = d'$, where $d_id_{i+1}(i < n)$ begins an indiscernible sequence. Assume that $bc \dashv d_0 \cdots d_n$ so $bc \dashv d_0 \cdots d_n$.

Claim. There are $e_i \models p (i < n)$ such that $d_id_{i+1} \dashv e_i$ and $e_id_i = e_id_{i+1}$.

Proof of Claim. Let $I = \langle d_id_{i+1} \cdots \rangle$ be an indiscernible sequence having a sufficiently large length. Due to the extension axiom, we can choose $e_i' = d_i$ with $e_i' \dashv I$. Since there are only boundedly many
types over $e_i$, one can find $d_j, d_j' (j < j')$ with $e_i d_j = e_i' d_j'$. Due to the indiscernibility of $I$, there is a map $f$ that maps $d_1 d_{i+1}$ to $d'_j d'_{j'}$. Then $e_i := f(e_i')$ satisfies $e_i d_i = e_i d_{i+1}$ as desired.

Again by extension we suppose $b c \downarrow d, d_{i+1}, e_1$, so that each $\{b, d_i, e_i\}, \{b, d_{i+1}, e_1\}$ is independent. Also each $\{b, c, e_{n-1}\}, \{c, d_n, e_{n-1}\}$ is independent ($\ast$).

There is $g_0 := g_0^+ - g_0^-$, where $g_0^+, g_0^-$ are 2-simplices with support $u := \{0, 1, 3\}$ such that $g_0^+(u) = [d_0, b, e_0]$ and $g_0^-(u) = [d_1, b, e_1]$; $\partial^2 g_0^+ = \partial^2 g_0^-$ (this follows from the above Claim); and $g_0^+$ extends $a_0$ (i.e., $\partial^2 g_0^+ = a_0$). Hence $\partial g_0 = a_0 = -\partial^2 g_0$.

Similarly, we can find $g_i := g_i^+ - g_i^-$ ($0 < i < n - 1$), where each $g_i^+, g_i^-$ is a 2-simplex with support $u$ such that $g_i^+(u) = [d_i, b, e_i]$ and $g_i^-(u) = [d_{i+1}, b, e_{i+1}]$; $\partial^2 g_i^+ = \partial^2 g_i^-$; and $\partial^2 g_{i+1}^+ = \partial^2 g_{i+1}^-$. Therefore we have

$$\partial(g_0 + \cdots + g_{n-2}) = a_0 - \partial^2 g_{n-2}.$$

Put $g_{n-1} := g_{n-1}^+ - a_{023} + a_{123}$, where $a_{23}$ is a 2-simplex with support $\{j, 2, 3\}$ extending $a_{12}$ such that $a_{023} \{0, 2, 3\} = \{d_n, c, e_{n-1}\}$, $a_{123} \{1, 2, 3\} = \{b, c, e_{n-1}\}$. Also $g_{n-1}^+$ is a 2-simplex with $g_{n-1}^+ \{0, 1, 3\} = \{d_{n-1}, b, e_{n-1}\}$ extending $\partial^2 g_{n-2}^-$, Moreover again by ($\ast$), we have $\partial^1 g_{n-1}^+ = -\partial^1 a_{023}$, Thus it follows

$$\partial g_{n-1} = \partial^2 g_{n-1}^+ - a_{02} + a_{12} = \partial^2 g_{n-2} - a_{02} + a_{12}.$$

Therefore $g := g_0 + \cdots + g_{n-1}$ satisfies $\partial g = f$ as desired.

\[ \text{\[ \text{3. Classification} \] \]}

In this section, we classify 2-chains having 1-shell boundaries using two operations, the crossing operation and the renaming support operation.

**Remark/Definition 3.1.** Suppose that an $n$-chain $c = \sum_i n_i f_i$ is given where each $f_i$ is an $n$-simplex. Assume that $j \in \text{supp}(c) \setminus \text{supp}(\partial(c))$. In this case we say $c$ has a vanishing support (in its boundary). Given $k \notin s := \text{supp}(c)$, we let $\sigma$ be a map sending $j$ to $k$ while fixing numbers in $s \setminus \{j\}$. Now as in 1.9, $\partial(\sigma^*(c)) = \sigma^*(\partial(c)) = \partial(c)$.

**Definition 3.2.**

(1) The crossing operation (or CR-operation): Let $\alpha$ and $\beta$ be 2-simplices with

$$\text{supp}(\alpha) = \{i_0, i_1, i_2\}, \text{supp}(\beta) = \{i_1, i_2, i_3\} (i_0 \neq i_3)$$

such that $\alpha \parallel P(\{i_1, i_2\}) = \beta \parallel P(\{i_1, i_2\}) = \gamma$. Suppose that $\partial(\alpha + \epsilon \beta)$ (c = 1 or -1) has no term $\gamma$ (i.e., $\gamma$ is cancelled out). Now by strong 2-amalgamation there is a 3-simplex $\delta$ with $\text{supp}(\delta) = \{i_0, i_1, i_2, i_3\}$ such that $\delta \parallel P(\{i_0, i_1, i_2\}) = \alpha$ and $\delta \parallel P(\{i_1, i_2, i_3\}) = \beta$. We take $\alpha' := \delta \parallel P(\{i_0, i_1, i_2\})$ and $\beta' := \delta \parallel P(\{i_1, i_2, i_3\})$. Then it follows $\partial(\alpha + \epsilon \beta) = \partial(\alpha' + \epsilon \beta')$. Replacing $\alpha + \epsilon \beta$ by $\alpha' + \epsilon \beta'$ is called the crossing operation. Hence from a 2-chain $c$, if we obtain $c'$ by the CR-operation (applied to two terms in $c$) then $\partial(c) = \partial(c')$ and $|c'| \leq |c|$.

(2) The renaming support operation (or RS-operation): This is basically what is described in 3.1 with $n = 2$. So let $c = \sum_i n_i f_i$ be a $2$-chain having a vanishing support, say $j \in \text{supp}(c) \setminus \text{supp}(\partial(c))$. Let $k \notin \text{supp}(c)$. Then as in Remark/Definition 3.1, we can change the support $j$ to $k$ and replace $c$ by some $c' := \sigma^*(c)$ so that $c$ and $c'$ have the same boundary. This replacement of $c$ by $c'$ is called the RS-operation. In general, if $d'$ is the result of $d$ by applying the RS-operation to a subsumand of $d$, then $\partial(d) = \partial(d')$ and $|d'| \leq |d|$.

**Remark/Definition 3.3.**

(1) In general, the CR-operation is not symmetric. For example suppose that $c = f_0 - f_1 + f_2$ is given where $f_i$ is a 2-simplex with $\text{supp}(f_i) = \{0, 1, 2, 3\} \setminus \{i\}$ such that $f_i \parallel P(\{k, 3\}) = f_j \parallel P(\{k, 3\}) (\{i, j, k\} = \{0, 1, 2\})$. Now assume that by the CR-operation, $f_0 - f_1$ is replaced by $f_4 - f_2$ where $\text{supp}(f_4) = \{0, 1, 2\}$ and $\partial f_4 = \partial c$ so that $c$ is replaced by $(f_4 - f_2) + f_2 = f_4$. But $c$ is not obtained from $f_4$ using the CR-operation (unless $f_4$ is redundantly written as $f_4 - f_2 + f_2$).
(2) Now we say a 2-chain \( c \) is *proper* if for any \( c' \) obtained from \( c \) by finitely many applications of the CR or RS-operation (to subsummands), we have \( |c| = |c'| \). Among proper 2-chains, now the CR and RS-operations are symmetric. Moreover clearly any 2-chain is reduced to a proper 2-chain by finitely many applications of the two operations.

We call proper 2-chains \( c \) and \( c' \) are *equivalent* (written \( c \sim c' \)) if \( c' \) is obtained from \( c \) by finitely many applications of the CR or RS-operation to some subsummands. Hence if proper \( c \) and \( c' \) are equivalent then \( \partial(c) = \partial(c') \) and \( |c| = |c'|. \)

Now we are ready to define the notions of two different types of 2-chain having a 1-shell boundary.

**Definition 3.4.** Let \( \alpha \) be a 2-chain having a 1-shell boundary.

1. We call \( \alpha \) *renameable type* (or *RN-type*) if a subsummand of \( \alpha \) has a vanishing support. If \( \alpha \) is not an RN-type 2-chain (so \( |\text{supp}(\alpha)| = 0 \)) we call \( \alpha \) *non-renameable* (NR-type).

2. \( \alpha \) is said to be *minimal* if it is proper, and for any proper \( \alpha' \) equivalent to \( \alpha \) there does not exist a subsummand \( \beta \) of \( \alpha' \) such that \( \partial(\beta) = 0 \).

For the notational simplicity, given a simplex \( f_i \) with \( u = \{j_0, \ldots, j_k\} \subseteq \text{supp}(f_i) \), we write \( f_i^{j_0, \ldots, j_k} \) to denote \( f_i \upharpoonright \mathcal{P}(u) \). Also given a chain \( c = \sum_{i \in I} n_i f_i \) (in its unique form), we write \( c^{j_0, \ldots, j_k} \) to denote \( \sum_{i \in I} n_i f_i \), where \( J := \{i \in I \mid \text{supp}(f_i) = \{j_0, \ldots, j_k\} \} \).

For the rest of this section, we fix a 1-shell boundary \( f_{12} - f_{02} + f_{01} \) with \( \text{supp}(f_{jk}) = \{j < k\} \).

**Definition 3.5.** Let \( \alpha \) be a 2-chain having the boundary \( f_{12} - f_{02} + f_{01} \). A subchain \( \beta = \sum_{i=0}^{m} \epsilon_i b_i \) (\( b_i \) 2-simplex) of \( \alpha \) is called a *chain-walk in \( \alpha \) from \( f_{01} \) to \( -f_{02} \) if

1. there are non-zero numbers \( k_0, \ldots, k_{m+1} \) (not necessarily distinct) such that \( k_0 = 1, k_{m+1} = 2 \), and for \( i \leq m \), \( \text{supp}(b_i) = \{k_i, k_{i+1}, 0\} \);
2. each \( \epsilon_i \in \{1, -1\} \); \((\partial \epsilon_0 b_0)^{0,1} = f_{01}, (\partial \epsilon_m b_m)^{0,2} = -f_{02}; \) and
3. for \( 0 \leq i < m \),
   \((\partial \epsilon_i b_i)^{0,k_{i+1}} + (\partial \epsilon_{i+1} b_{i+1})^{0,k_{i+1}} = 0. \)

Notice that such a representation is sensitive to its order, and a chain-walk can have distinct representations. Unless said otherwise a chain-walk is written in a form of a representation. A subchain of the chain-walk \( \beta \) of a form \( \beta' := \sum_{i=j}^{m'} \epsilon_i b_i \) for some \( 0 \leq j < m' \leq m \) is called a *section* of \( \beta \). A chain-walk \( \beta \) in \( \alpha \) is called *maximal* (in \( \alpha \)) if it has the maximal possible length. We say \( \alpha \) is *centered at 0* if a (so every) maximal chain-walk in \( \alpha \) from \( f_{01} \) to \( -f_{02} \) is, as a chain, equal to \( \alpha \).

Now a chain-walk in \( \alpha \) from \( -f_{02} \) to \( f_{12} \), and that \( \alpha \) is centered at 2, and so on are similarly defined.

**Lemma 3.6.** Let \( \alpha \) be a 2-chain with the 1-shell boundary \( f_{12} - f_{02} + f_{01} \). Let \( \beta = \sum_{i=0}^{m} \epsilon_i b_i \) be a chain-walk in \( \alpha \), say from \( -f_{02} \) to \( f_{12} \). Assume there is a section \( \beta' := \sum_{i=j}^{m'} \epsilon_i b_i \) of \( \beta \) such that for \( \text{supp}(b_i) = \{2, k_i, k_{i+1}\} \), either \( k_i \neq k_{m'+1} \) for all \( i \in \{j, \ldots, m'\} \); or \( k_i \neq k_j \) for all \( i \in \{j+1, \ldots, m'+1\} \). Then by finitely many applications of the CR-operation to \( \beta' \), we obtain a 2-simplex \( c \) with \( \text{supp}(c) = \{2, k_j, k_{m'+1}\} \) and \( \epsilon = 1 \) or \( -1 \) so that \( \beta'' := \sum_{i=0}^{j-1} \epsilon_i b_i + cc + \sum_{i=m'}^{m} \epsilon_i b_i \) is still a chain-walk from \( -f_{02} \) to \( f_{12} \).

**Theorem 3.7.** Let \( \alpha \) be a minimal 2-chain with the boundary \( f_{12} - f_{02} + f_{01} \).

1. Assume \( \alpha \) is of NR-type. Then \( |\alpha| = 1 \) or \( |\alpha| \geq 5 \). If \( |\alpha| \geq 5 \) then any chain-walk in \( \alpha \) from \( f_{01} \) to \( -f_{02} \) is of the form \( 2 \sum_{i=0}^{2n} (-1)^i a_i \) which is a chain equal to \( \alpha \) such that \( f_{12} = c_{12}^{1,n} \) for some \( 1 \leq j \leq n - 1 \).
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(2) $\alpha$ is of RN-type iff $\alpha$ is equivalent to a 2-chain

$$\alpha' = a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i + a_{2n}$$

$(n \geq 1)$ which is a chain-walk from $f_{01}$ to $-f_{02}$ such that supp$(a_{2n}) = \{0, 1, 2\}$ and $\partial^0 a_{2n} = f_{12}$, $\partial^1(a_{2n}) = -f_{02}$.

Proof. (1) This is easy to check.

(2) Here we give a brief sketch of the left to right.

$(\Rightarrow)$ Note that $|\alpha| \geq 3$.

Claim 1. There is $\alpha_1 \sim \alpha$ centered at 2 such that $|\text{supp}(\alpha_1)| > 3$.

Claim 2. There is a 2-chain $\alpha_2 \sim \alpha_1$ such that $\alpha_2$ has a 1-simplex term $c$ (with the coefficient 1) such that supp$(c) = \{0, 1, 2\}$, and $f_{12} - f_{02} = \partial^0(c) - \partial^1(c)$.

Then let us take a chain-walk $\gamma$ from $f_{01}$ to $-f_{02}$ in $\alpha_2$ terminating with $c$. By repeatedly applying the CR-operations to $\gamma$ (while $c$ unchanged), we obtain a desired $\alpha' \sim \alpha_2$ centered at 0 forming a chain-walk from $f_{01}$ to $-f_{02}$. Then we reorder the representation of the chain-walk $\alpha'$ if necessary. \( \square \)

The following theorem is proved using the notions of directed graph theory which are not covered in this note.

Theorem 3.8. Let $\alpha$ be a minimal 2-chain having the 1-shell boundary $f_{12} - f_{02} + f_{01}$. Then $\alpha$ is equivalent to a 2-chain which is a chain-walk from $f_{01}$ to $-f_{02}$ such that supp$(\alpha') = \{0, 1, 2\}$.

In the next section, we explore some of the consequences of this theorem.

4. APPLICATION : MATRIX EXPRESSION

In this section, we introduce the notion of a matrix expression, which determines whether a given minimal 2-chain having a 1-shell boundary is of RN-type.

For the rest, we fix a minimal 2-chain $\alpha$ of length $2n + 1$ with the 1-shell boundary $f_{12} - f_{02} + f_{01}$, and supp$(\alpha) = \{0, 1, 2\}$. For $\{0, 1, 2\} = \{i, j, k\}$, $f'_{i}$ denotes $f_{jk}$ $(j < k)$. Fix $I = \{0, 1, 2\}$ and $J = \{0, \ldots, n\}$. Also, we write $\epsilon a \in \alpha$ to denote that a 2-simplex term $\epsilon a$ is in $\alpha$.

Definition 4.1. Let $\sum_{j=0}^{2n} (-1)^{j} a_{j}$ be a representation of the given $\alpha$ which is a chain-walk from $f'_{i}$ to $-f'_{i}$. By a matrix expression of (the representation of) $\alpha$, we mean a function $M : I \times J \rightarrow J$ such that

1. for each $i \in I$, $M | \{i\} \times J : (\{i\} \times) J \rightarrow J$ is a permutation of $J$;
2. for each $i \in I$, $M(i, 0)$ is an index such that $f'_{i} = \partial^0 a_{2M(i, 0)}$;
3. for each $i \in I$, $j \in J \setminus \{0\}$, $M(i, j)$ is an index such that $\partial^i a_{2j-1} = \partial^i a_{2M(i, j)}$.

Interpret $M(i, j)$ as an entry $m_{ij}$ of a matrix in the $(i+1)$th row and the $(j+1)$th column, then $M = (m_{ij})_{i,j}$ is a $3 \times (n + 1)$ matrix.

Notice that matrix expressions are determined according to the choices of pairs of terms which cancel out each other.

Theorem 4.2. The following conditions are equivalent:

1. $\alpha$ is of RN-type.
2. There is a matrix expression $M$ for a representation $\alpha = \sum_{j=0}^{2n} (-1)^{j} a_{j}$ such that for some $0 \leq i_0 < i_1 \leq 2$, and non-empty $J_0 \subseteq \{1, \ldots, n\}$, $M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0)$ as image set under the function $M$. 

Proof. \((\Rightarrow)\) Let \(\alpha_1\) be a subchain of \(\alpha = \sum_{j=0}^{2n} (-1)^j a_j\) such that \(\partial^i \alpha_1 = 0\) for \(i \in I \setminus \{i_*\}\) and \(|\alpha_1| = 2m\), where \(i_* \in \{0, 1, 2\}\) is a vanishing support. Let \(a_{2M(i,j)} \in \alpha_1\) for each \(i \in I \setminus \{i_*\}\) and \(j \in J_0 := \{j \in J \mid -a_{2j-1} \in \alpha_1\}\). Then here \(M\) satisfies Definition 4.1 and \(M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0)\), where \(\{i_0, i_1, i_*\} = I\), as desired.

\((\Leftarrow)\) Suppose that \(M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0)\), say \(J_1\), where \(J_0 \subseteq \{1, \ldots, n\}\) and \(0 \leq i_0 < i_1 \leq 2\).

Let \(\alpha_1 := \sum_{j \in J_1} a_{2j} + \sum_{j \in J_0} -a_{2j-1}\), a subsummand of \(\alpha\). Then we have \(\partial^0 \alpha_1 = \partial^1 \alpha_1 = 0\), so \(\alpha_1\) has a vanishing support \(i_*\), where \(\{i_0, i_1, i_*\} = I\).

We end this note by stating some consequences of Theorem 4.2 which can be proved by using permutations induced from matrix expressions.

For a matrix expression \(M : I \times J \to J\), there is a triple \((\sigma_{01}, \sigma_{12}, \sigma_{02})\) of permutations of \(J\) such that \(\sigma_{ik}\) is a map sending the \((i+1)\text{th}\) row to the \((k+1)\text{th}\) row, i.e., \(\sigma_{ik}(m_{ij}) = m_{kj}\) for \(j \in J\), and \(0 \leq i < k \leq 2\). Notice that \(\sigma_{02} = \sigma_{12} \circ \sigma_{01}\).

Theorem 4.3. If \(n\) is odd, then \(\alpha\) is always of RN-type.

Theorem 4.4. Suppose that for \(\alpha\) as in Definition 4.1, one of the following holds:

1. \(\partial^\ell a_{2j_0} = \partial^\ell a_{2j_1} = 0\) for some \(0 < j_0 < j_1 \leq n\) and \(0 \leq \ell \leq 2\);

2. \(\partial^\ell a_{2j_0} = \partial^\ell a_{2j_1}\) for some \(0 \leq j_0 < j_1 \leq n\) and \(0 \leq \ell \leq 2\).

Then \(\alpha\) is of RN-type.

References

3. J. Goodrick, B. Kim, and A. Kolesnikov, Amalgamation functors and homology groups in model theory, Preprint.