数理解析研究所講究録
第 1889 巻 2014 年 13-20

Evolution equations for quadrature identities

九州大学 マス・フォア・インダストリ研究所　小野寺 有紹
Michiaki Onodera
Institute of mathematics for industry,
Kyushu University

Abstract. A new geometric flow describing the motion of a closed surface is introduced. We characterize a continuous family of quadrature surfaces by the geometric flow. This characterization reduces the uniqueness of quadrature surfaces to the unique solvability of the geometric flow. In fact, we prove the uniqueness under the geometric condition that each surface has positive mean curvature.

1 Introduction

A classical quadrature formula is an identity of the form

$$\int x^k \, d\mu = \int_\Omega x^k \, dx \qquad (k = 0, 1, \ldots, d), \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is a bounded open interval and $\mu = \sum_{j=1}^{l} \alpha_j \delta_{x_j}$ with δ_{x_j} being the Dirac measure supported at x_j. Here, $d \in \mathbb{N}$, $\alpha_j > 0$ and $x_j \in \Omega$ are called respectively the degree, weights and nodes of the formula. Such a formula is practically useful for the numerical computation of the integral of a polynomial, since the formula reduces the hard computation to simple point evaluations of the polynomial.

One may desire the construction of a quadrature formula of degree ∞, i.e., a pair (Ω, μ) such that the identity (1.1) holds for all $k = 0, 1, \ldots$. However, Weierstrass' polynomial approximation theorem implies that no such formulas exist for a bounded interval $\Omega \subset \mathbb{R}$. Interestingly, if we allow Ω to be a two-dimensional bounded domain, then the situation becomes different. In fact, we can prove that, for a given measure $\mu = \sum_{j=1}^{l} \alpha_j \delta_{x_j}$ with sufficiently large α_j, there exists a unique bounded domain $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ such that

$$\int z^k \, d\mu = \int_\Omega z^k \, dx \, dy \qquad (k = 0, 1, \ldots), \tag{1.2}$$

where $z = x + iy$, $i := \sqrt{-1}$. Note that the left-hand side of (1.2) is the same value as that of (1.1) if $x_j \in \mathbb{R}$ for $j = 1, 2, \ldots, l$. Furthermore, if Ω is assumed to be smooth and simply-connected, then (1.2) is equivalent to

$$\int f(z)\,d\mu = \int_{\Omega} f(z)\,dx\,dy$$

for all holomorphic functions f defined in a neighborhood of $\overline{\Omega}$, or equivalently,

$$\int h(x,y)\,d\mu = \int_{\Omega} h(x,y)\,dx\,dy$$

for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$. This observation motivates us to define the so-called quadrature domain of a general positive finite measure μ with compact support in $\mathbb{R}^N$ as follows. Note that, here and in what follows, we will denote by x a vector in $\mathbb{R}^N$.

Definition 1.1. *We call a domain $\Omega \subset \mathbb{R}^N$ a quadrature domain of a measure μ (for harmonic functions) if*

$$\int h\,d\mu = \int_{\Omega} h\,dx \tag{1.3}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$.

Remark 1.2. A direct generalization of quadrature formula (1.1) to the higher-dimensional space $\mathbb{R}^N$ ($N \geq 3$), called a cubature formula, is an identity of the form

$$\int p\,d\mu = \int_{\Omega} p\,dx \qquad (p \in \mathcal{P}_d^N), \tag{1.4}$$

where $\mathcal{P}_d^N$ is the vector space of all real-valued polynomials of degree at most d in $\mathbb{R}^N$. Note that the cubature formula (1.4) concerns all polynomials, while the generalized quadrature formula (1.3) concerns only harmonic polynomials, but of all the degrees.

Quadrature domains arise in various physical contexts, and they are closely related to complex analysis and potential analysis. The simplest example of a quadrature domain is the unit ball $B(0,1)$ in $\mathbb{R}^N$ with the corresponding measure being a point mass $\omega_N\delta_0$:

$$\omega_N h(0) = \int_{B(0,1)} h\,dx$$

for harmonic functions h, where ω_N is the volume of $B(0,1)$. This is nothing but the mean value property of harmonic functions. Thus, the identity (1.3) can be seen as a generalization of the mean value formula for harmonic functions. From this point of view, we are naturally lead to the notion of a variant of quadrature domain, called quadrature surface, as follows.

Definition 1.3. *We call a $(N-1)$-dimensional closed surface $\Gamma \subset \mathbb{R}^N$, enclosing a bounded domain Ω, a quadrature surface of a measure μ (for harmonic functions) if*

$$\int h\, d\mu = \int_\Gamma h\, d\mathcal{H}^{N-1} \tag{1.5}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

In the following section, we will see that quadrature surfaces have a physical interpretation, and this is one of the reasons why many efforts have been devoted to the investigation of these quadrature identities. Nevertheless, the uniqueness of quadrature surfaces is not clear, as compared with quadrature domains. The purpose of this paper is to establish a qualitative result on this uniqueness issue by using a geometric flow, which will be introduced later. For our main results, see Theorems 4.1 and 4.3 in section 4.

2 Physical interpretation of quadrature identities

In this section we mainly focus on quadrature surfaces, since we can analogously give a physical interpretation of quadrature domains.

One of the classical problems in potential theory is to specify a closed surface Γ for a prescribed electric charge density μ in such a way that the uniform electric charge distribution on Γ induces the same potential in a neighborhood of the infinity as μ does. To formulate the problem mathematically, let F be the fundamental solution of $-\Delta$ in $\mathbb{R}^N$, i.e.,

$$F(x) := \begin{cases} -\dfrac{1}{2\pi}\log|x| & (N=2), \\ \dfrac{1}{N(N-2)\omega_N|x|^{N-2}} & (N \geq 3), \end{cases} \tag{2.1}$$

and let $\mathcal{H}^{N-1}\lfloor\Gamma$ denote the Hausdorff measure restricted to the surface Γ. Then, the problem can be stated as follows: For a prescribed finite positive measure μ with compact support in $\mathbb{R}^N$, find a $(N-1)$-dimensional closed surface Γ enclosing a bounded domain Ω such that $F * \mu = F * \mathcal{H}^{N-1}\lfloor\Gamma$ in $\mathbb{R}^N \setminus \overline{\Omega}$, i.e.,

$$\int F(x-y)\, d\mu(y) = \int_\Gamma F(x-y)\, d\mathcal{H}^{N-1}(y) \quad \left(x \in \mathbb{R}^N \setminus \overline{\Omega}\right). \tag{2.2}$$

As a matter of fact, (2.2) is equivalent to the identity (1.5), i.e., that Γ is a quadrature surface of μ. Indeed, it is obvious that (1.5) implies (2.2). Conversely, if Γ satisfies (2.2), then by extending each harmonic function h to be smooth and have compact support in $\mathbb{R}^N$, we see that

$$\begin{aligned}\int h(y)\,d\mu(y) &= \int_{\mathbb{R}^N} \Delta h(x) \left(\int F(y-x)\,d\mu(y) \right) dx \\ &= \int_{\mathbb{R}^N} \Delta h(x) \left(\int_\Gamma F(y-x)\,d\mathcal{H}^{N-1}(y) \right) dx \\ &= \int_\Gamma h(y)\,d\mathcal{H}^{N-1}(y).\end{aligned}$$

Thus, (1.5) follows from (2.2). Therefore, a quadrature surface is a surface which produces the same electric potential as a given electric charge density μ.

As we have seen as an example of a quadrature domain, the mean value property of harmonic functions also implies that (1.5) holds when $\mu = N\omega_N\delta_0$ and $\Gamma = \partial B(0,1)$. An inverse problem, referred to as the "Potato Kugel" problem especially for quadrature domains, asks if there is no other domain or surface which produces the same potential as the one point mass. In other words, the problem asks if it is possible to determine the shape of potato only from its potential. This problem, both for domains and surfaces with a point mass, were affirmatively solved. However, if a general measure μ is concerned, then an example in the paper of Henrot [12] shows that the uniqueness of a quadrature surface does not hold in general, even for simple measures such as two point masses. On the other hand, there is a general result on the uniqueness of a quadrature domain as shown in the next section.

3 Previous studies on quadrature identities

The existence of a quadrature surface Γ of a prescribed μ has been studied by several authors with different approaches. Developing the idea of super/subsolutions of Beurling [4], Henrot [12] was able to prove that the existence of Γ is guaranteed when a supersolution and a subsolution are available. Gustafsson & Shahgholian [11] followed a variational approach developed by Alt & Caffarelli [1], namely, they consider the minimization problem for the functional

$$J(u) := \int_{\mathbb{R}^N} \left(|\nabla u|^2 - 2fu + \chi_{\{u>0\}} \right) dx,$$

and proved the existence and regularity of a minimizer u. Then, u is shown to satisfy the Euler-Lagrange equation

$$-\Delta u = f\lfloor\Omega - \mathcal{H}^{N-1}\lfloor\partial\Omega, \qquad \Omega = \{u>0\},$$

and thus $\Gamma = \partial\Omega$ is a quadrature surface of μ with $d\mu = f\,dx$.

Similarly, a quadrature domain has a variational characterization and can be obtained by solving an obstacle problem (see Sakai [18] and Gustafsson [10] for the detail). Moreover, the uniqueness of a quadrature domain follows from an argument based on the maximum principle. Indeed, it was shown by Sakai [17] that, if a quadrature domain Ω satisfies

$$F * (\mu - \chi_\Omega) > 0$$

everywhere in Ω, then there is no quadrature domain other than Ω. The above condition can be verified, in particular, when μ concentrates, relative to Ω.

However, as pointed out by Henrot [12], the uniqueness of a quadrature surface cannot be expected in general. He showed an example that the number of connected quadrature surfaces of $\mu(t) := t\delta_{(1,0)} + t\delta_{(-1,0)}$ in $\mathbb{R}^2$ changes according to the value of $t > 0$. The collapse of the uniqueness seems to indicate a bifurcation phenomenon of solutions to (1.5) with a parametrized measure $\mu = \mu(t)$. Hence, toward understanding of the uniqueness issue, we need to consider the corresponding family of surfaces $\Gamma = \Gamma(t)$. In this respect, it is natural to ask if there is a "flow" for surfaces $\{\Gamma(t)\}_{t>0}$ such that each $\Gamma(t)$ is a quadrature surface of a given parametrized measure $\mu(t)$. As a matter of fact, when $\mu(t) = t\delta_0 + \chi_{\Omega(0)}$ and $\Omega(t)$ is the corresponding quadrature domain, it is known that the Hele-Shaw flow, a model of interface dynamics in fluid mechanics, plays the desired role. This surprising connection between the two different physical problems was discovered by Richardson [16]. From this fact, the investigation of the evolution of quadrature domains is reduced to that of the Hele-Shaw flow, and the latter has been successfully proceeded by complex analysis and several methods in partial differential equations.

4 Geometric flow for quadrature surfaces

We are thus motivated to derive a flow having the corresponding property for quadrature surfaces, and eventually arrive at the following geometric flow:

$$\text{(4.1)} \qquad \begin{aligned} & v_n = p \quad \text{for } x \in \partial\Omega(t), \\ & \text{where} \begin{cases} -\Delta p = \mu & \text{for } x \in \Omega(t), \\ (N-1)Hp + \dfrac{\partial p}{\partial n} = 0 & \text{for } x \in \partial\Omega(t), \end{cases} \end{aligned}$$

where v_n is the growing speed of $\partial\Omega(t)$ in the outer normal direction and H is the mean curvature of $\partial\Omega(t)$. Here and in what follows, μ denotes a finite positive Radon measure with compact support in $\Omega(0)$. Note that, for each fixed time $t > 0$, the maximum principle applied to the elliptic boundary problem in (4.1) yields that $p > 0$ everywhere on $\partial\Omega(t)$ if H is positive. In other words, $\Omega(t)$ expands monotonically as long as the mean curvature of $\partial\Omega(t)$ is positive.

The following theorem shows that, as desired, for a given $\partial\Omega(0)$ as initial surface, the solution to (4.1) turns out to be a one-parameter family of quadrature surfaces. Moreover, we will see that (4.1) is the only possible flow having this property. Here, we call $\{\partial\Omega(t)\}_{0\le t<T}$ a $C^{3+\alpha}$ family of surfaces if each $\partial\Omega(t)$ is of $C^{3+\alpha}$ and its time derivative is of $C^{2+\alpha}$, namely, $\partial\Omega(t)$ can be locally represented as a graph of a function in the Hölder space $C^{3+\alpha}$ and its time derivative is in $C^{2+\alpha}$ (see Section 3).

Theorem 4.1. *Let $\{\partial\Omega(t)\}_{0\le t<T}$ be a $C^{3+\alpha}$ family of surfaces, and assume that each $\partial\Omega(t)$ has positive mean curvature. Then, the following are equivalent:*

(*i*) $\{\partial\Omega(t)\}_{0\le t<T}$ *is a solution to* (4.1);

(*ii*) *Each $\partial\Omega(t)$ is a quadrature surface of $\mu(t) := t\mu + \mathcal{H}^{N-1}\lfloor\partial\Omega(0)$, i.e.,*

$$\int_{\partial\Omega(0)} h\, d\mathcal{H}^{N-1} + t\int h\, d\mu = \int_{\partial\Omega(t)} h\, d\mathcal{H}^{N-1} \tag{4.2}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega(t)}$.

Remark 4.2. The exponent $3+\alpha$ naturally arises in the context of the Schauder theory for the oblique derivative problem (see Gilbarg & Trudinger [9]). Indeed, the regularity $H \in C^{1+\alpha}$ of the coefficient function H in the boundary condition is required for the existence of a solution $p \in C^{2+\alpha}(\overline{\Omega(t)})$ to the elliptic equation in (4.1). This implies that $\partial\Omega(t)$ is of $C^{3+\alpha}$. It is worth noting that, by taking appropriate coordinates, v_n can be regarded as the time derivative of a local function representation of $\partial\Omega(t)$. Hence, it is natural to impose the same regularity as $v_n = p \in C^{2+\alpha}$ on the time derivative of $\partial\Omega(t)$.

Theorem 4.1 enables us to reduce the uniqueness of a continuous family of quadrature surfaces $\Gamma(t)$ of $\mu(t)$ to the unique solvability of the geometric flow (4.1). In fact, the latter is guaranteed by the following theorem. Here, $\{\partial\Omega(t)\}_{0\le t<T}$ is called a $h^{3+\alpha}$ solution if it is a $h^{3+\alpha}$ family of surfaces and satisfies (4.1), where $h^{3+\alpha}$ is the so-called little Hölder space and is defined as the closure of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions in the topology of the Hölder space $C^{3+\alpha}$. Since our argument relies on the theory of maximal regularity of Da Prato and Grisvard [5], it is necessary to use $h^{3+\alpha}$, characterized as a continuous interpolation space, instead of $C^{3+\alpha}$.

Theorem 4.3. *There exists a unique $h^{3+\alpha}$ solution $\{\partial\Omega(t)\}_{0\le t<T}$ to* (4.1) *for any $h^{3+\alpha}$ initial surface $\partial\Omega(0)$ with positive mean curvature.*

Let us plot the points $(\Gamma, t) \in h^{3+\alpha} \times \mathbb{R}$ if Γ is a quadrature surface of $\mu(t)$. Theorem 4.3 shows that such points form a curve

$$t \mapsto (\partial\Omega(t), t) \quad (t \in [0,T))$$

in $h^{3+\alpha}\times\mathbb{R}$ starting from $(\partial\Omega(0), 0)$, if $\partial\Omega(0)$ has positive mean curvature. Moreover, as the parameter t increases, the curve does not split into two curves from any point $(\partial\Omega(t), t)$ unless $\partial\Omega(t)$ loses the positiveness of the mean curvature.

Corollary 4.4. *There is no curve*

$$s \mapsto (\Gamma(s), t(s)) \quad (s \in [0,\varepsilon))$$

of an $h^{3+\alpha}$ family of quadrature surfaces such that $(\Gamma(0), t(0)) = (\partial\Omega(0), 0)$, $\Gamma(s) \neq \partial\Omega(t(s))$ for $0 < s < \varepsilon$, and $t'(0) \ge 0$.

The proofs of Theorems 4.1, 4.3 and Corollary 4.4 relies on various theories in analysis, and are beyond the scope of this paper. For the interested reader, we refer to [15] for the detail.

References

[1] Alt, H. W.; Caffarelli, L. A., Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), 105–144.

[2] Amann, H., Linear and quasilinear parabolic problems. Vol. I. Abstract linear theory. Monographs in Mathematics, 89. *Birkhäuser Boston, Inc., Boston, MA*, 1995.

[3] Angenent, S. B., Nonlinear analytic semiflows. *Proc. Roy. Soc. Edinburgh Sect. A* **115** (1990), 91–107.

[4] Beurling, A., On free-boundary problems for the Laplace equation. *Sem. on Analytic Funcitons* **1**, Inst. for Advanced Study Princeton (1957), 248–263.

[5] Da Prato, G.; Grisvard, P., Equations d'évolution abstraites non linéaires de type parabolique. *Ann. Mat. Pura Appl. (4)* **120** (1979), 329–396.

[6] Escher, J.; Simonett, G., Maximal regularity for a free boundary problem. *NoDEA* **2** (1995), 463–510.

[7] Escher, J.; Simonett, G., Classical solutions for Hele-Shaw models with surface tension. *Adv. Differential Equations* **2** (1997), no. 4, 619–642.

[8] Escher, J.; Simonett, G., Classical solutions of multidimensional Hele-Shaw models. *SIAM J. Math. Anal.* **28** (1997), no. 5, 1028–1047.

[9] Gilbarg, D.; Trudinger, N. S., Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. *Springer-Verlag, Berlin*, 2001.

[10] Gustafsson, B., Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows. *SIAM J. Math. Anal.*, **16** (1985), no. 2, 279–300.

[11] Gustafsson, B.; Shahgholian, H., Existence and geometric properties of solutions of a free boundary problem in potential theory. *J. Reine Angew. Math.* **473** (1996), 137–179.

[12] Henrot, A., Subsolutions and supersolutions in free boundary problems. *Ark. Mat.* **32** (1994), 79–98.

[13] Lunardi, A., Analytic semigroups and optimal regularity in parabolic problems. Progress in Nonlinear Differential Equations and their Applications, 16. *Birkhäuser Verlag, Basel*, 1995.

[14] Ni, Wei-Ming; Takagi, Izumi, On the shape of least-energy solutions to a semilinear Neumann problem. Comm. Pure Appl. Math. **44** (1991), 819–851.

[15] Onodera, M., Geometric flows for quadrature identities. *preprint.*

[16] Richardson, S., Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. *J. Fluid Mech.* **56** (1972), 609–618.

[17] Sakai, M., Quadrature Domains. Lecture Notes in Mathematics, 934. *Springer-Verlag, Berlin-New York*, 1982.

[18] Sakai, M., Application of variational inequalities to the existence theorem on quadrature domains. *Trans. Amer. Math. Soc.* **276** (1983), 267–279.