数理解析研究所講究録
第 1890 巻 2014 年 85-94

内部波ビームの3次元的安定性

神戸大学 (Kobe University)　片岡　武 (Takeshi Kataoka)
Massachusetts Institute of Technology　Triantaphyllos R. Akylas

要旨

一様な密度成層流体中を伝播する内部波ビームの 3 次元撹乱に対する線形安定性を取り扱った。具体的には，撹乱の波長がビームの幅に比べて十分長い場合を仮定し，漸近理論を駆使して Euler 方程式系を基に変調安定性を調べた。その結果，一方向のみにエネルギーを伝える進行波ビームは，振幅がある値を超えると変調不安定となり，両方向にエネルギーを伝える定在波ビームは，任意の振幅において変調不安定となることが分かった。

1. 緒言

In an inviscid, incompressible, uniformly stratified fluid of constant Brunt-Väisälä frequency N_0, a plane internal wave has the wave frequency ω which is a function only of the angle θ between the wavenumber direction and the vertical[1]:

$$\omega = N_0 \sin\theta . \tag{1.1}$$

The internal wave beam involves plane waves with various wavenumbers l for a certain fixed angle θ, and the beam is localized in the wavenumber direction. Such localization is possible because internal waves essentially propagate perpendicular to the wave crest.

Internal wave beams can be readily produced from a two-dimensional oscillating source of a given frequency $\omega_0 (< N_0)$. The induced steady beam pattern consists of four straight lines stretching from the source with the angles $\pm\cos^{-1}(\omega_0/N_0)$ to the vertical. This well-known pattern is called 'St Andrew's Cross', and was first verified experimentally by Mowbray & Rarity[2] using vibration of a horizontal cylinder as an oscillating source.

In the present study, we examine the linear stability of these internal wave beams to long-wavelength three-dimensional perturbations. The stability of the internal wave beam was treated in the past only by Tabaei and Akylas[3], and they found that the wave beam is two-dimensionally stable. Here we examine the stability to three-dimensional perturbations, and found that they are, in fact, three-dimensionally unstable if their amplitude exceeds some threshold value for progressive beams and unstable for any amplitude for purely standing beams. This report is based on Kataoka and Akylas[4].

2. 基礎方程式

Consider three-dimensional internal wave disturbances in an inviscid, incompressible, uniformly stratified Boussinesq fluid of constant Brunt-Väisälä frequency N_0. For the purpose of studying the stability of an internal wave beam, it is convenient to work with the spatial coordinates (ξ,η,ζ), the along-beam, across-beam and horizontal transverse directions, respectively (Fig. 1). We use dimensionless variables throughout, employing the same scalings as in Tabaei & Akylas[3] (with the beam width as characteristic length, $1/N_0$ as time scale, and a typical value of the background density). The flow velocity in the (ξ,η,ζ) directions is denoted by $\boldsymbol{u}=(u,v,w)$. The governing equations are

$$\nabla\cdot\boldsymbol{u}=0, \tag{2.1a}$$
$$\rho_t+\boldsymbol{u}\cdot\nabla\rho=-u\sin\theta+v\cos\theta, \tag{2.1b}$$
$$u_t+\boldsymbol{u}\cdot\nabla\rho=-p_\xi+\rho\sin\theta, \tag{2.1c}$$
$$v_t+\boldsymbol{u}\cdot\nabla v=-p_\eta-\rho\cos\theta, \tag{2.1d}$$
$$w_t+\boldsymbol{u}\cdot\nabla w=-p_\zeta, \tag{2.1e}$$

where θ is an angle between the η axis and the vertical, t is the time, and ρ and p are the density and pressure perturbations from the background state, respectively, and the subscripts t, ξ, η and ζ denote partial differentiation with respect to these variables.

Equations (2.1) have the following exact solution representing finite-amplitude internal wave beam

$$\begin{cases} u=u_0(t,\eta)\equiv\left\{U(\eta)e^{-\mathrm{i}\sin\theta t}+\text{c.c.}\right\}, \\ v=w=0, \\ \rho=\rho_0(t,\eta)\equiv\left\{-\mathrm{i}U(\eta)e^{-\mathrm{i}\sin\theta t}+\text{c.c.}\right\}, \\ p=p_0(t,\eta)\equiv\left\{\mathrm{i}\cos\theta\int^{\eta}U(\eta')\mathrm{d}\eta' e^{-\mathrm{i}\sin\theta t}+\text{c.c.}\right\}, \end{cases} \tag{2.2}$$

where $U(\eta)$ is a given arbitrary function of η which decays rapidly as $\eta\to\pm\infty$ and c.c. denotes complex conjugate. In the present study we exclude the limiting cases of $\theta\to 0$ and $\pi/2$ for which the internal wave beam approaches a horizontal steady shear flow or becomes nearly vertical with frequency close to the Brunt-Väisälä frequency. Thus, we put

$$0<\theta<\frac{\pi}{2}. \tag{2.3}$$

Moreover, in order to avoid density inversions so that the internal wave beam (2.6) is statically stable, the derivative of the associated vertical particle displacement $\left\{\mathrm{i}U(\eta)e^{-\mathrm{i}\sin\theta t}+c.c.\right\}$ with respect to the vertical direction must not exceed unity in magnitude everywhere[5], i.e.

$$\left|\frac{\mathrm{d}U}{\mathrm{d}\eta}\right|<\frac{1}{2\cos\theta}. \tag{2.4}$$

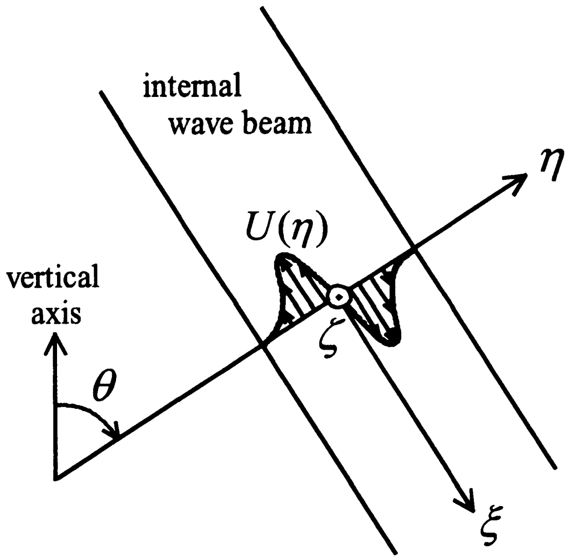

Fig. 1 Geometry.

We examine the linear stability of the above statically stable internal wave beam (2.2)—(2.4) to three-dimensional perturbations. To this end, employing Floquet theory, we write

$$\begin{Bmatrix} u \\ v \\ w \\ \rho \\ p \end{Bmatrix} = \begin{Bmatrix} u_0(t,\eta) \\ 0 \\ 0 \\ \rho_0(t,\eta) \\ p_0(t,\eta) \end{Bmatrix} + \begin{Bmatrix} \hat{u}(t,\eta) \\ \hat{v}(t,\eta) \\ \hat{w}(t,\eta) \\ \hat{\rho}(t,\eta) \\ \hat{p}(t,\eta) \end{Bmatrix} \exp[\sigma t + \mathrm{i}(k\xi + m\zeta)], \tag{2.5}$$

where $(\hat{u},\hat{v},\hat{w},\hat{\rho},\hat{p})$ are unknown functions of t and η which are periodic in t with the same period $2\pi/\sin\theta$ as that of the internal wave beam, σ is an unknown complex constant, and k and m are given real constants. Substituting (2.5) into (2.1) and linearizing with respect to the perturbations, we have the following set of equations for $(\hat{u},\hat{v},\hat{w},\hat{\rho},\hat{p})$:

$$\mathrm{i}k\hat{u} + \frac{\partial \hat{v}}{\partial \eta} + \mathrm{i}m\hat{w} = 0, \tag{2.6a}$$

$$\frac{\partial \hat{\rho}}{\partial t} + \sin\theta\,\hat{u} + \left(\frac{\partial \rho_0}{\partial \eta} - \cos\theta\right)\hat{v} = -(\sigma + \mathrm{i}ku_0)\hat{\rho}, \tag{2.6b}$$

$$\frac{\partial \hat{u}}{\partial t} - \sin\theta\,\hat{\rho} + \frac{\partial u_0}{\partial \eta}\hat{v} = -[\mathrm{i}k\hat{p} + (\sigma + \mathrm{i}ku_0)\hat{u}], \tag{2.6c}$$

$$\frac{\partial \hat{v}}{\partial t} + \cos\theta\,\hat{\rho} + \frac{\partial \hat{p}}{\partial \eta} = -(\sigma + \mathrm{i}ku_0)\hat{v}, \tag{2.6d}$$

$$\frac{\partial \hat{w}}{\partial t} + \mathrm{i}m\hat{p} = -(\sigma + \mathrm{i}ku_0)\hat{w}. \tag{2.6e}$$

In addition to being periodic in t with period $2\pi/\sin\theta$,

$$(\hat{u},\hat{v},\hat{w},\hat{\rho},\hat{p})(t) = (\hat{u},\hat{v},\hat{w},\hat{\rho},\hat{p})\left(t + \frac{2\pi}{\sin\theta}\right), \tag{2.7}$$

the perturbations must also decay in η,

$$(\hat{u},\hat{v},\hat{w},\hat{\rho},\hat{p}) \to 0 \quad \text{as} \quad \eta \to \pm\infty. \tag{2.8}$$

The above set of equations (2.6)—(2.8) constitutes an eigenvalue problem, σ being the eigenvalue parameter. When there is a solution $(\hat{u},\hat{v},\hat{w},\hat{\rho},\hat{p})$ with σ having a positive real part, the corresponding internal wave beam is linearly unstable. Since a solution for $k<0$ ($m<0$) is obtained from that for $k>0$ ($m>0$) by $(\hat{u},\hat{v},\hat{\rho},u_0,\rho_0,\eta) \to (-\hat{u},-\hat{v},-\hat{\rho},-u_0,-\rho_0,-\eta)$ ($\hat{w} \to -\hat{w}$), we set

$$k>0, \quad m>0. \tag{2.9}$$

3. 漸近解析 ($k \sim m^{3/2} << 1$)

Assuming now that the perturbation is long in the ξ and ζ directions, that is, k and m in (2.9) are small,

$$k = \varepsilon^3 \kappa, \quad m = \varepsilon^2, \tag{3.1}$$

where ε is a small positive parameter and κ is a positive $O(1)$ constant, we seek an asymptotic solution of (2.6)-(2.8) for small $0 < \varepsilon << 1$.

3.1. *Inner solution*

Putting aside the decaying boundary condition (2.8), we seek a solution of (2.6) which satisfies the periodicity condition (2.7) in t and varies by $O(1)$ in η, by introducing the following expansions

$$\begin{cases} \hat{u} = \hat{u}^{(0)} + \varepsilon\hat{u}^{(1)} + \cdots, \\ \hat{v} = \varepsilon^3\hat{v}^{(3)} + \varepsilon^4\hat{v}^{(4)} + \cdots, \\ \hat{w} = \varepsilon^2\hat{w}^{(2)} + \varepsilon^3\hat{w}^{(3)} + \cdots, \\ \hat{\rho} = \hat{\rho}^{(0)} + \varepsilon\hat{\rho}^{(1)} + \cdots, \\ \hat{p} = \hat{p}^{(0)} + \varepsilon\hat{p}^{(1)} + \cdots, \\ \sigma = \varepsilon^3\sigma^{(3)} \quad + \varepsilon^5\sigma^{(5)} + \cdots, \end{cases} \tag{3.2}$$

Substituting (3.1) and (3.2) into (2.6) and collecting the same-order terms in ε, we obtain a series of equations for $(\hat{u}^{(n)}, \hat{v}^{(n+3)}, \hat{w}^{(n+2)}, \hat{\rho}^{(n)}, \hat{p}^{(n)})$ $(n = 0, 1, 2, \cdots)$:

$$\mathrm{i}\kappa\hat{u}^{(n)} + \frac{\partial\hat{v}^{(n+3)}}{\partial\eta} = F^{(n)}, \tag{3.3a}$$

$$\frac{\partial\hat{\rho}^{(n)}}{\partial t} + \sin\theta\,\hat{u}^{(n)} = G^{(n)}, \tag{3.3b}$$

$$\frac{\partial\hat{u}^{(n)}}{\partial t} - \sin\theta\,\hat{\rho}^{(n)} = H^{(n)}, \tag{3.3c}$$

$$\frac{\partial\hat{p}^{(n)}}{\partial\eta} + \cos\theta\,\hat{\rho}^{(n)} = I^{(n)}, \tag{3.3d}$$

$$\frac{\partial\hat{w}^{(n+2)}}{\partial t} + \mathrm{i}\,\hat{p}^{(n)} = J^{(n)}, \tag{3.3e}$$

where the terms on the right-hand sides are inhomogeneous terms and given by

$$F^{(0)} = G^{(0)} = H^{(0)} = I^{(0)} = J^{(0)} = 0 \quad (n = 0) \tag{3.4a}$$

$$F^{(n)} = -\mathrm{i}\hat{w}^{(n+1)}, \;\; G^{(n)} = H^{(n)} = I^{(n)} = J^{(n)} = 0 \quad (n = 1, 2), \tag{3.4b}$$

$$\begin{cases} F^{(n)} = -\mathrm{i}\hat{w}^{(n+1)} \\ G^{(n)} = \left(\cos\theta - \dfrac{\partial\rho_0}{\partial\eta}\right)\hat{v}^{(n)} - (\sigma^{(3)} + \mathrm{i}\kappa u_0)\hat{\rho}^{(n-3)} \\ H^{(n)} = -\dfrac{\partial u_0}{\partial\eta}\hat{v}^{(n)} - \mathrm{i}\kappa\hat{p}^{(n-3)} - (\sigma^{(3)} + \mathrm{i}\kappa u_0)\hat{u}^{(n-3)} \quad (n = 3, 4). \\ I^{(n)} = -\dfrac{\partial\hat{v}^{(n)}}{\partial t} \\ J^{(n)} = -(\sigma^{(3)} + \mathrm{i}\kappa u_0)\hat{w}^{(n-1)} \end{cases} \tag{3.4c}$$

For $n = 0$, equations (3.3) are homogeneous and have the following nontrivial solution that satisfies the periodicity condition (2.7)

$$\begin{Bmatrix} \hat{u}^{(0)} \\ \hat{v}^{(3)} \\ \hat{w}^{(2)} \\ \hat{\rho}^{(0)} \\ \hat{p}^{(0)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \overline{V}^{(3)} \\ \overline{W}^{(2)} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \hat{U}_-^{(0)} \\ \hat{V}_-^{(3)} \\ \mathrm{i}\cot\theta\int^{\eta}\hat{U}_-^{(0)}\mathrm{d}\eta' \\ -\mathrm{i}\hat{U}_-^{(0)} \\ \mathrm{i}\cos\theta\int^{\eta}\hat{U}_-^{(0)}\mathrm{d}\eta' \end{Bmatrix} \mathrm{e}^{-\mathrm{i}\sin\theta t} + \begin{Bmatrix} \hat{U}_+^{(0)} \\ \hat{V}_+^{(3)} \\ \mathrm{i}\cot\theta\int^{\eta}\hat{U}_+^{(0)}\mathrm{d}\eta' \\ \mathrm{i}\hat{U}_+^{(0)} \\ -\mathrm{i}\cos\theta\int^{\eta}\hat{U}_+^{(0)}\mathrm{d}\eta' \end{Bmatrix} \mathrm{e}^{\mathrm{i}\sin\theta t}, \tag{3.5}$$

where $\overline{V}^{(3)}$ is constant and

$$\hat{V}_{\pm}^{(3)} = -\mathrm{i}\kappa\int^{\eta}\hat{U}_-^{(0)}\mathrm{d}\eta'. \tag{3.6}$$

Here $\hat{U}_{\pm}^{(0)}(\eta)$ and $\overline{W}^{(2)}(\eta)$ are as yet undetermined functions of η: Capital-letter variables with the hat and subscript $\pm$ are the complex amplitudes of components proportional to $\sim e^{\pm i\sin\theta t}$ that have the same frequency as the underlying beam, and variables with the overbar denote mean-flow components (which are independent of t); $\overline{V}^{(3)}$ and $\overline{W}^{(2)}$, in particular, represent $O(\varepsilon^3)$ and $O(\varepsilon^2)$ mean flows in the across-beam (η) and the transverse (ζ) directions, respectively. The higher harmonic components ($\sim e^{\pm 2i\sin\theta t}, e^{\pm 3i\sin\theta t},\cdots$) do not appear at this level.

For $n\geq 1$, the equations (3.3) are inhomogeneous, and the inhomogeneous terms $G^{(n)}$, $H^{(n)}$, $I^{(n)}$ and $J^{(n)}$ on the right-hand sides of (3.3) must satisfy the following solvability conditions to have a solution

$$\int_0^{2\pi/\sin\theta} e^{\pm i\sin\theta t}\left(\pm iG^{(n)}+H^{(n)}\right)dt=0, \tag{3.7a}$$

$$\int_0^{2\pi/\sin\theta}\left[i(\cot\theta H^{(n)}+I^{(n)})-\frac{\partial J^{(n)}}{\partial\eta}\right]dt=0. \tag{3.7b}$$

For $n=1$ and 2, the solvability conditions (3.6) are identically satisfied, and a solution of (3.3) satisfying (2.7) becomes the same form as (3.5) with the numbers in the parentheses at any superscripts being added by n and

$$\overline{V}^{(n+3)}=-i\int^{\eta}\overline{W}^{(n+1)}d\eta', \tag{3.8a}$$

$$\hat{V}_{\pm}^{(n+3)}=-i\kappa\int^{\eta}\hat{U}_{\pm}^{(n)}d\eta'+\cot\theta\int^{\eta}\int^{\eta'}\hat{U}_{\pm}^{(n-1)}d\eta''d\eta', \tag{3.8b}$$

$$(n=1,2).$$

For $n=3$ and 4, the solvability conditions (3.7) become the following six equations for $(\hat{U}_{-}^{(0)},\hat{U}_{+}^{(0)},\overline{W}^{(2)},\hat{U}_{-}^{(1)},\hat{U}_{+}^{(1)},\overline{W}^{(3)})$:

$$\sigma^{(3)}\hat{U}_{-}^{(0)}=\kappa\cos\theta\int^{\eta}\hat{U}_{-}^{(0)}d\eta'-\frac{dU}{d\eta}\overline{V}^{(3)}, \tag{3.9a}$$

$$\sigma^{(3)}\hat{U}_{+}^{(0)}=-\kappa\cos\theta\int^{\eta}\hat{U}_{+}^{(0)}d\eta'-\frac{dU^*}{d\eta}\overline{V}^{(3)}, \tag{3.9b}$$

$$\sigma^{(3)}\frac{d\overline{W}^{(2)}}{d\eta}=2\kappa\cot\theta\left(\frac{dU^*}{d\eta}\int^{\eta}\hat{U}_{-}^{(0)}d\eta'+\frac{dU}{d\eta}\int^{\eta}\hat{U}_{+}^{(0)}d\eta'\right), \tag{3.9c}$$

$$\sigma^{(3)}\hat{U}_{-}^{(1)}=\cos\theta\left(\kappa\int^{\eta}\hat{U}_{-}^{(1)}d\eta'+\frac{i}{2}\cot\theta\int^{\eta}\int^{\eta'}\hat{U}_{-}^{(0)}d\eta''d\eta'\right)+i\frac{dU}{d\eta}\int^{\eta}\overline{W}^{(2)}d\eta', \tag{3.9d}$$

$$\sigma^{(3)}\hat{U}_{+}^{(1)}=-\cos\theta\left(\kappa\int^{\eta}\hat{U}_{+}^{(1)}d\eta'+\frac{i}{2}\cot\theta\int^{\eta}\int^{\eta'}\hat{U}_{+}^{(0)}d\eta''d\eta'\right)+i\frac{dU^*}{d\eta}\int^{\eta}\overline{W}^{(2)}d\eta', \tag{3.9e}$$

$$\begin{aligned}\sigma^{(3)}\frac{d\overline{W}^{(3)}}{d\eta}=2\cot\theta&\left[\frac{dU^*}{d\eta}\left(\kappa\int^{\eta}\hat{U}_{-}^{(1)}d\eta'+\frac{i}{2}\cot\theta\int^{\eta}\int^{\eta'}\hat{U}_{-}^{(0)}d\eta''d\eta'\right)\right.\\&\left.+\frac{dU}{d\eta}\left(\kappa\int^{\eta}\hat{U}_{+}^{(1)}d\eta'+\frac{i}{2}\cot\theta\int^{\eta}\int^{\eta'}\hat{U}_{+}^{(0)}d\eta''d\eta'\right)\right]\end{aligned} \tag{3.9f}$$

where the asterisk denotes complex conjugate. These equations for the amplitudes, $\hat{U}_{\pm}^{(0)}(\eta)$ and $\hat{U}_{\pm}^{(1)}(\eta)$, of the primary harmonic perturbation and the induced transverse mean flow, $\overline{W}^{(2)}(\eta)$ and $\overline{W}^{(3)}(\eta)$, must be supplemented with suitable boundary conditions. Specifically,

$$\int^{\eta}\int^{\eta'}\hat{U}_{-}^{(0)}d\eta''d\eta'\to 0,\quad \int^{\eta}\int^{\eta'}\hat{U}_{+}^{(0)}d\eta''d\eta'\to 0,\quad \int^{\eta}\hat{U}_{-}^{(1)}d\eta'\to 0,\quad \int^{\eta}\hat{U}_{+}^{(1)}d\eta'\to 0, \tag{3.10}$$

$$\overline{W}^{(2)}\to 0,\quad \overline{W}^{(3)}\to\mp\frac{i\overline{V}^{(3)}}{\sin\theta}\quad (\eta\to\pm\infty),$$

where matching with the outer solution (3.13) obtained in Section 3.2 is already taken into account.

The conditions (3.9) ensure that the flow field associated with the primary-harmonic perturbation, as well as the $O(\varepsilon^2)$ transverse mean flow component vanishes far away from the beam. The induced mean flow at $O(\varepsilon^3)$, however, does not remain locally confined in the vicinity of the beam:

$$(\hat{u},\hat{v},\hat{w}) \to \varepsilon^3 \overline{V}^{(3)}\left(\cot\theta, 1, \frac{\mp \mathrm{i}}{\sin\theta}\right) \quad (\eta \to \pm\infty), \tag{3.11}$$

where $\overline{V}^{(3)}$ is constant. In order to construct an overall solution of (2.6)—(2.7) that satisfies the decaying condition (2.8) in η, we must seek an outer solution which decays slowly in η at infinity and is connected to (3.11) in the inner limit.

3.2. *Outer solution*

Introducing a reduced coordinate

$$Y = \varepsilon^2 \eta, \tag{3.12}$$

we look for a solution which varies by $O(1)$ in Y and is independent of t (mean flow) of the following orders

$$\hat{u} = \varepsilon^3 \hat{u}_O(Y), \quad \hat{v} = \varepsilon^3 \hat{v}_O(Y), \quad \hat{w} = \varepsilon^3 \hat{w}_O(Y), \quad \hat{\rho} = \varepsilon^6 \hat{\rho}_O(Y), \quad \hat{p} = \varepsilon^4 p_O(Y). \tag{3.13}$$

The orders of (3.13) are determined by (3.11) and balance of terms in (2.6) noting that $u_0, \rho_0 \to 0$ ($|\eta| \to \infty$). Substituting (3.1), (3.12)—(3.13) and $\sigma = \varepsilon^3 \sigma^{(3)}$ into (2.6) and collecting the same-order terms in ε of each equation, we obtain

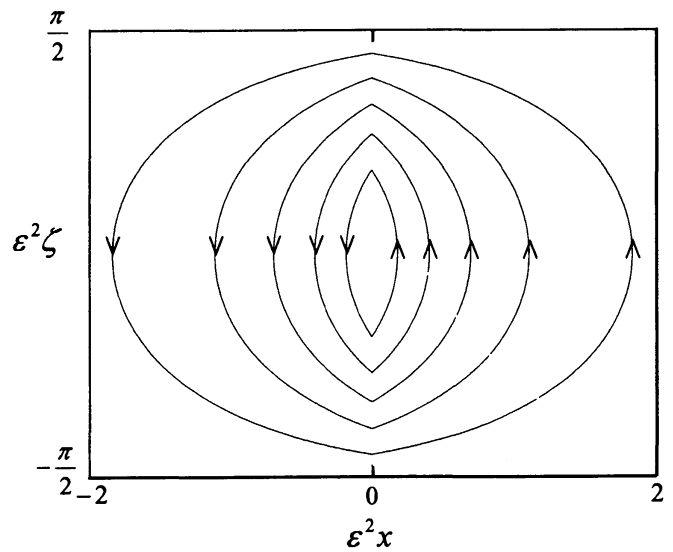

Fig. 2 Streamlines of the mean flow described by the outer solution (3.15) (with (2.5)). The abscissa $\varepsilon^2 x$ $[= \varepsilon^2(\xi\cos\theta + \eta\sin\theta)]$ is the horizontal direction perpendicular to the other horizontal transverse $\varepsilon^2\zeta$ direction (the ordinate) along the beam positioned at $x = 0$. Streamlines for $|\varepsilon^2\zeta| > \pi/2$ are symmetric with respect to $\varepsilon^2\zeta = \pm\pi/2$.

$$\begin{cases} \dfrac{\mathrm{d}\hat{v}_O}{\mathrm{d}Y} + \mathrm{i}\hat{w}_O = 0, \\ \sin\theta \hat{u}_O - \cos\theta \hat{v}_O = 0, \\ -\sin\theta \hat{\rho}_O + \sigma^{(3)} \hat{u}_O = 0, \\ \cos\theta \hat{\rho}_O + \dfrac{\mathrm{d}\hat{p}_O}{\mathrm{d}Y} + \sigma^{(3)} \hat{v}_O = 0, \\ \mathrm{i}\hat{p}_O + \sigma^{(3)} \hat{w}_O = 0. \end{cases} \tag{3.14}$$

These equations have a solution which decays as $|Y| \to \infty$ and is connected to (3.11) at $Y = 0$

$$\begin{Bmatrix} \hat{u}_O \\ \hat{v}_O \\ \hat{w}_O \\ \hat{\rho}_O \\ \hat{p}_O \end{Bmatrix} = \begin{Bmatrix} \cot\theta \\ 1 \\ -\mathrm{i}/\sin\theta \\ \sigma^{(3)} \cos\theta / \sin^2\theta \\ \sigma^{(3)} / \sin\theta \end{Bmatrix} \overline{V}^{(3)} \exp\left(-\frac{Y}{\sin\theta} \right) \quad \text{for } Y > 0, \tag{3.15a}$$

$$\begin{Bmatrix} \hat{u}_O \\ \hat{v}_O \\ \hat{w}_O \\ \hat{\rho}_O \\ \hat{p}_O \end{Bmatrix} = \begin{Bmatrix} \cot\theta \\ 1 \\ \mathrm{i}/\sin\theta \\ \sigma^{(3)} \cos\theta / \sin^2\theta \\ -\sigma^{(3)} / \sin\theta \end{Bmatrix} \overline{V}^{(3)} \exp\left(\frac{Y}{\sin\theta} \right) \quad \text{for } Y < 0, \tag{3.15b}$$

where $\overline{V}^{(3)}$ is constant. The flow described by (3.15) is purely horizontal because $\hat{u}_O / \hat{v}_O = \cot\theta$, and it forms a single circulating flow which traverses the beam because $\overline{V}^{(3)}$ is constant (figure 2).

Thus, we have constructed an overall solution of (2.6)—(2.7) which satisfies the decaying condition (2.8) under the supposition that there is a solution $(\hat{U}_-^{(0)}, \hat{U}_+^{(0)}, \overline{W}^{(2)}, \hat{U}_-^{(1)}, \hat{U}_+^{(1)}, \overline{W}^{(3)})$ of the eigenvalue problem (3.9)—(3.10). If the eigenvalue problem (3.9)—(3.10) has a solution whose eigenvalue $\sigma^{(3)}$ has a positive real part, the underlying beam is unstable. Its possibility is explored numerically in Section 4.

4. (3.9)-(3.10)の数値解

4.1. *Renormalization*

We let

$$\psi_\pm = \int^\eta \int^{\eta'} \hat{U}_\pm^{(0)} \mathrm{d}\eta'' \mathrm{d}\eta', \quad \psi_{S\pm} = \int^\eta \hat{U}_\pm^{(1)} \mathrm{d}\eta', \quad \varphi = -\mathrm{i}\tan\theta \int^\eta \overline{W}^{(2)} \mathrm{d}\eta', \quad \varphi_S = -\mathrm{i}\tan\theta \overline{W}^{(3)},$$
$$V = \tan\theta \overline{V}^{(3)}, \quad \tilde{\kappa} = 2\tan\theta\kappa, \quad \tilde{\sigma} = \frac{2\sin\theta}{\cos^2\theta}\sigma^{(3)}, \quad \tilde{U} = \frac{2}{\cos\theta}U, \tag{4.1}$$

and obtain a renormalized version of the eigenvalue problem (3.9)—(3.10):

$$\tilde{\sigma}\frac{\mathrm{d}^2\psi_-}{\mathrm{d}\eta^2} = \tilde{\kappa}\frac{\mathrm{d}\psi_-}{\mathrm{d}\eta} - \frac{\mathrm{d}\tilde{U}}{\mathrm{d}\eta}V, \tag{4.2a}$$

$$\tilde{\sigma}\frac{\mathrm{d}^2\psi_+}{\mathrm{d}\eta^2} = -\tilde{\kappa}\frac{\mathrm{d}\psi_+}{\mathrm{d}\eta} - \frac{\mathrm{d}\tilde{U}^*}{\mathrm{d}\eta}V, \tag{4.2b}$$

$$\tilde{\sigma}\frac{\mathrm{d}^2\varphi}{\mathrm{d}\eta^2} = -\mathrm{i}\tilde{\kappa}\left(\frac{\mathrm{d}\tilde{U}^*}{\mathrm{d}\eta}\frac{\mathrm{d}\psi_-}{\mathrm{d}\eta} + \frac{\mathrm{d}\tilde{U}}{\mathrm{d}\eta}\frac{\mathrm{d}\psi_+}{\mathrm{d}\eta} \right), \tag{4.2c}$$

$$\tilde{\sigma}\frac{\mathrm{d}\psi_{S-}}{\mathrm{d}\eta}=\tilde{\kappa}\psi_{S-}+\mathrm{i}\psi_{-}-\frac{\mathrm{d}\tilde{U}}{\mathrm{d}\eta}\varphi, \tag{4.2d}$$

$$\tilde{\sigma}\frac{\mathrm{d}\psi_{S+}}{\mathrm{d}\eta}=-\left(\tilde{\kappa}\psi_{S+}+\mathrm{i}\psi_{+}\right)-\frac{\mathrm{d}\tilde{U}^{*}}{\mathrm{d}\eta}\varphi, \tag{4.2e}$$

$$\tilde{\sigma}\frac{\mathrm{d}\varphi_{S}}{\mathrm{d}\eta}=-\mathrm{i}\left[\frac{\mathrm{d}\tilde{U}^{*}}{\mathrm{d}\eta}\left(\tilde{\kappa}\psi_{S-}+\mathrm{i}\psi_{-}\right)+\frac{\mathrm{d}\tilde{U}}{\mathrm{d}\eta}\left(\tilde{\kappa}\psi_{S+}+\mathrm{i}\psi_{+}\right)\right], \tag{4.2f}$$

with

$$\psi_{-}\to 0,\ \psi_{+}\to 0,\ \frac{\mathrm{d}\varphi}{\mathrm{d}\eta}\to 0,\ \psi_{S-}\to 0,\ \psi_{S+}\to 0,\ \varphi_{S}\to\frac{\mp V}{\sin\theta}\quad(\eta\to\pm\infty). \tag{4.3}$$

Equations (4.2)—(4.3) constitute the eigenvalue problem for $(\psi_{-},\psi_{+},\varphi,\psi_{S-},\psi_{S+},\varphi_{S})$, with $\tilde{\sigma}$ being the eigenvalue parameter. We solve this problem numerically.

The underlying beam profile $\tilde{U}(\eta)$ is chosen to be the same Gaussian streamfunction profiles as Tabaei et al.[5],

$$\tilde{U}(\eta)=\begin{cases}U_0\int_0^{\infty}A(l)\mathrm{e}^{\mathrm{i}l\eta}\mathrm{d}l & \text{(progressive beams)}, \quad (4.4\mathrm{a})\\ \dfrac{U_0}{2}\int_{-\infty}^{\infty}A(l)\mathrm{e}^{\mathrm{i}l\eta}\mathrm{d}l=-2U_0\eta e^{-2\eta^2} & \text{(standing beams)}, \quad (4.4\mathrm{b})\end{cases}$$

where U_0 is a positive parameter and $A(l)=\mathrm{i}le^{-l^2/8}/\sqrt{8\pi}$. Progressive beams describe uni-directional beams which involve plane waves with wavenumbers l of the same sign only, whereas standing beams include those of both signs. The profile $\tilde{U}(\eta)/U_0$ is shown in figure 3. Statically stable condition (2.4) becomes

$$U_0<\frac{1}{2\cos^2\theta}. \tag{4.5}$$

The internal wave beams (4.4) are statically stable for any θ if $U_0<0.5$. Even for the greater amplitudes, they are statically stable depending on the value of θ. In what follows, we present the stability results for progressive beams in Section 4.2 and standing beams in Section 4.3. For numerical method to solve (4.2)—(4.3), we use the finite-difference method for discretization and a standard QZ algorithm for the eigenvalue solver[6]. The parameters of (4.2)—(4.3) are θ, $\tilde{\kappa}$ and U_0.

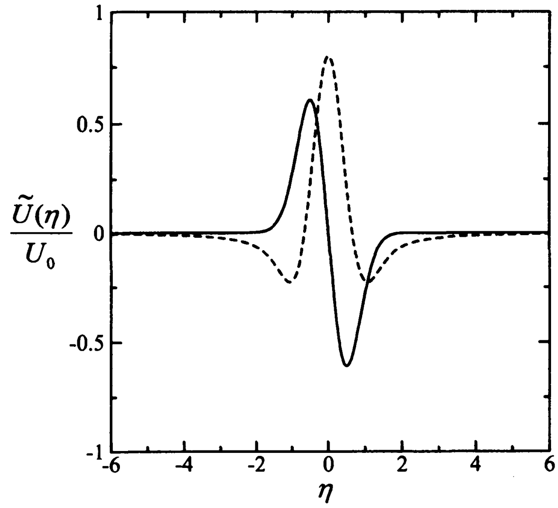

Fig. 3 Profiles $\tilde{U}(\eta)/U_0$ of the progressive beam (4.4a) (solid line: real part, dashed line: imaginary part) and the standing beam (4.4b) (solid line).

4.2. *Progressive beams*

Computed eigenvalues $\tilde{\sigma}$ with a positive real part versus $\tilde{\kappa}$ are plotted in figure 4 for $\theta = \pi/6$ and $\pi/3$. Amplitudes of the underlying beams are chosen to be $U_0 = 0.35$, 0.5 and 0.65 (these beams are all statically stable according to (4.5)). Figure 4(*a*) shows that progressive beams are unstable for $U_0 \geq 0.35$, and according to our numerical results, the critical amplitude of the instability is about $U_0 = 0.3$.

Figure 4 also shows that the growth rate $\mathrm{Re}[\tilde{\sigma}]$, which is an increasing function of $\tilde{\kappa}$ for small $\tilde{\kappa}$, reaches a peak at some finite $\tilde{\kappa}$ and finally falls to zero at the higher $\tilde{\kappa}$. Thus, the instability is three-dimensional (or oblique). The stability of the internal wave beam was first examined by Tabaei and Akylas[3] to longitudinal (two-dimensional) perturbation which corresponds to $\tilde{\kappa} \to \infty$ in the present notation, and found no instability. Their result is consistent with our result.

The imaginary part $\mathrm{Im}[\tilde{\sigma}]$ of the above complex eigenvalues is plotted in figure 4(*b*). It is always one-signed (negative) and the magnitude grows linearly in $\tilde{\kappa}$ with almost the same gradient for all cases.

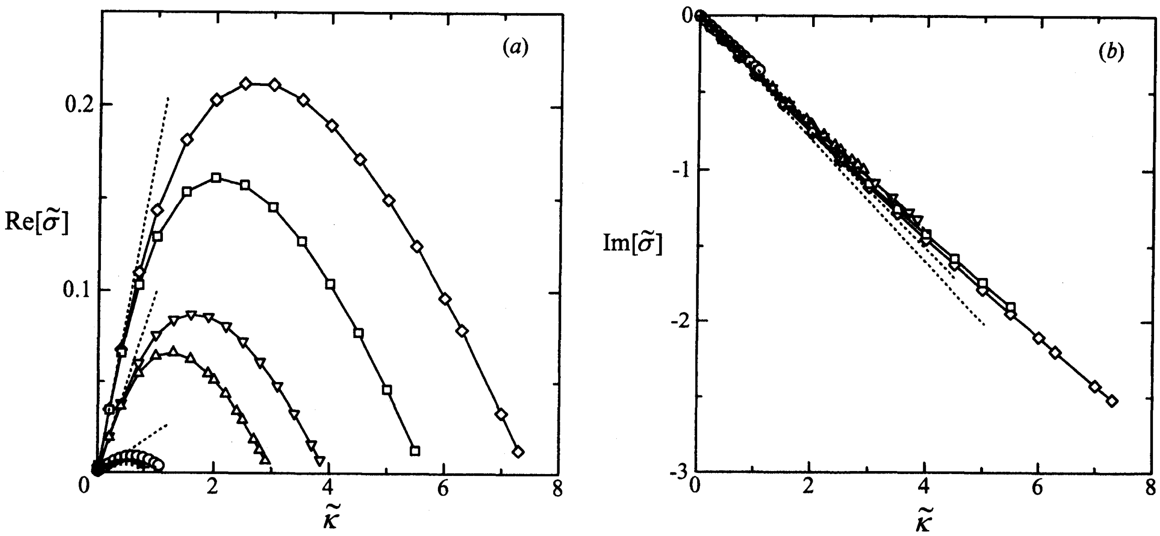

Fig. 4 Computed eigenvalues $\tilde{\sigma}$ with a positive real part versus $\tilde{\kappa}$ for the progressive beams (4.4a) with $\theta = \pi/6$ [$U_0 = 0.35$ (○), 0.5 (△) and 0.65 (□)] and $\pi/3$ [$U_0 = 0.35$ (+), 0.5 (▽) and 0.65 (◇)]: (*a*) $\mathrm{Re}[\tilde{\sigma}]$ versus $\tilde{\kappa}$; (*b*) $\mathrm{Im}[\tilde{\sigma}]$ versus $\tilde{\kappa}$. The dotted lines represent the corresponding gradients $\tilde{\sigma}/\tilde{\kappa}$ as $\tilde{\kappa} \to \infty$ for the solution of the smaller order $k = O(\varepsilon^4)$ (see [4]).

4.3. *Standing beams*

Eigenvalues $\tilde{\sigma}$ with a positive real part versus $\tilde{\kappa}$ are plotted in figure 5 for the statically stable beams with $U_0 = 0.1$, 0.4 and 0.65 when $\theta = \pi/6$ and $\pi/3$ (these beams are all statically stable according to (4.5)). In contrast to the case of progressive beams in which only complex eigenvalues appear, pure real eigenvalues solely appear in the standing-beam case.

Figure 5 shows that a standing beam is unstable for the small amplitude $U_0 = 0.1$. Indeed we have a surprising result that it is unstable even for very small amplitude $U_0 \ll 1$, that is, the eigenvalues $\tilde{\sigma}$ remain to be positive as $U_0 \to 0$. So the standing beam is unstable for any amplitude. Moreover the eigenvalues $\tilde{\sigma}$ go down to zero at finite $\tilde{\kappa}$, so that the instability is three-dimensional as in the case of progressive beams.

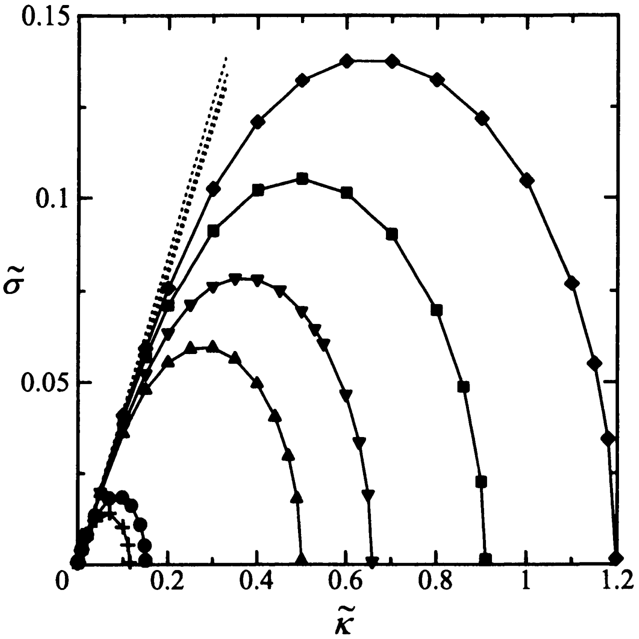

Fig. 5 Computed eigenvalues $\tilde{\sigma}$ versus $\tilde{\kappa}$ for the standing beams (4.4b) with $\theta=\pi/6$ [$U_0=0.1$ (●), 0.4 (▲) and 0.65 (■)] and $\pi/3$ [$U_0=0.1$ (+), 0.4 (▼) and 0.65 (◆)]. The dotted lines represent the corresponding gradients $\tilde{\sigma}/\tilde{\kappa}$ as $\tilde{\kappa}\to\infty$ for the solution of the smaller order $k=O(\varepsilon^4)$ (see [4]).

5. 結言

The linear stability to three-dimensional disturbances of a uniform, plane internal wave beam in a stratified fluid with constant buoyancy frequency is considered. The associated eigenvalue problem is solved asymptotically, assuming perturbations of long wavelength relative to the beam width. In this limit, instability occurs solely due to oblique perturbations and so it is three-dimensional. Propagating beams that transport energy in one direction, in particular, are found to be unstable to such oblique perturbations when the beam steepness exceeds a certain threshold value, whereas purely standing beams are unstable irrespective of their steepness.

参考文献

[1] Lighthill, M. J. 1978 *Waves in Fluids*. Cambridge University Press.
[2] Mowbray, D. E. & Rarity, B. S. H. 1967 A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid. *J. Fluid Mech.* **28**, 1–16.
[3] Tabaei, A. & Akylas, T. R. 2003 Nonlinear internal wave beams. *J. Fluid Mech.* **482**, 141-161.
[4] Kataoka, T. & Akylas, T. R. 2013 Stability of internal gravity wave beams to three-dimensional modulations. *J. Fluid Mech.* **736**, 67-90.
[5] Thorpe, S. A. 1987 On the reflection of a train of finite-amplitude internal waves from a uniform slope *J. Fluid Mech.* **178**, 279-302.
[6] Wilkinson, J. H. 1965 *The Algebraic Eigenvalue Problem*. Clarendon, Oxford.