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Existence for a PDE-model of a grain boundary motion involving solidification effect

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1 Introduction

Let $(0,T)$ be a time-interval with a fixed constant $0<T\in\mathbb{R}$. Let $1<N\in\mathbb{N}$ be a fixed number, let $\Omega\subset\mathbb{R}^{N}$ be a bounded domain with a smooth boundary $\partial\Omega$, and let $\nu_{\partial\Omega}$ be the unit outer normal on $\partial\Omega$. Besides, let us set $Q:=(0,T)\times\Omega$ and $\Sigma:=(0,T)\times\partial\Omega$.

In this paper, a PDE model of a grain boundary motion, involving a solidification effect, is considered. This mathematical model is denoted by (S), and formally described in a form of the following system of parabolic equations.

(S):

$$\begin{cases} w_{t}-\Delta w+\partial I_{[0,1]}(w)-c(w-u)+(w-\eta)+\nu\beta'(w)|\nabla\theta|^{2}\ni 0 \ \text{ in } Q, \\ \nabla w\cdot\nu_{\partial\Omega}=0 \ \text{ on } \Sigma, \\ w(0,x)=w_{0}(x), \quad x\in\Omega; \end{cases} \tag{1.1}$$

$$\begin{cases} \eta_{t}-\Delta\eta+(\eta-w)+\alpha'(\eta)|\nabla\theta|=0 \ \text{ in } Q, \\ \nabla\eta\cdot\nu_{\partial\Omega}=0 \ \text{ on } \Sigma, \\ \eta(0,x)=\eta_{0}(x), \quad x\in\Omega; \end{cases} \tag{1.2}$$

$$\begin{cases} \alpha_{0}(w,\eta)\,\theta_{t}-\operatorname{div}\left(\alpha(\eta)\dfrac{\nabla\theta}{|\nabla\theta|}+2\nu\beta(w)\nabla\theta\right)=0 \ \text{ in } Q, \\ \left(\alpha(\eta)\dfrac{\nabla\theta}{|\nabla\theta|}+2\nu\beta(w)\nabla\theta\right)\cdot\nu_{\partial\Omega}=0 \ \text{ on } \Sigma, \\ \theta(0,x)=\theta_{0}(x), \quad x\in\Omega. \end{cases} \tag{1.3}$$

The system (S) is derived as a gradient system of the following governing energy, called "free-energy":

$$
\begin{aligned}
[w,\eta,\theta] \in H^1(\Omega)^3 \quad \mapsto \quad \mathscr{F}_u(w,\eta,\theta) := {} & \frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \frac{1}{2}\int_\Omega |\nabla \eta|^2\,dx \\
& + \int_\Omega \left(I_{[0,1]}(w) - \frac{c}{2}(w-u)^2 \right) dx + \frac{1}{2}\int_\Omega (w-\eta)^2\,dx \qquad (1.4) \\
& + \int_\Omega \alpha(\eta)|\nabla \theta|\,dx + \nu \int_\Omega \beta(w)|\nabla \theta|^2\,dx.
\end{aligned}
$$

In the context, the unknowns $w = w(t,x)$ and $\eta = \eta(t,x)$ are order parameters, which indicate, respectively, "the solidification order" and "the crystalline orientation order" in a material, by using the values on $[0,1]$. Hence, the range constraint "$0 \leq w, \eta \leq 1$" is always imposed for these parameters, and in particular, the cases when $[w,\eta] = [1,1]$ and $[w,\eta] = [0,0]$ are supposed to reproduce "solidified-oriented phase" and "liquefied-disoriented phase", respectively. In the meantime, the unknown $\theta = \theta(t,x)$ is an order parameter to indicate the argument (mean-angle) of the crystalline orientation. The term $\partial I_{[0,1]}$ as in (1.1) is the subdifferential of the indicator function $I_{[0,1]}$ built in (1.4), i.e.:

$$
r \in \mathbb{R} \mapsto I_{[0,1]}(r) := \begin{cases} 0, & \text{if } r \in [0,1], \\ \infty, & \text{otherwise.} \end{cases} \qquad (1.5)
$$

The components $u \in \mathbb{R}$, $0 < c \in \mathbb{R}$ and $0 < \nu \in \mathbb{R}$ are fixed constants, and in particular, the value of u is supposed to be associated with the degree of relative temperature.

The components $\alpha_0 = \alpha_0(w,\eta)$, $\alpha = \alpha(\eta)$, $\beta = \beta(w)$, $w_0 = w_0(x)$, $\eta_0 = \eta_0(x)$ and $\theta_0 = \theta_0(x)$ are given functions, which are supposed to fulfill the following conditions.

(A1) $\alpha_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$ is a given positive-valued function.

(A2) $\alpha, \beta \in C^1(\mathbb{R})$ are given positive-valued convex functions, and the differentials $\alpha', \beta' \in C(\mathbb{R})$ satisfy that $\alpha'(0) = \beta'(0) = 0$. Hence, α and β are non-decreasing on $[0,\infty)$.

(A3) There exists a constant $\delta_* > 0$ such that:

$$
\min \left\{ \ \alpha_0(w,\eta), \alpha(\eta), \beta(w) \ \middle| \ [w,\eta] \in \mathbb{R}^2 \ \right\} \geq \delta_*.
$$

(A4) $w_0, \eta_0, \theta_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ are given initial data, and the triplet of the initial data $[w_0, \eta_0, \theta_0]$ is supposed to belong to a class $D_* \subset [H^1(\Omega) \cap L^\infty(\Omega)]^3$, defined as:

$$
D_* := \left\{ \ [\tilde{w}_0, \tilde{\eta}_0, \tilde{\theta}_0] \in [H^1(\Omega) \cap L^\infty(\Omega)]^3 \ \middle| \ 0 \leq \tilde{w}_0 \leq 1 \text{ and } 0 \leq \tilde{\eta}_0 \leq 1, \text{ a.e. in } \Omega \ \right\}.
$$

The derivation of (S) is based on the modelling method of Kobayashi et al. [18, 20, 21], and indeed, this system can be called a modified version of "ϕ-η-θ model" proposed in [18]. The main difference from the ϕ-η-θ model is in the choice of the double-well function, that is to characterize the bi-stable situations in phase transitions. More precisely, the double-well function as in the ϕ-η-θ model is the standard polynomial type, while we adopt another type of double-well function:

$$
w \in \mathbb{R} \mapsto I_{[0,1]}(w) - \frac{c}{2}(w-u)^2 \in [0,\infty], \qquad (1.6)
$$

in the formula (1.4) of free-energy. Incidentally, the above function has been one of representative expressions of double-well functions, in the modelling of phase transitions (cf. Visintin [31, Chapter VI]).

From the mathematical point of view, the indicator function $I_{[0,1]}$ as in (1.4) enables the immediate derivation of the range constraint property "$0 \leq w \leq 1$". But meanwhile, we should note that the term $\nu\beta'(w)|\nabla\theta|^2$ in (1.1) becomes nonstandard under the L^2-based setting in (1.4). So, we cannot expect to solve the system (S) by straightforward applications of some existing general theories of evolution equations (e.g. [14, 24]), even if we apply some generalized notions such as "L^2-subdifferentials".

Based on these, we set the goal in this paper to verify the existence of solutions to the system (S), which is stated in the form of the following Main Theorem.

Main Theorem (Existence result for the system (S)) *Under the assumptions (A1)-(A4), the system (S) admits at least one solution $[w, \eta, \theta]$, which is defined by the following conditions.*

(S0) $[w, \eta, \theta] \in W^{1,2}(0,T; L^2(\Omega))^3 \cap L^\infty(0,T; H^1(\Omega))^3 \cap L^\infty(Q)^3$, $\eta \in L^2(0,T; H^2(\Omega))$; $0 \leq w \leq 1$, $0 \leq \eta \leq 1$ *and* $|\theta| \leq |\theta_0|_{L^\infty(\Omega)}$, *a.e. in* Q.

(S1) w *solves (1.1) in the following variational sense:*

$$\begin{aligned}\int_\Omega \big(w_t(t) - c(w(t) - u) + (w - \eta)(t)\big)\,(w(t) - \varphi)\,dx \qquad&\\ + \int_\Omega \nabla w(t) \cdot \nabla(w(t) - \varphi)\,dx + \nu \int_\Omega (w(t) - \varphi)\beta'(w(t))|\nabla\theta(t)|^2\,dx&\\ + \int_\Omega I_{[0,1]}(w(t))\,dx \leq \int_\Omega I_{[0,1]}(\varphi)\,dx,&\end{aligned} \tag{1.7}$$

for any $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ *and a.e.* $t \in (0,T)$,

with the initial condition $w(0) = w_0$ *in* $L^2(\Omega)$.

(S2) η *solves (1.2) in the following variational sense:*

$$\begin{aligned}\int_\Omega \big(\eta_t(t) + (\eta - w)(t)\big)\,\psi\,dx + \int_\Omega \nabla\eta(t) \cdot \nabla\psi\,dx&\\ + \int_\Omega \psi\alpha'(\eta(t))|\nabla\theta(t)|\,dx = 0,&\end{aligned} \tag{1.8}$$

for any $\psi \in H^1(\Omega)$ *and a.e.* $t \in (0,T)$,

with the initial condition $\eta(0) = \eta_0$ *in* $L^2(\Omega)$.

(S3) θ *solves (1.3) in the following variational sense:*

$$\begin{aligned}\int_\Omega \alpha_0(w,\eta)(t)\,\theta_t(t)\,(\theta(t) - \omega)\,dx + 2\nu \int_\Omega \beta(w(t))\nabla\theta(t) \cdot \nabla(\theta(t) - \omega)\,dx&\\ + \int_\Omega \alpha(\eta(t))|\nabla\theta(t)|\,dx \leq \int_\Omega \alpha(\eta(t))|\nabla\omega|\,dx,&\end{aligned} \tag{1.9}$$

for any $\omega \in H^1(\Omega)$ *and a.e.* $t \in (0,T)$,

with the initial condition $\theta(0) = \theta_0$ *in* $L^2(\Omega)$.

Here is the content of this paper. In the next Section 2, some specific notations are prepared as preliminaries. In Section 3, we prove the existence and uniqueness for the approximating problems, which are prescribed as the time-discretization systems for (S). On that basis, our Main Theorem will be proved in Section 4. Finally, we overview the vision in the future of our study.

2 Preliminaries

First of all, we list the notations that are used throughout this paper.

Notation 1 (Notations in real analysis) For any $a_0, b_0 \in [-\infty, \infty]$, we define:

$$a_0 \vee b_0 := \max\{a_0, b_0\} \text{ and } a_0 \wedge b_0 := \min\{a_0, b_0\}.$$

Let $d \in \mathbb{N}$ be any fixed number. Then, we simply denote by $|x|$ and $x \cdot y$ the Euclidean norm of $x \in \mathbb{R}^d$ and the standard scalar product of $x, y \in \mathbb{R}^d$, respectively, i.e.:

$$|x| := \sqrt{x_1^2 + \cdots + x_d^2} \text{ and } x \cdot y := x_1 y_1 + \cdots + x_d y_d,$$
$$\text{for all } x = [x_1, \cdots, x_d], y = [y_1, \cdots, y_d] \in \mathbb{R}^d.$$

The d-dimensional Lebesgue measure is denoted by $\mathscr{L}^d$. Also, unless otherwise specified, the measure theoretical phrases, such as "a.e.", "dt" and "dx", and so on, are with respect to the Lebesgue measure in each corresponding dimension.

For a (Lebesgue) measurable function $f : B \longrightarrow [-\infty, \infty]$ on a Borel subset $B \subset \mathbb{R}^d$, we denote by $[f]^+$ and $[f]^-$, respectively, the positive part and the negative part of f, i.e.:

$$[f]^+(x) := f(x) \vee 0 \text{ and } [f]^-(x) := -(f(x) \wedge 0), \text{ for a.e. } x \in B.$$

Notation 2 (Notations in convex analysis) For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ its inner product.

For any proper lower semi-continuous (l.s.c. from now on) and convex function Ψ defined on a Hilbert space X, we denote by $D(\Psi)$ its effective domain and by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of Ψ, and it has a maximal monotone graph in the product space $X^2 := X \times X$. More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as a set of all elements $z_0^* \in X$ which satisfy the following variational inequality:

$$(z_0^*, z - z_0)_X \leq \Psi(z) - \Psi(z_0) \text{ for any } z \in D(\Psi).$$

The set $D(\partial\Psi) := \{ z \in X \mid \partial\Psi(z) \neq \emptyset \}$ is called the domain of $\partial\Psi$. We often use the notation "$[z_0, z_0^*] \in \partial\Psi$ in X^2", to mean that "$z_0^* \in \partial\Psi(z_0)$ in X with $z_0 \in D(\partial\Psi)$", by identifying the operator $\partial\Psi$ with its graph in X^2.

Remark 2.1 As a representative example, let us consider the following proper l.s.c. and convex function on $L^2(\Omega)$:

$$z \in L^2(\Omega) \mapsto \Psi_0(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 \, dx, & \text{if } z \in H^1(\Omega), \\ \infty, & \text{otherwise,} \end{cases}$$

that is the so-called functional of Dirichlet integral. Then, the subdifferential $\partial\Psi_0$ of this convex function is directly associated with the operator of Laplacian. More precisely, let us set:

$$D_{\mathrm{N}} := \{\ z \in H^2(\Omega) \mid \nabla z \cdot \nu_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\ \},$$

and let us denote by Δ_{N} the operator of Laplacian subject to the Neumann-zero boundary condition, i.e.:

$$\Delta_{\mathrm{N}} : z \in D_{\mathrm{N}} \subset L^2(\Omega) \mapsto \Delta z \in L^2(\Omega).$$

Then, it is known that (see Barbu [2] or Brézis [3], for example):

$$[z, z^*] \in \partial\Psi_0 \text{ in } L^2(\Omega)^2, \text{ iff. } (z^*, \varphi)_{L^2(\Omega)} = (\nabla z, \nabla\varphi)_{L^2(\Omega)^N} \text{ for any } \varphi \in H^1(\Omega), \quad (2.1)$$

and moreover,

$$z \in L^2(\Omega) \mapsto \partial\Psi_0(z) = \begin{cases} \{-\Delta_{\mathrm{N}} z\}, & \text{if } z \in D_{\mathrm{N}}, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (2.2)$$

In this light, the operators $\partial\Psi_0$ and $-\Delta_{\mathrm{N}}$ are usually identified.

Also, as another example, we mention about the subdifferential $\partial I_{[0,1]} \subset \mathbb{R}^2$ of the indicator function $I_{[0,1]}$, defined in (1.5). In this example, the subdifferential $\partial I_{[0,1]}$ is calculated as:

$$r \in \mathbb{R} \mapsto \partial I_{[0,1]}(r) = \begin{cases} 0, & \text{if } r \in (0,1), \\ [0,\infty), & \text{if } r = 1, \\ (-\infty, 0], & \text{if } r = -1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Remark 2.2 (Time-dependent subdifferentials) It is often useful to consider the subdifferentials under time-dependent settings of convex functions. With regard to this topic, certain general theories were established by a number of previous researchers (e.g. Kenmochi [14] and Ôtani [24]). So, referring to some of these (e.g. [14, Chapter 2]), we can see the following fact.

(Fact 0) Let E_0 be a convex subset in a Hilbert space X, let $I \subset [0,\infty)$ be a time-interval, and for any $t \in I$, let $\Psi^t : X \longrightarrow (-\infty, \infty]$ be a proper l.s.c. and convex function, such that $D(\Psi^t) = E_0$ for all $t \in I$. Based on this, let us define a convex function $\hat{\Psi}^I : L^2(I; X) \longrightarrow (-\infty, \infty]$, by putting:

$$\zeta \in L^2(I; X) \mapsto \hat{\Psi}^I(\zeta) := \begin{cases} \displaystyle\int_I \Psi^t(\zeta(t))\, dt, & \text{if } \Psi^{(\cdot)}(\zeta) \in L^1(I), \\ \infty, & \text{otherwise.} \end{cases}$$

Here, if $E_0 \subset D(\hat{\Psi}^I)$, i.e. if the function $t \in I \mapsto \Psi^t(z)$ is integrable for any $z \in E_0$, then it holds that:

$$[\zeta, \zeta^*] \in \partial\hat{\Psi}^I \text{ in } L^2(I; X)^2, \text{ iff.}$$
$$\zeta \in D(\hat{\Psi}^I) \text{ and } [\zeta(t), \zeta^*(t)] \in \partial\Psi^t \text{ in } X^2, \text{ a.e. } t \in I.$$

Notation 3 (Specific notations) For the solution $[w, \eta, \theta]$ to (S), we put $v := [w, \eta]$, for a simplicity. As well as, for the initial data $[w_0, \eta_0, \theta_0] \in D_*$, we put $v_0 := [w_0, \eta_0]$. In this regard, we add some specific notations, prescribed below.

For any pair of functions $\tilde{v} = [\tilde{w}, \tilde{\eta}] \in L^\infty(\Omega) \times L^2(\Omega)$, we denote by $\Phi(\tilde{v}; \cdot\,) = \Phi(\tilde{w}, \tilde{\eta}; \cdot\,)$ a proper l.s.c. and convex function on $L^2(\Omega)$, defined as:

$$z \in L^2(\Omega) \mapsto \Phi(\tilde{v}; z) = \Phi(\tilde{w}, \tilde{\eta}; z) := \begin{cases} \displaystyle\int_\Omega \alpha(\tilde{\eta})|\nabla z|\,dx + \nu \int_\Omega \beta(\tilde{w})|\nabla z|^2\,dx, \\ \qquad \text{if } z \in H^1(\Omega), \\ \infty, \quad \text{otherwise}, \end{cases}$$

and we denote by $\partial\Phi(\tilde{v}; \cdot\,)$ the subdifferential of $\Phi(\tilde{v}; \cdot\,)$ in the topology of $L^2(\Omega)$. Besides, we define a quadratic function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, by letting:

$$\tilde{v} = [\tilde{w}, \tilde{\eta}] \in \mathbb{R}^2 \mapsto g(\tilde{v}) \left(= g(\tilde{w}, \tilde{\eta}) \right) := \frac{1}{2}(\tilde{w} - \tilde{\eta})^2 \in \mathbb{R}. \tag{2.3}$$

Remark 2.3 By using the notations in Notation 3, the variational inequalities (1.7)-(1.8) can be reformulated as follows.

$$\begin{aligned} &\left(v_t(t), v(t) - \varpi\right)_{L^2(\Omega)^2} + \left(\nabla v(t), \nabla(v(t) - \varpi)\right)_{L^2(\Omega)^{2\times N}} \\ &\quad - c\left(w(t) - u, w(t) - \varphi\right)_{L^2(\Omega)} + \left([\nabla g](v(t)), v(t) - \varpi\right)_{L^2(\Omega)^2} \\ &\quad + \int_\Omega (\eta(t) - \psi)\alpha'(\eta(t))|\nabla\theta(t)|\,dx + \nu \int_\Omega (w(t) - \varphi)\beta'(w(t))|\nabla\theta(t)|^2\,dx \\ &\quad + \int_\Omega I_{[0,1]}(w(t))\,dx \le \int_\Omega I_{[0,1]}(\varphi)\,dx, \\ &\quad \text{for any } \varpi = [\varphi, \psi] \in [H^1(\Omega) \cap L^\infty(\Omega)] \times H^1(\Omega), \end{aligned} \tag{2.4}$$

where $[\nabla g]$ denotes the gradient of the binary (quadratic) function $g = g(\tilde{w}, \tilde{\eta})$.

Meanwhile, in the light of Notations 2-3 and Remarks 2.1-2.2, the variational inequalities (1.8) and (1.9) can be reformulated to the following forms of evolution equations:

$$\eta_t(t) - \varDelta_{\mathrm{N}}\eta(t) + (\eta - w)(t) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \text{ in } L^2(\Omega), \text{ a.e. } t \in (0, T), \tag{2.5}$$

and

$$\alpha_0(v(t))\theta_t(t) + \partial\Phi(v(t); \theta(t)) \ni 0 \text{ in } L^2(\Omega), \text{ a.e. } t \in (0, T),$$

respectively, where for any $\tilde{v} = [\tilde{w}, \tilde{\eta}] \in \mathbb{R}^2$, $\alpha_0(\tilde{v})$ is the abbreviation of $\alpha_0(\tilde{w}, \tilde{\eta})$. However, it must be noted that similar reformulations, by using the L^2-subdifferentials, are not available for (1.7) and (2.4), due to the L^1-perturbation term $\beta'(w)|\nabla\theta|^2$ ($\in L^\infty(0, T; L^1(\Omega))$).

Finally, we mention about the Mosco convergence, that is known as a representative notion of the functional-convergence.

Definition 2.1 (Mosco convergence: cf. [23]) Let X be an abstract Hilbert space. Let $\Psi : X \longrightarrow (-\infty, \infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n \,|\, n \in \mathbb{N}\}$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \longrightarrow (-\infty, \infty]$, $n \in \mathbb{N}$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco [23], as $n \to \infty$, iff. the following two conditions are fulfilled.

1° (the condition of lower-bound): $\liminf_{n\to\infty} \Psi_n(z_n^\dagger) \geq \Psi(z^\dagger)$, if $z^\dagger \in X$, $\{z_n^\dagger \,|\, n \in \mathbb{N}\} \subset X$, and $z_n^\dagger \to z^\dagger$ weakly in X as $n \to \infty$;

2° (the condition of optimality): for any $z^\ddagger \in D(\Psi)$, there exists a sequence $\{z_n^\ddagger \,|\, n \in \mathbb{N}\} \subset X$ such that $z_n^\ddagger \to z^\ddagger$ in X and $\Psi_n(z_n^\ddagger) \to \Psi(z^\ddagger)$, as $n \to \infty$.

Remark 2.4 As a basic matter of the Mosco-convergence, we can see the following fact (see [14, Chapter 2], for example).

(Fact 1) Let X, Ψ and $\{\Psi_n \,|n \in \mathbb{N}\}$ be as in Definition 2.1. Besides, let us assume that:

$$\Psi_n \to \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \to \infty,$$

and

$$\begin{cases} [z, z^*] \in X^2, \quad [z_n, z_n^*] \in \partial\Psi_n \text{ in } X^2,\ n \in \mathbb{N}, \\ z_n \to z \text{ in } X \text{ and } z_n^* \to z^* \text{ weakly in } X, \text{ as } n \to \infty. \end{cases}$$

Then, it holds that:

$$[z, z^*] \in \partial\Psi \text{ in } X^2, \text{ and } \Psi_n(z_n) \to \Psi(z), \text{ as } n \to \infty.$$

3 Approximating problem

In this section, we prove the existence and uniqueness for approximating problems of (S). As mentioned in Introduction, the approximating problems are settled as the time-discretization systems for (S). Hence, we denote by $0 < h < 1$ the index of time-step, and we denote by $(\mathrm{AP})_h$ the time-discretization systems for (S) prescribed as follows.

$(\mathrm{AP})_h$: for the initial data $[v_0, \theta_0] = [w_0, \eta_0, \theta_0] \in D_*$ with $v_0 = [w_0, \eta_0]$, find a sequence:

$$\{[v_i, \theta_i] = [w_i, \eta_i, \theta_i] \,|\, i \in \mathbb{N}\} \subset H^1(\Omega)^3 \text{ with } v_i = [w_i, \eta_i],\ i \in \mathbb{N},$$

such that:

$$\begin{aligned} &\frac{1}{h}(v_i - v_{i-1}, v_i - \varpi)_{L^2(\Omega)^2} + \big(\nabla v_i, \nabla(v_i - \varpi)\big)_{L^2(\Omega)^{2\times N}} \\ &-c\,(w_i - u, w_i - \varphi)_{L^2(\Omega)} + \big([\nabla g](v_i), v_i - \varpi\big)_{L^2(\Omega)^2} \\ &+ \int_\Omega (\eta_i - \psi)\alpha'(\eta_i)|\nabla\theta_{i-1}|\,dx + \nu\int_\Omega (w_i - \varphi)\beta'(w_i)|\nabla\theta_{i-1}|^2\,dx \\ &+ \int_\Omega I_{[0,1]}(w_i)\,dx \leq \int_\Omega I_{[0,1]}(\varphi), \\ &\text{for any } \varpi = [\varphi, \psi] \in [H^1(\Omega) \cap L^\infty(\Omega)] \times H^1(\Omega), \end{aligned} \tag{3.1}$$

$$0 \leq w_i \leq 1 \text{ and } 0 \leq \eta_i \leq 1 \text{ a.e. in } \Omega, \tag{3.2}$$

$$\begin{aligned} &\frac{1}{h}\big(\alpha_0(v_i)(\theta_i - \theta_{i-1}), \theta_i - \omega\big)_{L^2(\Omega)} + \Phi(v_i; \theta_i) \leq \Phi(v_i; \omega), \\ &\qquad \text{for any } \omega \in H^1(\Omega), \end{aligned} \tag{3.3}$$

and

$$|\theta_i| \leq |\theta_{i-1}|_{L^\infty(\Omega)} \text{ a.e. in } \Omega, \tag{3.4}$$

for $i = 1, 2, 3, \cdots$.

We call the sequence $\{[v_i, \theta_i] = [w_i, \eta_i, \theta_i] \,|\, i \in \mathbb{N}\} \subset D_*$ the solution to $(\mathrm{AP})_h$, or the approximating solution in short. Due to (3.4), the range of approximating solution is restricted into the following smaller class $D_*(\theta_0)$ than D_*:

$$D_*(\theta_0) := \left\{ \ [\tilde{w}, \tilde{v}, \tilde{\theta}] \in D_* \ \middle| \ |\tilde{\theta}| \leq |\theta_0|_{L^\infty(\Omega)} \text{ a.e. in } \Omega \ \right\}.$$

In what follows, we fix the time-step $0 < h < 1$, and prove the following theorem, concerned with the solvability of $(\mathrm{AP})_h$.

Theorem 1 (Solvability of the approximating problem) *There exists a small constant $h_0^\dagger \in (0,1)$, such that if $0 < h < h_0^\dagger$, then the approximating problem $(AP)_h$ admits a unique solution $\{[v_i, \theta_i] = [w_i, \eta_i, \theta_i] \,|\, i \in \mathbb{N}\} \subset D_*(\theta_0)$. Moreover, under $0 < h < h_0^\dagger$, the approximating solution $\{[v_i, \theta_i] = [w_i, \eta_i, \theta_i]\}$ fulfills the following inequality of energy-dissipation:*

$$\begin{aligned} \frac{1}{2h}|v_i - v_{i-1}|^2_{L^2(\Omega)^2} \ &+ \frac{1}{h}|\sqrt{\alpha_0(v_i)}(\theta_i - \theta_{i-1})|^2_{L^2(\Omega)} \\ &+ \mathscr{F}_u(v_i, \theta_i) \leq \mathscr{F}_u(v_{i-1}, \theta_{i-1}), \ i = 1, 2, 3, \cdots, \end{aligned} \tag{3.5}$$

where for any $[\tilde{v}, \tilde{\theta}] = [\tilde{w}, \tilde{\eta}, \tilde{\theta}] \in H^1(\Omega)^3$, $\mathscr{F}_u(\tilde{v}, \tilde{\theta})$ is the abbreviation of $\mathscr{F}_u(\tilde{w}, \tilde{\eta}, \tilde{\theta})$.

For the proof of this theorem, we prepare some auxiliary lemmas.

Lemma 3.1 *Let us fix $\theta_0^\dagger \in H^1(\Omega)$ and $v_0^\dagger = [w_0^\dagger, \eta_0^\dagger] \in H^1(\Omega)^2$, and let us consider the following auxiliary problem, to find a pair of functions $v = [w, \eta] \in H^1(\Omega)^2$ such that:*

$$\begin{aligned} &\frac{1}{h}(v - v_0^\dagger, v - \varpi)_{L^2(\Omega)^2} + \big(\nabla v, \nabla(v - \varpi)\big)_{L^2(\Omega)^{2\times N}} \\ &-c\left(w - u, w - \varphi\right)_{L^2(\Omega)} + \left([\nabla g](v), v - \varpi\right)_{L^2(\Omega)^2} \\ &+ \int_\Omega (\eta - \psi)\alpha'(\eta)|\nabla\theta_0^\dagger|\, dx + \nu \int_\Omega (w - \varphi)\beta'(w)|\nabla\theta_0^\dagger|^2\, dx \\ &+ \int_\Omega I_{[0,1]}(w)\, dx \leq \int_\Omega I_{[0,1]}(\varphi), \\ &\textit{for any } \varpi = [\varphi, \psi] \in [H^1(\Omega) \cap L^\infty(\Omega)] \times H^1(\Omega). \end{aligned} \tag{3.6}$$

Then, there exists a small constant $0 < h_0^\dagger < 1$, such that if $0 < h < h_0^\dagger$, then the problem (3.6) admits a unique solution $v = [w, \eta] \in H^1(\Omega)^2$.

Proof. Let us assume that:

$$0 < h < h_0^\dagger := \frac{1}{1+c}. \tag{3.7}$$

Then, a functional $\Psi_0^\dagger : H^1(\Omega)^2 \longrightarrow (-\infty, \infty]$, defined as:

$$\begin{aligned} v = [w, \eta] \in H^1(\Omega)^2 \mapsto \Psi_0^\dagger(v) = \Psi_0^\dagger(w, \eta) &:= \frac{1}{2h}|v - v_0^\dagger|^2_{L^2(\Omega)^2} + |\nabla v|^2_{L^2(\Omega)^{2\times N}} \\ &-\frac{c}{2}|w - u|^2_{L^2(\Omega)} + \int_\Omega g(v)\, dx + \int_\Omega I_{[0,1]}(w)\, dx \\ &+ \int_\Omega \left(\alpha(\eta)|\nabla\theta_0^\dagger| + \nu\beta(w)|\nabla\theta_0^\dagger|^2\right)\, dx, \end{aligned} \tag{3.8}$$

will be proper l.s.c., coercive and strictly convex on $H^1(\Omega)^2$. Additionally, the minimization problem for $\Psi_0^\dagger$ is equivalent to the problem (3.6). Therefore, the existence and uniqueness for (3.6) will be a straightforward consequence of the general theory of convex analysis (e.g. [7, Chapter 2]). ■

Lemma 3.2 *For arbitrary $\theta_0^\dagger \in H^1(\Omega)$, $w^\dagger \in L^2(\Omega)$ and $\check{\eta}_0, \hat{\eta}_0 \in H^1(\Omega)$, let $\check{\eta}, \hat{\eta} \in H^2(\Omega)$ be functions, such that:*

$$\frac{1}{h}(\check{\eta} - \check{\eta}_0) - \Delta_{\mathrm{N}}\check{\eta} + (\check{\eta} - w^\dagger) + \alpha'(\check{\eta})|\nabla\theta_0^\dagger| \le 0, \ \textit{a.e. in } \Omega, \tag{3.9}$$

and

$$\frac{1}{h}(\hat{\eta} - \hat{\eta}_0) - \Delta_{\mathrm{N}}\hat{\eta} + (\hat{\eta} - w^\dagger) + \alpha'(\hat{\eta})|\nabla\theta_0^\dagger| \ge 0, \ \textit{a.e. in } \Omega. \tag{3.10}$$

Then,

$$|[\check{\eta} - \hat{\eta}]^+|_{L^2(\Omega)}^2 \le |[\check{\eta}_0 - \hat{\eta}_0]^+|_{L^2(\Omega)}^2. \tag{3.11}$$

Hence, in particular, if $\check{\eta}_0 \le \hat{\eta}_0$ a.e. in Ω, then $\check{\eta} \le \hat{\eta}$ a.e. in Ω.

Proof. Let us take the difference between (3.9) and (3.10), and multiply the both sides of the result by $[\check{\eta} - \hat{\eta}]^+$. Then, we have:

$$\begin{aligned}\frac{1}{h}|[\check{\eta} - \hat{\eta}]^+|_{L^2(\Omega)}^2 + |[\check{\eta} - \hat{\eta}]^+|_{H^1(\Omega)}^2 + \int_\Omega [\check{\eta} - \hat{\eta}]^+(\alpha'(\check{\eta}) - \alpha'(\hat{\eta}))|\nabla\theta_0^\dagger|\, dx \\ \le \frac{1}{h}(\check{\eta}_0 - \hat{\eta}_0, [\check{\eta} - \hat{\eta}]^+)_{L^2(\Omega)} \le \frac{1}{h}|[\check{\eta}_0 - \hat{\eta}_0]^+|_{L^2(\Omega)}|[\check{\eta} - \hat{\eta}]^+|_{L^2(\Omega)}.\end{aligned}$$

Based on this, the assertion (3.11) is obtained by using (A2) and Young's inequality. ■

Corollary 3.1 *Let us assume that $0 < h < h_0^\dagger$ with the constant $h_0^\dagger \in (0,1)$ given in (3.7). For arbitrary $\theta_0^\dagger \in H^1(\Omega)$ and $v_0^\dagger = [w_0^\dagger, \eta_0^\dagger] \in H^1(\Omega)^2$, let $v = [w, \eta] \in H^1(\Omega)^2$ be the unique solution to the auxiliary problem (3.6). Here, if:*

$$0 \le \eta_0^\dagger \le 1 \ \textit{a.e. in } \Omega, \tag{3.12}$$

then:

$$0 \le w \le 1 \ \textit{and} \ 0 \le \eta \le 1 \ \textit{a.e. in } \Omega, \quad \textit{and} \quad \eta \in D_{\mathrm{N}} \subset H^2(\Omega).$$

Proof. Since $v = [w, \eta] \in H^1(\Omega)$ is the minimizer of the convex function $\Psi_0^\dagger$ given in (3.8), the inequality of the range constraint:

$$0 \le w \le 1 \text{ a.e. in } \Omega, \tag{3.13}$$

is immediately seen from the effect of the indicator function $I_{[0,1]}$. So, putting $\varphi = w$ in (3.6), and having (2.1)-(2.2) in mind, we infer that $\eta \in D_{\mathrm{N}}$, and

$$\frac{1}{h}(\eta - \eta_0^\dagger) - \Delta_{\mathrm{N}}\eta + (\eta - w) + \alpha'(\eta)|\nabla\theta_0^\dagger| = 0 \text{ in } L^2(\Omega).$$

On the other hand, it is easily checked from (A2) and (3.12)-(3.13) that:

$$\frac{1}{h}(0 - \eta_0^\dagger) - \Delta_{\mathrm{N}}0 + (0 - w) + \alpha'(0)|\nabla\theta_0^\dagger| \le 0, \text{ a.e. in } \Omega,$$

and

$$\frac{1}{h}(1-\eta_0^\dagger) - \varDelta_{\mathrm{N}} 1 + (1-w) + \alpha'(1)|\nabla\theta_0^\dagger| \geq 0, \text{ a.e. in } \Omega.$$

Now, the assertion "$\eta \geq 0$ a.e. in Ω" (resp. "$\eta \leq 1$ a.e. in Ω") will be obtained by applying Lemma 3.2 as the case when $\check{\eta}_0 = \hat{\eta}_0 = \eta_0^\dagger$, $w^\dagger = w$, $\check{\eta} = 0$ and $\hat{\eta} = \eta$ (resp. $\check{\eta}_0 = \hat{\eta}_0 = \eta_0^\dagger$, $w^\dagger = w$, $\check{\eta} = \eta$ and $\hat{\eta} = 1$). ∎

Lemma 3.3 *Let $v^\dagger = [w^\dagger, \eta^\dagger] \in [H^1(\Omega) \cap L^\infty(\Omega)] \times H^1(\Omega)$ be a fixed pair of functions, and let $\theta_0^\dagger \in H^1(\Omega)$ be a fixed function. Then, the following variational inequality:*

$$\frac{1}{h}\left(\alpha_0(v^\dagger)(\theta-\theta_0^\dagger), \theta-\omega\right)_{L^2(\Omega)} + \Phi(v^\dagger;\theta) \leq \Phi(v^\dagger;\omega), \text{ for any } \omega \in H^1(\Omega), \tag{3.14}$$

admits a unique solution $\theta \in H^1(\Omega)$.

Proof. As easily seen, the variational inequality (3.14) is equivalent to the minimization problem for a proper l.s.c. and convex function on $L^2(\Omega)$, defined as:

$$\theta \in L^2(\Omega) \mapsto \frac{1}{2h}|\sqrt{\alpha_0(v^\dagger)}(\theta-\theta_0^\dagger)|^2_{L^2(\Omega)} + \Phi(v^\dagger;\theta).$$

Here, by virtue of (A3), we can show that this convex function is coercive and strictly convex on $L^2(\Omega)$.

Hence, this lemma will be obtained by applying the general theory of convex analysis (e.g. [7, Chapter 2]), immediately. ∎

Remark 3.1 Note that the variational inequality (3.14) can be reformulated to a form of inclusion:

$$\frac{1}{h}\sqrt{\alpha_0(v^\dagger)}(\theta-\theta_0^\dagger) + \partial\Phi(v^\dagger;\theta) \ni 0 \text{ in } L^2(\Omega),$$

with the use of the subdifferential $\partial\Phi(v^\dagger;\,\cdot\,)$.

Lemma 3.4 (T-monotonicity) *Let $v^\dagger = [w^\dagger, \eta^\dagger] \in [H^1(\Omega) \cap L^\infty(\Omega)] \times H^1(\Omega)$ be a fixed pair of functions. Then, it holds that:*

$$\begin{gathered}(\vartheta_1^* - \vartheta_2^*, [\vartheta_1 - \vartheta_2]^+)_{L^2(\Omega)} \geq 0, \\ \text{if } [\vartheta_k, \vartheta_k^*] \in \partial\Phi(v^\dagger;\cdot\,) \text{ in } L^2(\Omega)^2,\ k=1,2.\end{gathered} \tag{3.15}$$

Proof. This lemma can be proved by applying the theory of T-monotonicity (cf. [3, 16]). According to the general theory, we need to start with checking that:

$$\begin{aligned}
&\Phi(v^\dagger;\omega_1 \wedge \omega_2) + \Phi(v^\dagger;\omega_1 \vee \omega_2) \\
&= \int_\Omega \alpha(\eta^\dagger)|\nabla(\omega_1 \wedge \omega_2)|\,dx + \nu\int_\Omega \beta(w^\dagger)|\nabla(\omega_1 \wedge \omega_2)|^2\,dx \\
&\qquad + \int_\Omega \alpha(\eta^\dagger)|\nabla(\omega_1 \vee \omega_2)|\,dx + \nu\int_\Omega \beta(w^\dagger)|\nabla(\omega_1 \vee \omega_2)|^2\,dx \\
&= \sum_{k=1}^2 \left[\int_\Omega \alpha(\eta^\dagger)|\nabla\omega_k|\,dx + \nu\int_\Omega \beta(w^\dagger)|\nabla\omega_k|^2\,dx\right] \\
&= \Phi(v^\dagger;\omega_1) + \Phi(v^\dagger;\omega_2), \text{ for all } \omega_k \in H^1(\Omega),\ k=1,2.
\end{aligned}$$

Based on this, taking arbitrary $[\vartheta_k, \vartheta_k^*] \in \partial\Phi(v^\dagger; \cdot)$ in $L^2(\Omega)$, $k = 1, 2$, the assertion (3.15) of this lemma is verified as follows.

$$\begin{aligned}(\vartheta_1^* - \vartheta_2^*, [\vartheta_1 - \vartheta_2]^+)_{L^2(\Omega)} &= (\vartheta_1^*, \vartheta_1 - \vartheta_1 \wedge \vartheta_2)_{L^2(\Omega)} + (\vartheta_2^*, \vartheta_2 - \vartheta_1 \vee \vartheta_2)_{L^2(\Omega)} \\ &\geq \Phi(v^\dagger; \vartheta_1) + \Phi(v^\dagger; \vartheta_2) - \big(\Phi(v^\dagger; \vartheta_1 \wedge \vartheta_2) + \Phi(v^\dagger; \vartheta_1 \vee \vartheta_2)\big) = 0.\end{aligned}$$

■

Lemma 3.5 *Let $v^\dagger = [w^\dagger, \eta^\dagger] \in [H^1(\Omega) \cap L^\infty(\Omega)] \times H^1(\Omega)$ be a fixed pair of functions, and let $\check{\theta}_0, \hat{\theta}_0 \in H^1(\Omega)$ be fixed functions. Let $[\check{\theta}, \check{\theta}^*], [\hat{\theta}, \hat{\theta}^*] \in L^2(\Omega)^2$ be pairs of functions, such that:*

$$[\check{\theta}, \check{\theta}^*] \in \partial\Phi(v^\dagger; \cdot), \quad [\hat{\theta}, \hat{\theta}^*] \in \partial\Phi(v^\dagger; \cdot) \text{ in } L^2(\Omega)^2,$$

and

$$\begin{cases} \dfrac{1}{h}\alpha_0(v^\dagger)(\check{\theta} - \check{\theta}_0) + \check{\theta}^* \leq 0 \text{ a.e. in } \Omega, \\ \dfrac{1}{h}\alpha_0(v^\dagger)(\hat{\theta} - \hat{\theta}_0) + \hat{\theta}^* \geq 0 \text{ a.e. in } \Omega, \end{cases} \tag{3.16}$$

respectively. Then:

$$|\sqrt{\alpha_0(v^\dagger)}[\check{\theta} - \hat{\theta}]^+|_{L^2(\Omega)}^2 \leq |\sqrt{\alpha_0(v^\dagger)}[\check{\theta}_0 - \hat{\theta}_0]^+|_{L^2(\Omega)}^2.$$

Moreover, it follows from (A3) that if $\check{\theta}_0 \leq \hat{\theta}_0$ a.e. in Ω, then $\check{\theta} \leq \hat{\theta}$ a.e. in Ω.

Proof. We can prove this lemma by taking the difference between the inequalities in (3.16), multiplying the both sides of the result by $[\check{\theta} - \hat{\theta}]^+$, and applying Lemma 3.4. ■

Corollary 3.2 *Let $v^\dagger = [w^\dagger, \eta^\dagger] \in [H^1(\Omega) \cap L^\infty(\Omega)] \times H^1(\Omega)$ be a fixed pair of functions, and let $\theta_0^\dagger \in H^1(\Omega)$ be a fixed function. Let $\theta \in H^1(\Omega)$ be the solution to the variational inequality (3.14). Then, it holds that:*

$$|\theta| \leq |\theta_0^\dagger|_{L^\infty(\Omega)} \text{ a.e. in } \Omega.$$

Proof. As easily seen:

$$[|\theta_0^\dagger|_{L^\infty(\Omega)}, 0] \in \partial\Phi(v^\dagger;\ \cdot\ ) \text{ and } [-|\theta_0^\dagger|_{L^\infty(\Omega)}, 0] \in \partial\Phi(v^\dagger;\ \cdot\ ) \text{ in } L^2(\Omega)^2,$$

and

$$-|\theta_0|_{L^\infty(\Omega)} \leq \theta_0^\dagger \leq |\theta_0^\dagger|_{L^\infty(\Omega)} \text{ a.e. in } \Omega.$$

Therefore, with (A3) and Remark 3.1 in mind, the condition "$\theta \leq |\theta_0^\dagger|_{L^\infty(\Omega)}$ a.e. in Ω" (resp. "$\theta \geq -|\theta_0^\dagger|_{L^\infty(\Omega)}$ a.e. in Ω") will be verified by applying Lemma 3.5 as the case when $\check{\theta}_0 = \theta_0^\dagger$, $\check{\theta} = \theta$ and $\hat{\theta}_0 = \hat{\theta} = |\theta_0^\dagger|_{L^\infty(\Omega)}$ (resp. $\check{\theta}_0 = \check{\theta} = -|\theta_0^\dagger|_{L^\infty(\Omega)}$, $\hat{\theta}_0 = \theta_0^\dagger$ and $\hat{\theta} = \theta$). ■

Proof of Theorem 1. Let us assume $0 < h < h_0^\dagger$ with the constant given in (3.7). Then, on the basis of the above lemmas, the existence and uniqueness for $(\mathrm{AP})_h$ is verified through the following steps.

(step 0) let $i = 1$, and fix $[v_0, \theta_0] = [w_0, \eta_0, \theta_0] \in D_*$;

(step 1) obtain a unique solution $v_i = [w_i, \eta_i] \in H^1(\Omega)^2$ to (3.1), by applying Lemma 3.1 as the case when $\theta_0^\dagger = \theta_{i-1}$ and $v_0^\dagger = v_{i-1}$;

(step 2) verify the range constraint property (3.2) with the regularity $\eta_i \in D_{\mathrm{N}}$, by applying Corollary 3.1 as the case when $\theta_0^\dagger = \theta_{i-1}$ and $v_0^\dagger = v_{i-1}$;

(step 3) obtain a unique solution $\theta_i \in H^1(\Omega)$ to (3.3), by applying Lemma 3.3 as the case when $v^\dagger = v_i$ and $\theta_0^\dagger = \theta_{i-1}$;

(step 4) verify the L^∞-estimate (3.4), by applying Corollary 3.2 as the case when $v^\dagger = v_i$, $\theta_0^\dagger = \theta_{i-1}$ and $\theta = \theta_i$;

(step 5) let the value of the index i proceed to the next one, i.e. $i \leftarrow i + 1$, and return to (step 1).

Next, we verify the inequality of energy-dissipation (3.5). Let us put $\varpi = v_{i-1}$ in (3.1). Then, by relying on the convexities of functionals, it is deduced that:

$$
\begin{aligned}
&\frac{1}{h}|v_i - v_{i-1}|^2_{L^2(\Omega)^2} + \frac{1}{2}|\nabla v_i|^2_{L^2(\Omega)^{2\times N}} - \frac{1}{2}|\nabla v_{i-1}|^2_{L^2(\Omega)^{2\times N}} \\
&-c\int_\Omega (w_i - u)(w_i - w_{i-1})\,dx + \int_\Omega g(v_i)\,dx - \int_\Omega g(v_{i-1})\,dx \\
&+\int_\Omega \alpha(\eta_i)|\nabla\theta_{i-1}|\,dx - \int_\Omega \alpha(\eta_{i-1})|\nabla\theta_{i-1}|\,dx \\
&+\nu\int_\Omega \beta(w_i)|\nabla\theta_{i-1}|^2\,dx - \nu\int_\Omega \beta(w_{i-1})|\nabla\theta_{i-1}|^2\,dx \\
&+\int_\Omega I_{[0,1]}(w_i)\,dx \le \int_\Omega I_{[0,1]}(w_{i-1})\,dx, \text{ for } i = 1, 2, 3, \cdots.
\end{aligned}
$$

Additionally, noting that:

$$
\begin{aligned}
&-c(w_i - u)(w_i - w_{i-1}) \\
&\quad = -\frac{c}{2}|w_i - u|^2 + \frac{c}{2}|w_{i-1} - u|^2 - \frac{c}{2}|w_i - w_{i-1}|^2, \text{ a.e. in } \Omega, \\
&\quad \text{for } i = 1, 2, 3, \cdots,
\end{aligned}
$$

and having (2.3) and (3.7) in mind, we further compute that:

$$
\begin{aligned}
&\frac{1}{2h}|v_i - v_{i-1}|^2_{L^2(\Omega)^2} + \frac{1}{2}|\nabla v_i|^2_{L^2(\Omega)^{2\times N}} - \frac{c}{2}|w_i - u|^2_{L^2(\Omega)} + \frac{1}{2}|w_i - \eta_i|^2_{L^2(\Omega)} \\
&\quad + \int_\Omega \alpha(\eta_i)|\nabla\theta_{i-1}|\,dx + \nu\int_\Omega \beta(w_i)|\nabla\theta_{i-1}|^2\,dx + \int_\Omega I_{[0,1]}(w_i)\,dx \\
&\le \frac{1}{2}|\nabla v_{i-1}|^2_{L^2(\Omega)^{2\times N}} - \frac{c}{2}|w_{i-1} - u|^2_{L^2(\Omega)} + \frac{1}{2}|w_{i-1} - \eta_{i-1}|^2_{L^2(\Omega)} \\
&\quad + \int_\Omega \alpha(\eta_{i-1})|\nabla\theta_{i-1}|\,dx + \nu\int_\Omega \beta(w_{i-1})|\nabla\theta_{i-1}|^2\,dx + \int_\Omega I_{[0,1]}(w_{i-1})\,dx, \\
&\text{for } i = 1, 2, 3, \cdots.
\end{aligned}
\tag{3.17}
$$

On the other hand, let us put $\omega = \theta_{i-1}$ in (3.3). Then, we have:

$$\begin{aligned}&\frac{1}{h}|\sqrt{\alpha_0(v_i)}(\theta_i - \theta_{i-1})|^2_{L^2(\Omega)} + \int_\Omega \alpha(v_i)|\nabla\theta_i|\,dx + \nu\int_\Omega \beta(v_i)|\nabla\theta_i|^2\,dx \\ &\qquad - \int_\Omega \alpha(\eta_i)|\nabla\theta_{i-1}|\,dx - \nu\int_\Omega \beta(w_i)|\nabla\theta_{i-1}|^2\,dx \le 0, \text{ for } i = 1,2,3,\cdots.\end{aligned} \tag{3.18}$$

Now, the inequality of energy-dissipation (3.5) will be obtained by taking the sum of (3.17) and (3.18). ∎

4 Proof of the Main Theorem

Throughout this section, we assume that $0 < h < h_0^\dagger$ with the constant as in (3.7), and we denote by $\{[v_i, \theta_i] = [w_i, \eta_i, \theta_i] \,|\, i \in \mathbb{N}\} \subset H^1(\Omega)^3$ the solution to the approximating problem $(\mathrm{AP})_h$ with the initial data $[v_0, \theta_0] = [w_0, \eta_0, \theta_0] \in D_*$.

Based on these, let us define three kinds of time-interpolations $[\widehat{v}_h, \widehat{\theta}_h] = [\widehat{w}_h, \widehat{\eta}_h, \widehat{\theta}_h] \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))^3$, $[\overline{v}_h, \overline{\theta}_h] = [\overline{w}_h, \overline{\eta}_h, \overline{\theta}_h] \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))^3$, $[\underline{v}_h, \underline{\theta}_h] = [\underline{w}_h, \underline{\eta}_h, \underline{\theta}_h] \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))^3$ with the abbreviations $\overline{v}_h = [\overline{w}_h, \overline{\eta}_h]$, $\underline{v}_h = [\underline{w}_h, \underline{\eta}_h]$ and $\widehat{v}_h = [\widehat{w}_h, \widehat{\eta}_h]$, by putting:

$$\begin{cases} \bullet\ [\overline{v}_h(t), \overline{\theta}_h(t)] = [\overline{w}_h(t), \overline{\eta}_h(t), \overline{\theta}_h(t)] := [v_i, \theta_i] = [w_i, \eta_i, \theta_i] \text{ in } L^2(\Omega)^3, \\ \quad \text{if } t \in ((i-1)h, ih] \text{ for some } i \in \mathbb{N}, \text{ and } [\overline{v}_h(0), \overline{\theta}_h(0)] := [v_0, \theta_0] \\ \quad \text{in } L^2(\Omega)^3, \\ \bullet\ [\underline{v}_h(t), \underline{\theta}_h(t)] = [\underline{w}_h(t), \underline{\eta}_h(t), \underline{\theta}_h(t)] := [v_{i-1}, \theta_{i-1}] = [w_{i-1}, \eta_{i-1}, \theta_{i-1}] \\ \quad \text{in } L^2(\Omega)^3, \text{ if } t \in [(i-1)h, ih) \text{ for some } i \in \mathbb{N}, \\ \bullet\ [\widehat{v}_h(t), \widehat{\theta}_h(t)] = [\widehat{w}_h(t), \widehat{\eta}_h(t), \widehat{\theta}_h(t)] := [v_i, \theta_i] + \left(\frac{t}{h} - i\right)[v_i - v_{i-1}, \theta_i - \theta_{i-1}] \\ \quad \text{in } L^2(\Omega)^3, \text{ if } t \in [(i-1)h, ih) \text{ for some } i \in \mathbb{N}, \end{cases} \tag{4.1}$$

for all $t \ge 0$. Then, from (1.4), (3.2), (3.4) and (3.5), it is inferred that:

$$\begin{cases} \bullet\ \{[\widehat{v}_h, \widehat{\theta}_h] = [\widehat{w}_h, \widehat{\eta}_h, \widehat{\theta}_h] \,|\, 0 < h < h_0^\dagger\} \text{ is bounded in} \\ \quad W^{1,2}(0,T; L^2(\Omega))^3 \cap L^\infty(0,T; H^1(\Omega))^3, \\ \bullet\ \{[\overline{v}_h, \overline{\theta}_h] = [\overline{w}_h, \overline{\eta}_h, \overline{\theta}_h], [\underline{v}_h, \underline{\theta}_h] = [\underline{w}_h, \underline{\eta}_h, \underline{\theta}_h] \,|\, 0 < h < h_0^\dagger\} \\ \quad \text{is bounded in } L^\infty(0,T; H^1(\Omega))^3, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \{\overline{v}_h(t,x), \underline{v}_h(t,x), \widehat{v}_h(t,x) \,|\, 0 < h < h_0^\dagger\} \subset [0,1]^2, \\ \{\overline{\theta}_h(t,x), \underline{\theta}_h(t,x), \widehat{\theta}_h(t,x) \,|\, 0 < h < h_0^\dagger\} \subset [-|\theta_0|_{L^\infty(\Omega)}, |\theta_0|_{L^\infty(\Omega)}], \end{cases} \tag{4.3}$$
$$\text{for a.e. } x \in \Omega \text{ and any } t \in [0,T].$$

Therefore, by applying the compactness theory of Aubin's type (cf. [29]), we find a sequence:

$$h_0^\dagger > h_1 > \cdots > h_n \searrow 0 \text{ as } n \to \infty,$$

and a triplet of functions $[v, \theta] = [w, \eta, \theta] \in C([0,T]; L^2(\Omega))^3$ with the abbreviation $v = [w, \eta]$, such that:

$$[v, \theta] \in W^{1,2}(0,T; L^2(\Omega))^3 \cap L^\infty(0,T; H^1(\Omega))^3, \tag{4.4}$$

$$[v(t,x),\theta(t,x)] \in [0,1]^2 \times [-|\theta_0|_{L^\infty(\Omega)}, |\theta_0|_{L^\infty(\Omega)}], \quad \text{for a.e. } x \in \Omega \text{ and any } t \in [0,T], \tag{4.5}$$

$$\left\{\begin{array}{l} \widehat{v}_n = [\widehat{w}_n, \widehat{\eta}_n] := \widehat{v}_{h_n} \to v \\ \quad \text{in } C([0,T];L^2(\Omega))^2, \text{ weakly in } W^{1,2}(0,T;L^2(\Omega))^2, \\ \quad \text{weakly-}* \text{ in } L^\infty(0,T;H^1(\Omega))^2 \text{ and pointwise sense a.e. in } Q, \\ \overline{v}_n = [\overline{w}_n, \overline{\eta}_n] := \overline{v}_{h_n} \to v \text{ and } \underline{v}_n = [\underline{w}_n, \underline{\eta}_n] := \underline{v}_{h_n} \to v \\ \quad \text{weakly-}* \text{ in } L^\infty(0,T;H^1(\Omega))^2 \text{ and pointwise sense a.e. in } Q, \end{array}\right. \text{ as } n \to \infty, \tag{4.6}$$

and

$$\left\{\begin{array}{l} \widehat{\theta}_n := \widehat{\theta}_{h_n} \to \theta \\ \quad \text{in } C([0,T];L^2(\Omega)), \text{ weakly in } W^{1,2}(0,T;L^2(\Omega)), \\ \quad \text{weakly-}* \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ and pointwise sense a.e. in } Q, \\ \overline{\theta}_n := \overline{\theta}_{h_n} \to \theta \text{ and } \underline{\theta}_n := \underline{\theta}_{h_n} \to \theta \\ \quad \text{weakly-}* \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ and pointwise sense a.e. in } Q, \end{array}\right. \text{ as } n \to \infty. \tag{4.7}$$

Now, for the proof of the Main Theorem, we prepare some additional lemmas.

Lemma 4.1 (Mosco convergence) *Let $I \subset (0,T)$ be any open interval. Let $\hat{\Phi}^I : L^2(I;L^2(\Omega)) \longrightarrow [0,\infty]$ and $\hat{\Phi}_n^I : L^2(I;L^2(\Omega)) \longrightarrow [0,\infty]$, $n \in \mathbb{N}$, be functionals, defined as:*

$$\zeta \in L^2(I;L^2(\Omega)) \mapsto \hat{\Phi}^I(\zeta) := \int_I \Phi(v(t);\zeta(t))\,dt,$$

and

$$\zeta \in L^2(I;L^2(\Omega)) \mapsto \hat{\Phi}_n^I(\zeta) := \int_I \Phi(\overline{v}_n(t);\zeta(t))\,dt, \ n \in \mathbb{N}, \tag{4.8}$$

by using $v = [w,\eta] \in L^2(0,T;L^2(\Omega))^2$ and $\overline{v}_n = [\overline{w}_n, \overline{\eta}_n] \in L^2(0,T;L^2(\Omega))^2$, $n \in \mathbb{N}$, as in (4.4)-(4.6). Then, the following two items hold.

(I) $\hat{\Phi}^I$ and $\hat{\Phi}_n^I$, $n \in \mathbb{N}$, are proper l.s.c and convex functions on $L^2(I;L^2(\Omega))$, such that $D(\hat{\Phi}^I) = D(\hat{\Phi}_n^I) = L^2(I;H^1(\Omega))$, for all $n \in \mathbb{N}$.

(II) $\hat{\Phi}_n^I \to \hat{\Phi}^I$ on $L^2(I;L^2(\Omega))$, in the sense of Mosco, as $n \to \infty$.

Proof. Since the item (I) is a straightforward consequence from (A2)-(A3), Notation 3 and (4.2)-(4.5), we can concentrate to the proof of the item (II).

For the verification of the condition of lower-bound, let us take a sequence $\{\zeta_n^\dagger \,|\, n \in \mathbb{N}\} \subset L^2(I;L^2(\Omega))$ with a function $\zeta^\dagger \in L^2(I;L^2(\Omega))$ and a subsequence $\{\zeta_{n_k}^\dagger \,|\, k \in \mathbb{N}\} \subset \{\zeta_n^\dagger\}$, to suppose the following non-trivial situation:

$$\left\{\begin{array}{l} \zeta_n^\dagger \to \zeta^\dagger \text{ weakly in } L^2(I;L^2(\Omega)) \text{ as } n \to \infty, \\ \liminf_{n\to\infty} \hat{\Phi}_n^I(\zeta_n^\dagger) = \lim_{k\to\infty} \hat{\Phi}_{n_k}^I(\zeta_{n_k}^\dagger) < \infty. \end{array}\right. \tag{4.9}$$

Then, due to (A3) and (4.8)-(4.9), the subsequence $\{\theta_{n_k}^\dagger\}$ must be bounded in $L^2(I;H^1(\Omega))$. So, taking a subsequence if necessary, we may also suppose that:

$$\zeta_{n_k}^\dagger \to \zeta^\dagger \text{ weakly in } L^2(I;H^1(\Omega)) \text{ as } k \to \infty.$$

Additionally, having (A3), (4.3) and (4.5)-(4.6) in mind, we can see that:

$$\left\{\begin{array}{l} \alpha(\overline{v}_{n_k})\nabla\zeta^\dagger_{n_k} \to \alpha(v)\nabla\zeta^\dagger, \\ \sqrt{\beta(\overline{v}_{n_k})}\nabla\zeta^\dagger_{n_k} \to \sqrt{\beta(v)}\nabla\zeta^\dagger, \end{array}\right. \text{weakly in } L^2(I;L^2(\Omega)^N) \text{ as } k \to \infty.$$

From the above convergence, the condition of lower-bound is confirmed as follows.

$$\begin{aligned} &\liminf_{n\to\infty} \hat{\Phi}^I_n(\zeta^\dagger_n) = \lim_{k\to\infty} \hat{\Phi}^I_{n_k}(\zeta^\dagger_{n_k}) \\ &= \liminf_{k\to\infty} \left(\left|\alpha(\overline{v}_{n_k})\nabla\zeta^\dagger_{n_k}\right|_{L^1(I;L^1(\Omega)^N)} + \nu \left|\sqrt{\beta(\overline{v}_{n_k})}\nabla\zeta^\dagger_{n_k}\right|^2_{L^2(I;L^2(\Omega)^N)} \right) \\ &\geq \left|\alpha(v)\nabla\zeta^\dagger\right|_{L^1(I;L^1(\Omega)^N)} + \nu \left|\sqrt{\beta(v)}\nabla\zeta^\dagger\right|^2_{L^2(I;L^2(\Omega)^N)} \\ &= \hat{\Phi}^I(\zeta^\dagger). \end{aligned}$$

In the meantime, taking into account (4.3), (4.5)-(4.6) and Lebesgue's dominated convergence theorem, it is inferred that:

$$\begin{aligned} &|\hat{\Phi}^I_n(\zeta^\ddagger) - \hat{\Phi}^I(\zeta^\ddagger)| \\ &\leq \int_I \int_\Omega |\alpha(\overline{\eta}_n) - \alpha(\eta)||\nabla\zeta^\ddagger|\, dxdt + \nu \int_I \int_\Omega |\beta(\overline{w}_n) - \beta(w)||\nabla\zeta^\ddagger|^2\, dxdt \\ &\to 0 \text{ as } n \to \infty, \text{ for any } \zeta^\ddagger \in D(\hat{\Phi}^I(\,\cdot\,)). \end{aligned}$$

This implies the validity of the condition of optimality, for the Mosco convergence $\hat{\Phi}^I_n \to \hat{\Phi}^I$ on $L^2(I;L^2(\Omega))$, as $n \to \infty$. ∎

Lemma 4.2 *In addition to the assumptions as in Lemma 4.1, let us assume that* $\zeta^\ddagger \in L^2(I;H^1(\Omega))$, $\{\zeta^\ddagger_n \,|\, n \in \mathbb{N}\} \subset L^2(I;H^1(\Omega))$, *and*

$$\zeta^\ddagger_n \to \zeta^\ddagger \text{ in } L^2(I;L^2(\Omega)) \text{ and } \hat{\Phi}^I_n(\zeta^\ddagger_n) \to \hat{\Phi}^I(\zeta^\ddagger), \text{ as } n \to \infty. \tag{4.10}$$

Then, $\zeta^\ddagger_n \to \zeta^\ddagger$ *in* $L^2(I;H^1(\Omega))$ *as* $n \to \infty$.

Proof. This lemma is proved by using the following elementary fact:

(†) if $m^\dagger \in \mathbb{N}$, $a^\dagger_\ell \in \mathbb{R}$, $\{a^\dagger_{\ell,n} \,|\, n \in \mathbb{N}\} \subset \mathbb{R}$, $\liminf_{n\to\infty} a^\dagger_{\ell,n} \geq a^\dagger_\ell$, for $\ell = 1, \cdots, m^\dagger$, and

$$\limsup_{n\to\infty} \sum_{\ell=1}^{m^\dagger} a^\dagger_{\ell,n} \leq \sum_{\ell=1}^{m^\dagger} a^\dagger_\ell, \text{ then } \lim_{n\to\infty} a^\dagger_{\ell,n} = a^\dagger_\ell, \text{ for } \ell = 1, \cdots, m^\dagger.$$

In the light of (A3) and (4.10), we may suppose that:

$$\zeta^\ddagger_n \to \zeta^\ddagger \text{ weakly in } L^2(I;H^1(\Omega)), \text{ as } n \to \infty, \tag{4.11}$$

by taking a subsequence if necessary. Subsequently, from (A3), (4.3), (4.5)-(4.6) and (4.11), we can see that:

$$\left\{\begin{array}{l} \nabla\zeta^\ddagger_n \to \nabla\zeta^\ddagger, \\ (\alpha(\overline{v}_n) - \delta_*)\nabla\zeta^\ddagger_n \to (\alpha(v) - \delta_*)\nabla\zeta^\ddagger, \\ \sqrt{\beta(\overline{v}_n) - \delta_*}\nabla\zeta^\ddagger_n \to \sqrt{\beta(v) - \delta_*}\nabla\zeta^\ddagger, \end{array}\right. \text{weakly in } L^2(I;L^2(\Omega)^N) \text{ as } n \to \infty.$$

Based on this, it is observed that:

$$
\begin{cases}
\displaystyle \liminf_{n\to\infty} \int_I \int_\Omega |\nabla \zeta_n^\ddagger(t)|\, dxdt \geq \int_I \int_\Omega |\nabla \zeta^\ddagger(t)|\, dxdt, \\
\displaystyle \liminf_{n\to\infty} \int_I \int_\Omega |\nabla \zeta_n^\ddagger|^2\, dxdt \geq \int_I \int_\Omega |\nabla \zeta^\ddagger(t)|^2\, dxdt,
\end{cases}
\tag{4.12}
$$

and

$$
\begin{cases}
\displaystyle \liminf_{n\to\infty} \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \alpha(\overline{\eta}_n(t)) - 1\right) |\nabla \zeta_n^\ddagger(t)|\, dxdt \\
\displaystyle \qquad \geq \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \alpha(\eta(t)) - 1\right) |\nabla \zeta^\ddagger(t)|\, dxdt, \\
\displaystyle \liminf_{n\to\infty} \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \beta(\overline{w}_n(t)) - 1\right) |\nabla \zeta_n^\ddagger(t)|^2\, dxdt \\
\displaystyle \qquad \geq \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \beta(w(t)) - 1\right) |\nabla \zeta^\ddagger(t)|^2\, dxdt.
\end{cases}
\tag{4.13}
$$

Additionally, it follows from (4.10) that:

$$
\begin{aligned}
&\limsup_{n\to\infty} \left[\int_I \int_\Omega |\nabla \zeta_n^\ddagger(t)|\, dxdt + \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \alpha(\overline{\eta}_n(t)) - 1\right) |\nabla \zeta_n^\ddagger(t)|\, dxdt \right. \\
&\qquad \left. +\nu \int_I \int_\Omega |\nabla \zeta_n^\ddagger(t)|^2\, dxdt + \nu \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \beta(\overline{w}_n(t)) - 1\right) |\nabla \zeta_n^\ddagger(t)|^2\, dxdt\right] \\
&= \frac{1}{\delta_*} \lim_{n\to\infty} \hat{\Phi}_n^I(\zeta_n^\ddagger) = \frac{1}{\delta_*}\hat{\Phi}^I(\zeta^\ddagger) \\
&= \int_I \int_\Omega |\nabla \zeta^\ddagger(t)|\, dxdt + \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \alpha(\eta(t)) - 1\right) |\nabla \zeta^\ddagger(t)|\, dxdt \\
&\qquad +\nu \int_I \int_\Omega |\nabla \zeta^\ddagger(t)|^2\, dxdt + \nu \int_I \int_\Omega \left(\frac{1}{\delta_*}\cdot \beta(w(t)) - 1\right) |\nabla \zeta^\ddagger(t)|^2\, dxdt.
\end{aligned}
\tag{4.14}
$$

By virtue of (4.12)-(4.14), we can apply the fact (†) to infer that:

$$
\nu |\nabla \zeta_n^\ddagger|^2_{L^2(I;L^2(\Omega))^N} \to \nu |\nabla \zeta^\ddagger|^2_{L^2(I;L^2(\Omega)^N)} \text{ as } n \to \infty. \tag{4.15}
$$

The strong convergence of $\{\zeta_n^\ddagger\}$ in $L^2(I; H^1(\Omega))$ will be obtained as a consequence of (4.10), (4.15) and the uniform convexity of the topology. ∎

Proof of the Main Theorem. Note that (4.4)-(4.5) imply that the triplet $[w, \eta, \theta]$ mostly fulfills the condition (S0) in the Main Theorem, except for the regularity $\eta \in L^2(0, T; H^2(\Omega))$. So, our objective is to verify that the limiting triplet $[v, \theta] = [w, \eta, \theta]$ fulfills the conditions (S1)-(S3) with the remaining regularity $\eta \in L^2(0, T; H^2(\Omega))$.

Let us fix any open interval $I \subset (0,T)$. Then, due to (3.1)-(3.3) and (4.1), the triplets $[\overline{v}_n, \overline{\theta}_n] = [\overline{w}_n, \overline{\eta}_n, \overline{\theta}_n]$, $[\underline{v}_n, \underline{\theta}_n] = [\underline{w}_n, \underline{\eta}_n, \underline{\theta}_n]$, $[\widehat{v}_n, \widehat{\theta}_n] = [\widehat{w}_n, \widehat{\eta}_n, \widehat{\theta}_n]$ must fulfill that:

$$\begin{aligned}
&\int_I \big((\widehat{v}_n)_t(t), \overline{v}_n(t) - \varpi\big)_{L^2(\Omega)^2}\,dt + \int_I \big(\nabla\overline{v}_n(t), \nabla(\overline{v}_n(t) - \varpi)\big)_{L^2(\Omega)^{2\times N}}\,dt \\
&-c\int_I \big(\overline{w}_n(t) - u, \overline{w}_n(t) - \varphi\big)_{L^2(\Omega)}\,dt + \int_I \big([\nabla g](\overline{v}_n(t)), \overline{v}_n(t) - \varpi\big)_{L^2(\Omega)^2}\,dt \\
&+\int_I\int_\Omega \big((\overline{\eta}_n(t) - \psi)\alpha'(\overline{\eta}_n(t))|\nabla\underline{\theta}_n(t)| + \nu(\overline{w}_n(t) - \varphi)\beta'(\overline{w}_n(t))|\nabla\underline{\theta}_n(t)|^2\big)\,dxdt \qquad (4.16)\\
&+\int_I\int_\Omega I_{[0,1]}(\overline{w}_n(t))\,dxdt \le \int_I\int_\Omega I_{[0,1]}(\varphi)\,dxdt, \\
&\qquad\text{for any } \varpi = [\varphi,\psi] \in [H^1(\Omega)\cap L^\infty(\Omega)] \times H^1(\Omega) \text{ and any } n \in \mathbb{N},
\end{aligned}$$

and

$$[\overline{\theta}_n, -\alpha_0(\overline{v}_n)(\widehat{\theta}_n)_t] \in \partial\hat{\Phi}_n^I \ \text{ in } L^2(I; L^2(\Omega))^2, \ \text{ for any } n \in \mathbb{N}.$$

Here, from (Fact 1) in Remark 2.4, (4.6)-(4.8) and Lemma 4.1, it follows that:

$$[\theta, -\alpha_0(v)\theta_t] \in \partial\hat{\Phi}^I \text{ in } L^2(I; L^2(\Omega))^2, \tag{4.17}$$

and

$$\hat{\Phi}_n^I(\overline{\theta}_n) \to \hat{\Phi}^I(\theta) \text{ as } n \to \infty. \tag{4.18}$$

In the light of (Fact 0) of Remark 2.2, (I) of Lemma 4.1 and (4.17), we can see that the triplet $[v,\theta] = [w,\eta,\theta]$ fulfills the condition (S3).

Next, from (4.6)-(4.7), (4.18) and Lemma 4.2, it is inferred that:

$$\overline{\theta}_n \to \theta,\ \underline{\theta}_n \to \theta \text{ and } \widehat{\theta}_n \to \theta \text{ in } L^2(I; H^1(\Omega)), \text{ as } n \to \infty. \tag{4.19}$$

In addition, with (2.3), (4.3)-(4.7) and (4.19) in mind, letting $n \to \infty$ in (4.16) yields that:

$$\begin{aligned}
&\int_I \big(v_t(t), v(t) - \varpi\big)_{L^2(\Omega)^2}\,dt + \int_I \big(\nabla v(t), \nabla(v(t) - \varpi)\big)_{L^2(\Omega)^{2\times N}}\,dt \\
&-c\int_I \big(w(t) - u, w(t) - \varphi\big)_{L^2(\Omega)}\,dt + \int_I \big([\nabla g](v(t)), v(t) - \varpi\big)_{L^2(\Omega)^2}\,dt \\
&+\int_I\int_\Omega \big((\eta(t) - \psi)\alpha'(\eta(t))|\nabla\theta(t)| + \nu(w(t) - \varphi)\beta'(w(t))|\nabla\theta(t)|^2\big)\,dxdt \\
&+\int_I\int_\Omega I_{[0,1]}(w(t))\,dxdt \le \int_I\int_\Omega I_{[0,1]}(\varphi)\,dxdt, \\
&\text{for any } \varpi = [\varphi,\psi] \in [H^1(\Omega)\cap L^\infty(\Omega)] \times H^1(\Omega).
\end{aligned}$$

Since the choice of the open interval $I \subset (0,T)$ is arbitrary, we can verify the remaining conditions (S1) and (S2) on the basis of the above inequality and Remark 2.3.

The regularity $\eta \in L^2(0,T; H^2(\Omega))$ will be seen by taking into account the reformulation (2.5) in Remark 2.3. ∎

5 Vision in the future

Finally, we mention about the vision in the future of our study. As the future prospective, we have two research issues, listed below.

1. Unification of the solving method. As mentioned in Introduction, the system (S) is a modified version of "ϕ-η-θ model", proposed in [18], that is aimed to reproduce the grain boundary motion involving the solidification effect. Hence, the system (S) consists of two parts: the part of the so-called Allen-Cahn type equation (1.1) for the solid-liquid phase transition; the part of Kobayashi-Warren-Carter type system $\{(1.2)$-$(1.3)\}$ for the grain boundary motion, originated from [20, 21].

Naturally, this study is a part of the previous works [9, 11, 12, 13, 17, 18, 19, 20, 21, 22, 27, 28, 33, 34], that dealt with the Kobayashi-Warren-Carter type systems. In particular, we note that the time-discretization approach as in Section 3 comes from the ideas conceived in [22], and we further note that this approach include a strong possibility to unify the solving methods for various problems, associated with the Kobayashi-Warren-Carter type system.

In the meantime, it should be noted that there are now a number of previous works concerned with the mathematical studies of phase transitions (cf. [1, 4, 5, 6, 8, 10, 15, 25, 26, 30, 31]), and most of these adopted the double-well functions that belonged to either of the following three cases.

(Case 1) The standard polynomial type (cf. [1, 4, 8, 31]):

$$w \in \mathbb{R} \mapsto \frac{1}{4}w^2(w-1)^2 - u\left(\frac{w^3}{3} - \frac{w^2}{2}\right) \in \mathbb{R}.$$

(Case 2) The case with logarithmic constraint (cf. [10, 30]):

$$w \in (0,1) \mapsto \frac{1}{2}\left(w \log w + (1-w)\log(1-w)\right) - \frac{1}{2}(w-u)^2 \in \mathbb{R}.$$

(Case 3) The case with nonsmooth constraint (cf. [5, 6, 15, 26, 31]): this is just the case adopted in our paper, i.e. the case when the double-well function is provided by (1.6).

In view of these, we are thinking that we need to develop some mathematical theory which provides a unified solving method of a wide scope of coupled systems, including the Allen-Cahn type equations in (Case 1)-(Case 3) and the Kobayashi-Warren-Carter type systems.

2. Extension of the theory to non-isothermal situations. In the systems based on ϕ-η-θ model, the (relative) temperature u is fixed as a constant. It implies that any ϕ-η-θ type model, including our system (S), can respond to only restricted situation, i.e. the isothermal situation.

In this light, the next stage of this study will be on the following non-isothermal mathematical model, originated from Warren-Kobayashi-Lobkovski-Carter [32]:

$$(u-w)_t - \Delta u = 0 \text{ in Q}, \tag{5.1}$$

$$w_t - \Delta w + w(w-1)(w-u-1/2) + \alpha'(w)|\nabla\theta| + \nu\beta'(w)|\nabla\theta|^2 = 0 \text{ in Q}, \tag{5.2}$$

$$\alpha_0(w)\theta_t - \operatorname{div}\left(\alpha(w)\frac{\nabla\theta}{|\nabla\theta|} + 2\nu\beta(w)\nabla\theta\right) = 0 \text{ in } Q, \tag{5.3}$$

subject to the suitable initial-boundary condition.

The above model can be said as an enhanced (generalized) version of ϕ-η-θ model, because it is formulated as a coupled system of the heat equation (5.1), the Allen-Cahn type equation (5.2), and the singular type diffusion equation (5.3) similar to (1.3). However, we must note the point that the orientation order η disappears in the non-isothermal model, because it is identified with the solidification order w.

From physical point of view, the identification $\eta = w$ could be a possible and reasonable simplification. But, from mathematical point of view, it might be better to develop a powerful theory which provide the unified solving method for non-isothermal models, regardless of the simplification $\eta = w$.

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