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MATRIX MEANS OF FINITE ORDERS

DINH TRUNG HOA, TOAN M. HO, AND HIROYUKI OSAKA[A]

ABSTRACT. Using the same idea of the definition of means for positive operators by Kubo and Ando, for each natural number n, we can define means of two positive definite matrices of order n and try to study the canonical map: we can describe an one to one corresponding from the class of matrix connections of order n to the class of positive n-monotone functions on $(0,\infty)$ and the range of this corresponding covers the class of interpolation functions of order $2n$. In particular, the space of symmetric connections is isomorphic to the space of symmetric positive n-monotone functions. Moreover, we show that, for each n, the class of n-connections extremely contains that of $(n+2)$- connections.

1. INTRODUCTION

Averaging operations are useful in science. The arithmetic, geometric and harmonic means are the three best-known ones. When we try to generalize these operations on positive definite matrices, there are no difficulty for the arithmetic and harmonic means while the geometric one needs some more efforts: since $A^{\frac{1}{2}}B^{\frac{1}{2}}$ is positive definite if $AB = BA$ and even not symmetric in general for positive definite matrices A, B. Trying to generalize these notions, Fumio Kubo and Tsuyoshi Ando [11] introduced the connections of positive operators on an infinite dimensional Hilbert space H via three axioms. Furthermore, they showed that there is an affine order-isomorphism from the class of means to the class of positive operator monotone functions f on $(0,\infty)$ with $f(1) = 1$. This theory has found a number of applications in operator theory. In particular, Petz [16] connected the theory of monotone metrics with the theory of connections and means by Kubo and Ando. He proved that an operator monotone function $f : (0,\infty) \longrightarrow \mathbb{R}$ satisfying the functional equation

$$f(t) = tf(t^{-1}), \quad t > 0 \tag{1}$$

related to a Morozova-Chentsov function which gives a monotone metric on the manifold of $n \times n$ density matrices.

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Restricting the definition of operator connections from [11] on the set of positive semidefinite matrices of order n, in this paper, we get a concept of matrix connections of order n (or n-connections). The following natural question is one of the motivations of our study: *Will there exist an affine order-isomorphism from the class of n-connections onto the class of positive matrix monotone functions (or of interpolation functions of order n)?*

A *n-monotone* function on $[0, \infty)$ is a function which preserves the order on the set of all $n \times n$ positive semi-definite matrices. Moreover, if f is n-monotone for all $n \in \mathbb{N}$, then f is called *operator monotone*. Note that this is equivalent to that f is a Pick function. Each n-connection σ induces a n-monotone function (so an interpolation function of order n) f on $(0, \infty)$ via the identity $f(t)I_n = I_n\sigma(tI_n)$. This corresponding is an one to one map from the class of n-connections to the class of positive n-monotone functions on $(0, \infty)$ (hence, to the set of interpolation functions of order n).

On the other hand, an interpolation function of order n is a positive function f on $(0, \infty)$ such that for each n-subset $S = \{\lambda_i\}_{i=1}^n$ of $(0, \infty)$ there exists a Pick function h on $(0, \infty)$ interpolating f at S. Using the integral representation of the Pick function, we know that $f(A)$ has the integral representation on $[0, \infty]$ for any positive matrix A (Theorem 3.4). Applying this representation, we can get a 'local' integral formula for a connection of order n corresponding to a n-monotone function on $(0, \infty)$. Furthermore, this 'local' formula also establishes, for each interpolation function f of order $2n$, a connection σ of order n corresponding to the given interpolation function f. Therefore, it shows that the map from the n-connections to the n-monotone functions is one to one with the range containing the interpolation functions of order $2n$. Moreover, we also show that the class of 1-connections is isomorphic to the class of interpolation functions of order 2 and as much as properties we know in the space of n-connections also hold in the space C_{2n} of interpolation functions of order $2n$ (Proposition 4.1 and Proposition 3.8). This gives a hope that the class of n-connections is isomorphic to the class C_{2n}.

Using the definition of symmetric connections, we can also give a corresponding concept for interpolation functions and n-monotone functions. It is shown that the space of n-connections is strictly subset of the space of positive n-monotone functions on $(0, \infty)$ (Corollary 3.9). However, restricting on the symmetric functions, the space of symmetric n-monotone functions is the same as that of symmetric n-connections (Theorem 3.10).

In this short note, we state the results obtaining when we study the connections of finite order positive definite matrices and sketch the ideas of some proofs. For details of the proofs, please see [10].

2. Means of positive linear operators in the sense of Kubo-Ando [11]

Taking an axiomatic approach, Fumio Kubo and Tsuyoshi Ando introduced the generalization of above notions namely connections and means of positive operators as follows.

Definition 2.1 ([11]). A connection σ is a binary operation $A\sigma B$ for positive semi-definite operators A, B on an infinite dimensional Hilbert spaces H, which satisfies the following:

(I) Monotonicity: $A \leq C$ and $B \leq D \Rightarrow A\sigma B \leq C\sigma D$.
(II) Transformer inequality: $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
(III) Continuous from above: $A_n \downarrow A$, $B_n \downarrow B \Rightarrow A_n\sigma B_n \downarrow A\sigma B$.

A mean is a connection with normalization condition, that is $I\sigma I = I$ (I is the identity).

We can show that $I\sigma(xI)$ is a scalar for any positive number x. Hence one can define a function $f(x)$ on $(0, \infty)$ by

$$f(x)I = I\sigma(xI).$$

Then f is monotone and positive on $(0, \infty)$, and, $f(A) = I\sigma A$ for any positive definite operator A, so, f is operator-monotone since σ is monotone. Furthermore, with help of (2), we have the relation

$$A\sigma B = A^{\frac{1}{2}} f[A^{\frac{-1}{2}} B A^{\frac{-1}{2}}] A^{\frac{1}{2}}.$$

On the other hand, the Lowner theory on operator-monotone functions gives an integral representation of operator-monotone functions as follows.

Theorem 2.2 ([11]). *The map which associates each positive Radon measure m on $[0, \infty]$ to a positive operator-monotone function f defined on $(0, \infty)$ by*

$$f(x) = \int_{[0,\infty]} \frac{x(1+t)}{x+t} dm(t) \quad \text{for } x > 0 \tag{2}$$

is an isomorphism from the class of positive Radon measures on $[0, \infty]$ onto the class of positive operator-monotone functions.

3. Means and connections of order n

3.1. **Interpolation functions.** Throughout the paper, let us denote $\mathbb{R}_+$ the subset $(0, \infty)$ of the real line $\mathbb{R}$, M_n the algebra of square matrices of order n with coefficients in $\mathbb{C}$ and M_n^+ the cone of positive semi-definite matrices in M_n. In this section we study some properties of interpolation functions of order n and their local integral representations.

Definition 3.1 ([1]). A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called an *interpolation function of order n* if for any $T, A \in M_n$ with $A > 0$ and $T^*T \leq 1$

$$T^*AT \leq A \quad \Longrightarrow \quad T^*f(A)T \leq f(A).$$

We denote by $\mathcal{C}_n$ the class of all interpolation functions of order n on $\mathbb{R}_+$. Similarly, we can define an interpolation class $\mathcal{C}_n(I)$ for an interval $I \subset \mathbb{R}$ (I: open, closed, half-open) (see [15, Definition 1]). From the above definition it is straightforward to check that $\mathcal{C}_n \circ \mathcal{C}_n \subset \mathcal{C}_n$ (or $\mathcal{C}_n \circ \mathcal{C}_n(I) \subset \mathcal{C}_n(I)$ for any interval I).

Let P' be a set of all positive Pick functions on $\mathbb{R}_+$, i.e., functions of the form

$$h(s) = \int_{[0,\infty]} \frac{(1+t)s}{1+ts}\, d\rho(t), \quad s > 0,$$

where ρ is some positive Radon measure on $[0, \infty]$.

Remark 3.2. For $n \in \mathbb{N}$ denote by P'_n the set of all strictly positive n-monotone functions. We have the following properties can be found in [9], [14], [1], [2], [3] or [4]:

(i) $P' = \cap_{n=1}^{\infty} P'_n$, $P' = \cap_{n=1}^{\infty} \mathcal{C}_n$;
(ii) $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$;
(iii) $P'_{n+1} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq P'_n$, $P'_n \subsetneq \mathcal{C}_n$
(iv) $\mathcal{C}_{2n} \subsetneq P'_n$ [15].

The following useful characterization of a function in $\mathcal{C}_n$ is due to Donoghue (see [6], [7]), and to Ameur (see [1]).

Theorem 3.3. ([4, Corollary 2.4]) *A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{C}_n$ if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ there exists a P'-function h such that $f(\lambda_i) = h(\lambda_i)$ for $i = 1, \ldots, n$.*

We have a local integral representation of every function in $\mathcal{C}_n$ and that the representation will be used in study of matrix connections in the next sections.

Theorem 3.4. ([2, Theorem 7.1]) *Let A be a positive definite matrix in M_n and $f \in \mathcal{C}_n$. Then there exists a positive Radon measure $\rho_{\sigma(A)}$ on $[0, \infty]$ such that*

$$f(A) = \int_{[0,\infty]} A(1+s)(A+s)^{-1} d\rho_{\sigma(A)}(s), \tag{3}$$

where $\sigma(A)$ is the set of eigenvalues of A.

3.2. **Interpolation functions and Means of positive matrices.** Similarly to Definition 2.1 of connections of positive operators on infinite dimensional Hilbert space, we can give the definition of positive matrices of order n: A binary operation σ on M_n^+, $(A, B) \mapsto A\sigma B$ is called a *matrix connection of order n* (or *n-connection*) if it satisfies three axioms an in Definition 2.1. Note that the order relation $A \leq B$

always means the positivity of $B - A$. That $A_1 \geq A_2 \geq \dots$ and A_n converges strongly to A is denoted by $A_n \downarrow A$.

In [11], there is an affine order-isomorphism from the set of connections onto the set of operator monotone functions. In this section, we describe the similar relation between the connections of order n and $P'_n \supsetneq \mathcal{C}_{2n}$. Note that every positive semi-definite matrix can be obtained as a limit of a decreasing sequence of positive definite matrices, from now on, we can always assume that connections are defined on positive definite matrices.

For any n-connection σ, the matrix $I_n\sigma(tI_n)$ is a scalar (see the proof of [11, Theorem 3.2]), and so we can define a function f on $(0, \infty)$ by

$$f(t)I_n = I_n\sigma(tI_n),$$

where I_n is the identity in M_n. As in the proof of [11, Theorem 3.2], using the property (I) of the definition of connection, f is a n-monotone function on $(0, \infty)$.

Now we can state the main theorem.

Theorem 3.5. *For any natural number n there is an injective map Σ from the set of matrix connections of order n to $P'_n \supset \mathcal{C}_{2n}$ associating each connection σ to the function f_σ such that $f_\sigma(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. Furthermore, the range of this map contains $\mathcal{C}_{2n}$.*

Sketch the proof. **Step 1.** Let f be a function belonging to C_n. Define a binary operation σ on positive definite matrices in M_n by:

$$A\sigma B = A^{\frac{1}{2}} f[A^{\frac{-1}{2}} B A^{\frac{-1}{2}}] A^{\frac{1}{2}} \quad \forall A, B > 0. \tag{4}$$

It is straightforward to check that σ satisfies the axiom (III) of the definition of n-connection.

Step 2. To check the axioms (I) and (II), we first show that for $f \in C_n$, there exists a Radon measure $\rho = \rho_{\sigma(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})}$ on $[0, \infty]$ such that

$$f[A^{\frac{-1}{2}} B A^{\frac{-1}{2}}] = \int_{[0,\infty]} A^{\frac{-1}{2}} B A^{\frac{-1}{2}} (1+s)(A^{\frac{-1}{2}} B A^{\frac{-1}{2}} + s)^{-1} d\rho(s),$$

where $\sigma(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})$ is the set of eigenvalues of $A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$. Substituting this equality into (4), we obtian

$$A\sigma B = \int_{[0,\infty]} \frac{1+s}{s} \{(sA) : B\} d\rho(s).$$

step 3. Now for any $f \in C_{2n}$, there exists a Radon measure ρ on $\sigma(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) \cup \sigma(C^{\frac{-1}{2}} D C^{\frac{-1}{2}})$ such that

$$A\sigma B = \int_{[0,\infty]} \frac{1+s}{s} \{(sA) : B\} d\rho(s),$$

$$C\sigma D = \int_{[0,\infty]} \frac{1+s}{s} \{(sC) : D\} d\rho(s).$$

Since $\{(sA):B\} \leq \{(sC):D\}$ for $A \leq C$ and $B \leq D$, the conditions (I), (II) hold. $\square$

Remark 3.6. Since $P_n' \subsetneq \mathcal{C}_n$, the map associating each connection of order n to a function in $\mathcal{C}_n$ as above is not surjective.

3.3. **Decreasing inclusion of the connections of order n.** Via the usual embedding of M_n into M_{n+1}, it is straightforward to check that the classes of connections of order n is decreasing. It is natural to ask the following question: *Is there a matrix mean σ_n of the order n on M_n such that σ_n is not of order $n+1$?*

The following observation gives partially affirmative data to the above question.

Proposition 3.7.

(1) *For any $n \geq 2$ there is a matrix mean σ_n of order n which is not of order $n+2$.*
(2) *There is a matrix mean σ_1 of order 1 which is not of order 2.*

Proof. Denote by Σ_n the image of the class of connections of order n via the map in Theorem 3.5 for each n. Therefore, Σ_n is isomorphic to the class of n-connections (so the sequence $\{\Sigma_n\}$ is decreasing) and $\Sigma_n \subseteq P_n'$. From now on, we can identify the space of n-connections with Σ_n.

(1): On account of Remark 3.2 and Theorem 3.5, we obtain the following inclusion:

$$\begin{aligned}\Sigma_{n+2} \subseteq P_{n+2}' \subseteq \mathcal{C}_{2(n+1)+1} \subseteq \mathcal{C}_{2(n+1)} \subseteq \Sigma_{n+1}\\ \subseteq P_{n+1}' \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq \Sigma_n.\end{aligned}$$

And since $P_{n+2}' \subsetneq P_{n+1}'$, we imply that $\Sigma_{n+2} \subsetneq \Sigma_n$.

(2): Using Remark 3.2 again and the characterization of matrix connections of order one (Subsection 3.5 below), we get

$$\Sigma_2 \subseteq P_2' \subseteq \mathcal{C}_3 \subseteq \mathcal{C}_2 = \Sigma_1.$$

By [4, Proposition 3.14] $P_2' \neq \mathcal{C}_3$, we then have the statement. $\square$

3.4. **Symmetric connections.** As the same in [11], we can recall some notations and properties of connections as follows. Let σ be a n-connection. The *transpose* σ', the *adjoint* σ^* and the *dual* $\sigma^\perp$ of σ are defined by

$$A\sigma' B = B\sigma A, \quad A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}, \quad \sigma^\perp = \sigma'^*.$$

A connection is called *symmetric* if it equals to its transpose. Denoted by Σ_n^{sym} the set of n-monotone representing functions of symmetric n-connections, i.e., Σ_n^{sym} is the image of the set of all symmetric n-connections via the canonical map in Theorem 3.5.

(1) $\sigma + \sigma'$ and $\sigma(:)\sigma'$ are symmetric.

(2) $\omega_l(\sigma)\omega_r = \sigma$; $\omega_r(\sigma)\omega_l = \sigma'$, where $A\omega_l B = A$ and $A\omega_r B = B$.

(3) The n-monotone representing function of the n-connection $\sigma(\tau)\rho$ is $f(x)g[h(x)/f(x)]$, where f, g, h are the representing functions of σ, τ, ρ in Theorem 3.5, respectively, and $A\sigma(\tau)\rho B = (A\sigma B)\tau(A\rho B)$.

(4) σ is symmetric if and only if its n-monotone representing function f is *symmetric*, that is, $f(x) = xf(x^{-1})$.

Each n-connection corresponds to a positive n-monotone function belonging to Σ_n by Theorem 3.5. Therefore, combining with the observation above, we get the following.

Proposition 3.8. *Let $f(x), g(x), h(x)$ belong to Σ_n. Then the following statements hold true:*

(i) $k(x) = xf(x^{-1})$, $f^*(x) = f(x^{-1})^{-1}$, $\frac{x}{f(x)}$, $f(x)g[h(x)/f(x)]$, $af(x) + bg(x)$ *all belong to* Σ_n;

(ii) $f(x) + k(x)$, $\frac{f(x)k(x)}{f(x)+k(x)}$ *all belong to* Σ_n^{sym}.

We know that (from Theorem 3.5):

Corollary 3.9.

$$C_{2n} \subset \Sigma_n \subsetneq P'_n.$$

But if restricting our attention to the class of the symmetric, we get the following equality.

Theorem 3.10.

$$\Sigma_n^{sym} = P_n'^{sym},$$

where $P_n'^{sym}$ is the set of all symmetric functions in P'_n.

However,

Example 3.11. *Let $p(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ be a polynomial which belongs to $P'_2(0,\alpha)$ but does not belong to $P'_3(0,\alpha)$ for some $\alpha > 0$ (see [12]). Let $q(x)$ be the symmetrization of p by*

$$q(x) = p(x) + xp(x^{-1}).$$

Then q is symmetric. However, we can show that q does not belong to $P'_2(0,\alpha)$.

3.5. **Matrix means of order one.** We recall the results in [4] for the sets C_1, C_2 as follows.

- C_1 is the set of all positive functions on $(0,\infty)$.
- C_2 consists of all quasi-concave functions (i.e., $f(s) \le f(t)\max\{1, \frac{s}{t}\}$ for all $s, t > 0$).

For any connection σ of order 1, then the corresponding function f belongs to $\mathcal{C}_2$. Indeed, for any numbers $0 < t \leq s$, we have

$$\begin{aligned} f(t)\max\{1, \frac{s}{t}\} &= (1\sigma t)\frac{s}{t} = \frac{s}{t}\sigma s \\ &\geq 1\sigma s = f(s), \text{ and,} \\ f(s)\max\{1, \frac{t}{s}\} &= (1\sigma s) \\ &\geq 1\sigma t = f(t). \end{aligned}$$

Thus, we can characterize connections of order 1 completely:

a) Every connection σ of order one can be determined uniquely by:
$$x\sigma y = xf(\frac{y}{x}) \quad \forall x, y > 0,$$
where f is an interpolation function in C_2.

b) Every function f in C_2 can be represented uniquely by:
$$f(x) = 1\sigma x \quad \forall x > 0,$$
where σ is a connection of order 1.

From this result, we can easily get a functions in $\mathcal{C}_2$ from the corresponding connections and vise versa. For example, the functions in $\mathcal{C}_2$ which correspond to arithmetic mean, harmonic mean and the geometric mean are $\frac{1+x}{2}$, $\frac{2}{1+x}$ and $x^{\frac{1}{2}}$; and any (positive) linear combination of these functions also belongs to $\mathcal{C}_2$.

If we take the function $f(x) = 2\frac{x}{1+x} + \left(\frac{x}{1+x}\right)^2 \in \mathcal{C}_2 \backslash \mathcal{C}_3$ in [4, Example 3.13], we have a connection σ_f of order 1 which is not of order 2 as follows:

$$\begin{aligned} x\sigma_f y &= xf(\frac{y}{x}) \\ &= 2\frac{xy}{x+y} + \frac{xy^2}{(x+y)^2} \end{aligned}$$

for $x, y \in \mathbb{R}^+$.

4. Toward the conjecture $\mathcal{C}_{2n} = \Sigma_n$

We know that $\mathcal{C}_{2n} \subseteq \Sigma_n \subseteq P_n'$ and $\mathcal{C}_2 = \Sigma_1$. Therefore, we may give a conjecture that, for any positive integer n,

$$\mathcal{C}_{2n} = \Sigma_n \text{ and } \Sigma_n^{sym} = \mathcal{C}_{2n}^{sym}.$$

Even we still do not know whether $\mathcal{C}_{2n} = \Sigma_n$ or not, but they have some similar properties. In particular, the properties of the space Σ_n represented in Proposition 3.8 also hold true when we replace Σ_n by $\mathcal{C}_{2n}$. That is,

Proposition 4.1. *The statements in Proposition 3.8 hold if we replace Σ_n by $\mathcal{C}_{2n}$.*

Note that Proposition 4.1 still holds true in the space C_n.

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Center of Research and Development, Duy Tan University, K7/25 Quang Trung, Da Nang, Viet Nam
E-mail address: trunghoa.math@gmail.com

Institute of Mathematics, 18 Hoang Quoc Viet, Hanoi, Vietnam
E-mail address: hmtoan@math.ac.vn

Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan
E-mail address: osaka@se.ritsumei.ac.jp